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REPRESENTATIONS OVER GREEN ALGEBRAS OF WEAK HOPF ALGEBRAS BASED ON TAFT ALGEBRAS

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Abstract. In this paper, we study the Green ring $r(\mathfrak{w}_n^0)$ of the weak Hopf algebra $\mathfrak{w}_n^0$ based on Taft Hopf algebra $H_n(q)$. Let $R(\mathfrak{w}_n^0) := r(\mathfrak{w}_n^0)\otimes_{\mathbb{Z}}\mathbb{C}$ be the Green algebra corresponding to the Green ring $r(\mathfrak{w}_n^0)$. We first determine all finite dimensional simple modules of the Green algebra $R(\mathfrak{w}_n^0)$, which is based on the observations of the roots of the generating relations associated with the Green ring $r(\mathfrak{w}_n^0)$. Then we show that the nilpotent elements in $r(\mathfrak{w}_n^0)$ can be written as a sum of finite dimensional indecomposable projective $\mathfrak{w}_n^0$-modules. The Jacobson radical $J(r(\mathfrak{w}_n^0))$ of $r(\mathfrak{w}_n^0)$ is a principal ideal, and its rank equals $n-1$. Furthermore, we classify all finite dimensional non-simple indecomposable $R(\mathfrak{w}_n^0)$-modules. It turns out that $R(\mathfrak{w}_n^0)$ has n^2-n+2 simple modules of dimension 1, and n non-simple indecomposable modules of dimension 2.

1. Introduction

Let H be a finite dimensional (weak) Hopf algebra over $\mathbb{C}$. The Green ring $r(H)$ (see [7]) is an abelian group generated by the isomorphism classes $[V]$ of finite dimensional H-modules V modulo the relations $[M\oplus V]=[M]+[V]$. For any H-modules V and M, the multiplication of $r(H)$ is given by the tensor product, that is, $[M][V]=[M\otimes V]$. Then $r(H)$ is an associative ring with identity $[\mathbb{C}]$, where $\mathbb{C}$ is the trivial H-module. Notice that $r(H)$ is an associative ring with a $\mathbb{Z}$-basis $\{[V]\,|\,V\in \operatorname{ind}(H)\}$, where $\operatorname{ind}(H)$ denotes the set of finite dimensional indecomposable H-modules up to isomorphism.

Recently, many researches of Green rings of (weak) Hopf algebras have been done. Chen et al. described explicitly the generators and generating relations of the Green rings of Taft algebras in [3]. Then in [9], Li and Zhang investigated the Green rings of generalized Taft algebras, and determined all finite dimensional indecomposable modules of the corresponding Green algebras. In

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[12] and [13], Wang et al. studied the Green rings of finite dimensional pointed rank one Hopf algebras of nilpotent type and non-nilpotent type, respectively.

In [8], Li gave a version of definition of weak Hopf algebras. Then, under some conditions, Aizawa and Isaac classified the weak Hopf algebras corresponding to $U_q(sl_n)$ in [1]. In [5], Cheng and Li gave some results of a weak Hopf algebra associated with $U_q(sl_2)$. Cheng [4] determined the structures of weak Hopf algebras based on Sweedler's Hopf algebra. Su and Yang in [10], introduced two classes of weak Hopf algebras $\mathfrak{w}^s_{n,d}$ of type s ($s = 0, 1$) based on generalized Taft Hopf algebras $H_{n,d}(q)$, and described the Green rings of them.

In this paper, the Green ring $r(\mathfrak{w}^0_n)$ of the weak Hopf algebra $\mathfrak{w}^0_n$ based on Taft Hopf algebra $H_n(q)$ has been studied. We describe explicitly the Jacobson radical $J(r(\mathfrak{w}^0_n))$ of $r(\mathfrak{w}^0_n)$, and classify all non-isomorphic finite dimensional indecomposable modules over the Green algebra $R(\mathfrak{w}^0_n)$. This paper is organized as follows. In Section 2, we recall the definitions of Taft algebra $H_n(q)$ and the weak Hopf algebra $\mathfrak{w}^0_n$ corresponding to $H_n(q)$. Moreover, the classification of finite dimensional indecomposable $\mathfrak{w}^0_n$-modules, and the Clebsch-Gordan rules of them will be reviewed. In Section 3, we determine all finite dimensional simple modules over the Green algebra $R(\mathfrak{w}^0_n)$, which is based on the observation of the roots of the generating relations associated with $r(\mathfrak{w}^0_n)$. Then the Jacobson $J(r(\mathfrak{w}^0_n))$ of $r(\mathfrak{w}^0_n)$ will be described. Furthermore, we classify all finite dimensional non-simple indecomposable $R(\mathfrak{w}^0_n)$-modules.

2. Preliminaries

Throughout this paper, the letters $\mathbb{Z}$ and $\mathbb{C}$ stand for the ring of integers and the field of complex numbers, respectively. Unless otherwise stated, all algebras, (weak) Hopf algebras and modules are defined over $\mathbb{C}$, all modules are left modules and finite dimensional; dim, $\otimes$ and Hom stand for $\dim_{\mathbb{C}}$, $\otimes_{\mathbb{C}}$ and $\mathrm{Hom}_{\mathbb{C}}$, respectively.

Let q be an n-th primitive root of unity. The Taft algebra $H_n(q)$ is generated by two elements G and X subject to the relations (see [11]):

$$G^n = 1,\ X^n = 0,\ XG = qGX.$$

The comultiplication Δ, counit ε and the antipode S are respectively given by

$$\begin{aligned} &\Delta(G) = G \otimes G, && \varepsilon(G) = 1, && S(G) = G^{-1} = G^{n-1}, \\ &\Delta(X) = X \otimes G + 1 \otimes X, && \varepsilon(X) = 0, && S(X) = -q^{-1}G^{n-1}X. \end{aligned}$$

Notice that $\dim(H_n(q)) = n^2$ and $\{G^iX^j \mid 0 \le i, j \le n-1\}$ forms a $\mathbb{C}$-basis for $H_n(q)$.

We continue to review the definition of the weak Hopf algebra $\mathfrak{w}^0_n$, which is the generalization of Taft algebra $H_n(q)$. Let B_0 be a bialgebras generated by two elements g and x subject to the relations (see [2] and [6])

$$g = g^{n+1},\ xg = qgx,\ x^n = 0.$$

The coalgebra structure is given by

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1,$$
$$\Delta(x) = g^n \otimes x + x \otimes g, \qquad \varepsilon(x) = 0,$$

As a bialgebra, $\mathfrak{w}_n^0 = B_0/(x-g^n x)$, where $(x-g^n x)$ is a bi-ideal of B_0 generated by $x - g^n x$.

Recall that a bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ is called a weak Hopf algebra if H is equipped with a weak antipode $T \in \mathrm{Hom}(H, H)$ such that $T * id * T = T$ and $id * T * id = id$, where $*$ is the convolution map in $\mathrm{Hom}(H, H)$. In [10], Su and Yang proved that the map $T : \mathfrak{w}_n^0 \to \mathfrak{w}_n^0$ defined as follows:

$$1 \mapsto 1,\ g \mapsto g^{n-1},\ x \mapsto -q^{-1} g^{n-1} x$$

is the weak antipode of $\mathfrak{w}_n^0$. Thus, $\mathfrak{w}_n^0$ is a weak Hopf algebra with $\dim(\mathfrak{w}_n^0) = n^2 + 1$, and the set

$$\{g^i x^j g^n, 1 - g^n \mid 0 \leq i, j \leq n-1\}$$

forms a PBW basis for $\mathfrak{w}_n^0$. Moreover, $\mathfrak{w}_n^0/(1-g^n) = H_n(q)$ as Hopf algebras (see [10, Proposition 3.4]).

In the following, we list all non-isomorphic indecomposable $\mathfrak{w}_n^0$-modules, which was studied in [10].

$M(l, i)$ $(0 \leq l \leq n-1,\ i \in \mathbb{Z}_n)$: Let $M(l, i)$ be a $\mathbb{C}$-vector space with a basis $\{v_0^{(i)}, \dots, v_l^{(i)}\}$. The action of $\mathfrak{w}_n^0$ on $M(l, i)$ is given by

$$g \cdot v_j^{(i)} = q^{i-j} v_j^{(i)},\ 0 \leq j \leq l,$$
$$x \cdot v_j^{(i)} = \begin{cases} v_{j+1}^{(i)}, & 0 \leq j \leq l-1, \\ 0, & j = l. \end{cases}$$

It is noted that $M(l, i)$ is simple if and only if $l = 0$, and $M(l, i)$ is projective if and only if $l = n-1$.

N: It is a 1-dimensional vector space ($N = \mathbb{C}$ as vector spaces), and the action of $\mathfrak{w}_n^0$ on N is determined by

$$g \cdot 1 = x \cdot 1 = 0.$$

Notice that N is simple and projective.

The set $\{M(l, i), N \mid 0 \leq l \leq n-1, i \in \mathbb{Z}_n\}$ forms a complete set of non-isomorphic indecomposable modules over $\mathfrak{w}_n^0$ (see [10, Proposition 4.1]).

The decomposition formulas of the tensor product of two indecomposable $\mathfrak{w}_n^0$-modules are listed as follows (see [10, Theorem 4.2]).

Let $0 \leq u, v, l \leq n-1$ and $i, j \in \mathbb{Z}_n$. Then as $\mathfrak{w}_n^0$-modules, we have

1. (a) If $u + v \leq n-1$, then

$$M(u, i) \otimes M(v, j) \cong M(v, j) \otimes M(u, i) \cong \bigoplus_{k=0}^{\min\{u,v\}} M(u+v-2k, i+j-k).$$

(b) If $u+v \geq n$, set $t = u+v-(n-1)$, then

$$\begin{aligned}
&M(u,i)\otimes M(v,j)\\
\cong\ &M(v,j)\otimes M(u,i)\\
\cong\ &\bigoplus_{k=0}^{t} M(n-1,i+j-k) \oplus \bigoplus_{k=t+1}^{\min\{u,v\}} M(u+v-2k,i+j-k).
\end{aligned}$$

2. $M(l,i)\otimes N \cong N\otimes M(l,i) \cong \underbrace{N\oplus\cdots\oplus N}_{l+1 \text{ copies}}$.

3. $N\otimes N \cong N$.

Let $F_t(y,z)$ be a kind of generalized Fibonacci polynomial defined by

$$F_{t+2}(y,z) = zF_{t+1}(y,z) - yF_t(y,z)$$

for $t \geq 0$, while $F_0(y,z)=0$, $F_1(y,z)=1$. The generalized Fibonacci polynomial $F_t(y,z)$ has the following general form:

$$F_t(y,z) = \sum_{i=0}^{[\frac{t-1}{2}]} (-1)^i \binom{t-1-i}{i} y^i z^{t-1-2i},$$

where $\left[\frac{t-1}{2}\right]$ denotes the biggest integer which is not bigger than $\frac{t-1}{2}$.

It is easy to see that the Green ring $r(\mathfrak{w}_n^0)$ is commutative. Furthermore, it follows from [10, Theorem 5.5] that $r(\mathfrak{w}_n^0) \cong \mathbb{Z}[x,y,z]/I$ as a ring isomorphism, where I is the ideal generated by the relations

$$x^n-1,\ (y-x-1)F_n(x,y),\ xz-z,\ yz-2z,\ z^2-z.$$

3. The representation theory of $R(\mathfrak{w}_n^0)$

The aim of this section is to classify all indecomposable modules over the Green algebra $R(\mathfrak{w}_n^0) = r(\mathfrak{w}_n^0)\otimes_{\mathbb{Z}}\mathbb{C}$ up to isomorphism.

By Section 2, we know that $r(\mathfrak{w}_n^0)$ is commutative, and $r(\mathfrak{w}_n^0) \cong \mathbb{Z}[x,y,z]/(x^n-1,(y-x-1)F_n(x,y),xz-z,yz-2z,z^2-z)$. We first consider all non-isomorphic simple modules over $R(\mathfrak{w}_n^0)$, which is closely related to the solutions of the following system of equations in $\mathbb{C}$

$$\left\{\begin{array}{l} x^n-1=0,\\ (y-x-1)F_n(x,y)=0,\\ xz-z=0,\\ yz-2z=0,\\ z^2-z=0. \end{array}\right. \tag{3.1}$$

Thanks to the work in [9], we can easily obtain the solutions of the system (3.1). In [9], the authors considered the solutions of the following system of equations in $\mathbb{C}$:

$$\left\{\begin{array}{l} x^n-1=0,\\ (y-x-1)F_n(x,y)=0. \end{array}\right. \tag{3.2}$$

It turns out that when $n \geq 2$, the system (3.2) has $n^2 - n + 1$ distinct solutions given by

$$\Phi = \{(1, 2)\} \cup \{(\omega_k, \sigma_{k,j}) \,|\, 0 \leq k \leq n-1, 1 \leq j \leq n-1\},$$

where $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ is an n-th root of unity in $\mathbb{C}$, and $\sigma_{k,j} = 2\sqrt{\omega_k}\cos\frac{j\pi}{n}$, $1 \leq j \leq n-1$ (see [9, Lemma 4.4]).

Proposition 3.1. *The system of equations* (3.1) *has* n^2-n+2 *distinct solutions in* $\mathbb{C}$*, which are determined by*

$$\Omega = \{(1, 2, 0), (1, 2, 1)\} \cup \{(\omega_k, \sigma_{k,j}, 0) \,|\, 0 \leq k \leq n-1, 1 \leq j \leq n-1\}.$$

Proof. It follows from a straightforward verification. □

Since the Green algebra $R(\mathfrak{w}_n^0)$ is commutative, each simple $R(\mathfrak{w}_n^0)$-module over $\mathbb{C}$ is 1-dimensional. Proposition 3.1 states that the system (3.1) has $n^2 - n + 2$ distinct solutions given by Ω. For each solution $\omega = (\alpha, \beta, \gamma) \in \Omega$, one can define a simple $R(\mathfrak{w}_n^0)$-module $\mathbb{C}_{\alpha,\beta,\gamma}$ on the vector space $\mathbb{C}$ by $x \cdot 1 = \alpha$, $y \cdot 1 = \beta$ and $z \cdot 1 = \gamma$.

It is clear that $\omega \mapsto \mathbb{C}_\omega$ ($\omega \in \Omega$) gives a one to one correspondence between the set of solutions for the system (3.1) and the set of the isomorphism classes of simple $R(\mathfrak{w}_n^0)$-modules. Hence one gets the following result.

Proposition 3.2. *The set* $\{\mathbb{C}_\omega \,|\, \omega \in \Omega\}$ *forms a complete set of non-isomorphic simple modules over* $R(\mathfrak{w}_n^0)$.

Proof. It follows from Proposition 3.1. □

Theorem 3.3. *The Jacobson radical* $J(r(\mathfrak{w}_n^0))$ *of* $r(\mathfrak{w}_n^0)$ $(n \geq 2)$ *has a* $\mathbb{Z}$*-basis*

$$\{[M(n-1, i)] - [M(n-1, i+1)] \,|\, 0 \leq i \leq n-2\}.$$

Proof. On one hand, for each $0 \leq i \leq n-2$, it is easy to check that $([M(n-1, i)] - [M(n-1, i+1)])^2 = 0$, which implies that $[M(n-1, i)] - [M(n-1, i+1)]$ is an nilpotent element. Hence $[M(n-1, i)] - [M(n-1, i+1)] \in J(r(\mathfrak{w}_n^0))$ for each $0 \leq i \leq n-2$. On the other hand, by Wedderburn-Artin Theorem, one can easily know that the dimension of $J(r(\mathfrak{w}_n^0)) \otimes_{\mathbb{Z}} \mathbb{C}$ $\dim(J(r(\mathfrak{w}_n^0))) = (n^2 + 1) - (n^2 - n + 2) = n - 1$. The $\mathbb{C}$-linear independence of $[M(n-1, i)] - [M(n-1, i+1)]$, $0 \leq i \leq n-2$, is obvious. Thus, the proof is completed. □

Corollary 3.4. *The Jacobson radical* $J(r(\mathfrak{w}_n^0))$ *is a principal ideal of* $r(\mathfrak{w}_n^0)$ *generated by the element* $[M(n-1, 0)] - [M(n-1, 1)]$.

Proof. For any $0 \leq i \leq n-2$, we have

$$[M(n-1, i)] - [M(n-1, i+1)] = [M(0, i)]([M(n-1, 0)] - [M(n-1, 1)]).$$

Notice that $[M(0, i)]$ is invertible in $r(\mathfrak{w}_n^0)$. Hence, we complete the proof. □

In the sequel, we will classify all non-isomorphic non-simple indecomposable $R(\mathfrak{w}_n^0)$-modules. Recall that for each $0 \le k \le n-1$, $\omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ given in Proposition 3.1 is an n-th root of unity. Let $V(k)$ be a 2-dimensional $\mathbb{C}$-vector space with a basis $\{v_1, v_2\}$. Define the action of $R(\mathfrak{w}_n^0)$ on $V(k)$ as follows:

$$x \cdot v_i = \omega_k v_i,\ y \cdot v_1 = (1+\omega_k)v_1,\ y \cdot v_2 = v_1 + (1+\omega_k)v_2,\ z \cdot v_i = 0.$$

Lemma 3.5. *Let $0 \le k \le n-1$. Then $V(k)$ is a non-simple indecomposable module of $R(\mathfrak{w}_n^0)$. Furthermore, $V(k) \cong V(t)$ if and only if $k = t$.*

Proof. It is easy to check that $V(k)$ is a module of $R(\mathfrak{w}_n^0)$. By $y \cdot v_1 = (1+\omega_k)v_1$ and $y \cdot v_2 = v_1 + (1+\omega_k)v_2$, the action of y corresponds to the following Jordan block matrix

$$\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix}.$$

Hence $V(k)$ is indecomposable. Note that $\mathbb{C}v_1 \cong \mathbb{C}_{\omega_k, 1+\omega_k, 0}$, which follows that $V(k)$ is reducible by Proposition 3.1. The rest of the proof is straightforward. □

In the following, we will show that $V(k)$ $(0 \le k \le n-1)$ are all non-isomorphic non-simple indecomposable modules over $R(\mathfrak{w}_n^0)$.

Theorem 3.6. *Let V be a non-simple indecomposable $R(\mathfrak{w}_n^0)$-module. Then there exists k, $0 \le k \le n-1$, such that $V \cong V(k)$.*

Proof. We first show $\dim(V) = 2$. Assume that the dimension of V is greater than 2. By the commutativity of the generators x, y and z, $x^n = 1$ and $z^2 = z$, there exists a basis $\{v_1, v_2, \dots, v_t\}$ $(t > 2)$ of V such that the matrices X and Z corresponding to the actions of x and z on V are diagonal, and the matrix Y corresponding to the action of y on V is Jordan type, respectively. It is easy to see that Y is a Jordan block matrix, otherwise, V is not indecomposable. Suppose that

$$X = \begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_{t-1} & \\ & & & & x_t \end{pmatrix}, \quad Y = \begin{pmatrix} y & 1 & & & \\ & y & 1 & & \\ & & \ddots & \ddots & \\ & & & y & 1 \\ & & & & y \end{pmatrix}.$$

By a straightforward computation, we get

$$XY = \begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_{t-1} & \\ & & & & x_t \end{pmatrix} \begin{pmatrix} y & 1 & & & \\ & y & 1 & & \\ & & \ddots & \ddots & \\ & & & y & 1 \\ & & & & y \end{pmatrix}$$

$$= \begin{pmatrix} x_1 y & x_1 & & & \\ & x_2 y & x_2 & & \\ & & \ddots & \ddots & \\ & & & x_{t-1} y & x_{t-1} \\ & & & & x_t y \end{pmatrix},$$

and

$$YX = \begin{pmatrix} y & 1 & & & \\ & y & 1 & & \\ & & \ddots & \ddots & \\ & & & y & 1 \\ & & & & y \end{pmatrix} \begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & \ddots & & \\ & & & x_{t-1} & \\ & & & & x_t \end{pmatrix}$$
$$= \begin{pmatrix} x_1 y & x_2 & & & \\ & x_2 y & x_3 & & \\ & & \ddots & \ddots & \\ & & & x_{t-1} y & x_t \\ & & & & x_t y \end{pmatrix}.$$

Since $xy = yx$, one can easily get that $x_1 = x_2 = \cdots = x_t$, i.e., $X = \omega_k E_t$, where $0 \leq k \leq n-1$ and E_t is the $t \times t$ identity matrix.

Note that the matrix Y must satisfy the following matrix equation

$$(Y - (1+\omega_k)E_t) \prod_{j=1}^{n-1} (Y - \sigma_{k,j} E_t) = 0,$$

where $\{\sigma_{k,j} \mid 1 \leq j \leq n-1\}$ is the set of roots of the generalized Fibonacci polynomial $F_n(\omega_k, y)$.

Since the matrix $Y - \sigma_{k,j} E_t$ is a Jordan block for each $1 \leq j \leq n-1$, $Y - \sigma_{k,j} E_t$ is invertible if $y \neq \sigma_{k,j}$. Furthermore, $\sigma_{k,j}$ for $1 \leq j \leq n-1$ are distinct and $\sigma_{k,k} = 1 + \omega_k$. It follows that $t = 2$, $\omega_k \neq 1$ and $y = 1 + \omega_k$. Now, we already get that $X = \omega_k E_2$, E_2 is the 2×2 identity matrix, and

$$Y = \begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix}.$$

Since the matrix Z is diagonal and $Z^2 = Z$, we only need to consider the following four conditions:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix} \neq 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is contrary with $yz = 2z$.

And

$$\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 1+\omega_k & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix}=\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 0 & 1+\omega_k \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+\omega_k & 1 \\ 0 & 1+\omega_k \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 1+\omega_k \end{pmatrix},$$

which contradict with $yz = zy$. Hence

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the proof is completed. □

Corollary 3.7. *The set $\{V(k) \mid 0 \le k \le n-1\}$ forms a complete set of non-simple indecomposable $R(\mathfrak{w}_n^0)$-modules up to isomorphism.*

Corollary 3.8. *The set $\{\mathbb{C}_\omega, V(k) \mid \omega \in \Omega, 0 \le k \le n-1\}$ forms a complete set of non-isomorphic indecomposable modules of $R(\mathfrak{w}_n^0)$.*

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