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SINGULARITY FORMATION FOR A NONLINEAR VARIATIONAL SINE-GORDON EQUATION IN A MULTIDIMENSIONAL SPACE

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Abstract. We study a multidimensional nonlinear variational sine-Gordon equation, which can be used to describe long waves on a dipole chain in the continuum limit. By using the method of characteristics, we show that a solution of a nonlinear variational sine-Gordon equation with certain initial data in a multidimensional space has a singularity in finite time.

1. Introduction

The multidimensional nonlinear variational sine-Gordon equation is

$$u_{tt} - c(u)\nabla \cdot (c(u)\nabla u) + \frac{\omega^2}{2}\sin(2u) = 0, \tag{1.1}$$

where ω is a constant. Here the wave speed $c(u) > 0$ satisfies

$$c^2(u) = a\cos^2 u + b\sin^2 u \tag{1.2}$$

for some constants $a > 0$ and $b > 0$. If $a = b$, then the equation (1.1) reduces to the multidimensional nonlinear Klein-Gordon equation. In fact, this equation originates from the study of long waves on a dipole chain in the continuum limit which occurs in an anisotropic system [16]. The wave in a massive liquid crystal director field is the example of the system as well. The equation (1.1) can be regarded as the sine-Gordon version of the nonlinear variational wave equation

$$u_{tt} - c(u)\nabla \cdot (c(u)\nabla u) = 0$$

which is used in the theory of nematic liquid crystals. Refer to [1–5,8–11,13–15] for more information and mathematical results on the nonlinear variational

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wave equation and its sine-Gordon version. In particular, singularity of solutions were studied in [6, 7, 12].

In this paper we are concerned with the singularity formation of smooth solutions for the multidimensional nonlinear variational sine-Gordon equations. Let $u = u(t, r)$ and $r = |x|$, where $x = (x_1, \dots, x_l)$. Then (1.1) transforms into

$$u_{tt} - c(u)(c(u)u_r)_r - \frac{(l-1)c^2(u)u_r}{r} + \frac{\omega^2}{2}\sin(2u) = 0. \tag{1.3}$$

The wave speed $c(\cdot) \in C^2$ is assumed to satisfy

$$0 < c_0 \le c(\cdot) \le c_1, \quad |c'(\cdot)| \le c_1 \tag{1.4}$$

for some constants $c_0 > 0$ and $c_1 > 0$.

We notice that a constant solution of (1.3) becomes a critical point of $c(\cdot)$ which causes some difficulty in using the characteristic method employed in [6, 7] for the singularity formation. Thus we have to find another type of solutions to make use of the characteristic method. In this paper, we overcome the difficulty by finding a proper function that is not a critical point of the wave speed. In fact, it is known that there exists a unique solution $y = Y(t)$, which is a spatial independent solution of (1.3), for a second order differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{2}\omega^2 \sin(2y(t)) = 0$$

with the data

$$y(0) = k_0 \quad \text{and} \quad \frac{dy}{dt}(0) = 0.$$

Then $Y(t) \approx k_0 \cos(\omega t)$ for sufficiently small $0 < k_0 \ll 1$, $0 < Y(t) \le k_0$ and $c'(k_0) > 0$ for a certain time period. Let such a time period be $[0, t_0)$. Let us define

$$\frac{c'(Y(0))}{4} = \frac{c'(k_0)}{4} := c_2.$$

Let us state the singularity formation which is the main result of the paper.

Theorem 1.1 (Main theorem)**.** *Let us assume that* (1.3) *with initial data*

$$\begin{cases} u(0, r) = Y(0) + \varepsilon\psi\left(\frac{r - r_0}{\varepsilon}\right), \\ u_t(0, r) = (-c(u(0, r)) + \varepsilon)u_r(0, r) \end{cases} \tag{1.5}$$

has a smooth solution $u(t, r) \in C^1([0, T) \times \mathbb{R}^+)$. Here ε and r_0 are positive constants to be determined. If $c'(k_0) > 0$, and ψ satisfies

$$\psi(\cdot) \in C_c^1((-1, 1)),\ \psi \not\equiv 0,\ \text{and}\ \psi'(0) < -2\max\left\{\frac{3}{2c_0 r_0^d}, \frac{16c_1^2 3^d}{r_0 c_0 c_2}\right\}, \tag{1.6}$$

where $d = \frac{l-1}{2} > 0$, then $T < \infty$.

2. The proof of main theorem

The purpose of this section is to establish the singularity formation of (1.1). We prove Theorem 1.1 by using the method of [6,7]. More precisely, by deriving the energy equation of the Riemann variables, we show that the blow-up result for the equation (1.1) occurs at a certain finite time by the characteristic method.

Let us define $d = \frac{l-1}{2} > 0$ and take r_0 small enough so that $r_0/c_1 \ll t_0$. Let us introduce new variables

$$U := (u_t + c(u)u_r)r^d, \quad S := (u_t - c(u)u_r)r^d, \tag{2.1}$$

which yields that

$$u_t = \frac{U+S}{2r^d}, \quad u_r = \frac{U-S}{2c(u)r^d}.$$

Equation (1.3) transforms into the system of equations for U and S:

$$\begin{cases} U_t - c(u)U_r = \dfrac{c'}{4cr^d}(U^2 - S^2) - \frac{dc}{r}S - \dfrac{r^d\omega^2\sin(2u)}{2}, \\ S_t + c(u)S_r = \dfrac{c'}{4cr^d}(S^2 - U^2) + \frac{dc}{r}U - \dfrac{r^d\omega^2\sin(2u)}{2}. \end{cases} \tag{2.2}$$

From (2.2) we can obtain the following equation of conservative form

$$\frac{\partial}{\partial t}(U^2 + S^2 + 2r^{2d}\omega^2\sin^2 u) + \frac{\partial}{\partial r}[c(u)(S^2 - U^2)] = 0. \tag{2.3}$$

According to (1.5), the initial data for the Riemann variables are given by

$$U(0,r) = \varepsilon r^d u_r(0,r),\ S(0,r) = (-2c\big(u(0,r)\big) + \varepsilon)r^d u_r(0,r). \tag{2.4}$$

As a consequence, the blow-up of the smooth solutions to Cauchy problem (1.3) with (1.5) is transformed into the blow-up of the smooth solution to (2.2) with (2.4).

Next, let us introduce an energy function which is uniformly bounded by its initial energy. For the time being, let $\varepsilon < \frac{3}{4}r_0$. From (1.6) and (2.4) one has

$$U(0,r) = 0 = S(0,r)$$

for all $r \in \big[0,\infty) \setminus [r_0 - \varepsilon, r_0 + \varepsilon\big]$. Let us define the energy function $E(t)$ as follows:

$$E(t) = \int_0^{+\infty} \big[U^2(t,r) + S^2(t,r) + 2r^{2d}\omega^2\sin^2 u(t,r)\big]dr. \tag{2.5}$$

From integration (2.3) on t, we notice that E is time-independent, which implies $E(t) = E(0)$. Furthermore, we can obtain

$$\begin{aligned} E(0) &= \int_0^{+\infty} \big(U^2(0,r) + S^2(0,r) + 2r^{2d}\omega^2\sin^2 u(0,r)\big)dr \\ &= \int_{r_0-\varepsilon}^{r_0+\varepsilon} \Big\{\Big[\varepsilon^2 + (\varepsilon - 2c(u(0,r)))^2\Big]\Big(\psi'(\frac{r-r_0}{\varepsilon})\Big)^2 + 2\omega^2\sin^2 u(0,r)\Big\}r^{2d}dr. \end{aligned}$$

Then

$$E(0) \le (2r_0)^{2d}\Big\{4\omega^2\varepsilon + \big(\varepsilon^2 + (2c_1+\varepsilon)^2\big)\int_{r_0-\varepsilon}^{r_0+\varepsilon}\Big(\psi'(\frac{r-r_0}{\varepsilon})\Big)^2 dr\Big\}$$

$$\le \Big\{4\omega^2 + \big[1+(2c_1+1)^2\big]\int_{-1}^{1}(\psi'(z))^2 dz\Big\}(2r_0)^{2d}\varepsilon := Mr_0^{2d}\varepsilon \tag{2.6}$$

for some positive constant M.

Let $(r_1, 0)$ and $(r_2, 0)$ be two points with $0 < r_1 < r_2$. We define two characteristic curves passing through them, respectively. One is a positive characteristic curve $r^+(t)$ emitting from $(r_1, 0)$ and the other is a negative characteristic curve $r^-(t)$ emitting from $(r_2, 0)$ as follows:

$$\frac{dr^+(t)}{dt} = c\big(u\big(t, r^+(t)\big)\big), \quad r^+(0) = r_1,$$

and

$$\frac{dr^-(t)}{dt} = -c\big(u\big(t, r^-(t)\big)\big), \quad r^-(0) = r_2.$$

From (1.4), let us choose r_1 and r_2 so that

$$r_2 - r_1 \le \frac{2c_0\big(r_0-\varepsilon\big)}{c_1}.$$

Then it follows that two characteristic curves $r^+(t)$ and $r^-(t)$ will intersect at the point (r_m, t_m) with $r_1 < r_m < r_2$ and $t_m < \frac{r_0-\varepsilon}{c_1}$. Applying the Green's formula for equation (2.3) to a region enclosed by the characteristic curves $r = r^{\pm}(t)$ and r-axis which is depicted in Figure 1, we have

$$\begin{aligned}&\int_{r_1}^{r_m}(U^2(t^+(r), r) + r^{2d}\omega^2\sin^2 u(t^+(r), r))dr\\ &+\int_{r_m}^{r_2}(S^2(t^-(r), r) + r^{2d}\omega^2\sin^2 u(t^-(r), r))dr\\ &= \frac{1}{2}\int_{r_1}^{r_2}\Big[U^2(0,r) + S^2(0,r) + 2r^{2d}\omega^2\sin^2 u(0,r)\Big]dr \le Mr_0^{2d}\varepsilon\end{aligned} \tag{2.7}$$

by the energy estimate (2.6).

Let $r = \hat{r}(t)$ with

$$\frac{d\hat{r}(t)}{dt} = c\big(u\big(t, \hat{r}(t)\big)\big), \quad \hat{r}(0) = r_0 \quad (r_1 < r_0 < r_2)$$

be another positive characteristic curve starting from $(r_0, 0)$. We will show that $c'(u)$ is always positive on the curve $r = \hat{r}(t)$ if ε is sufficiently small. Let us define $\partial_1 := \partial_t + c(u)\partial_r$. From (2.1), one obtain

$$\partial_1 u\big(t, \hat{r}(t)\big) = \frac{U(t, \hat{r}(t))}{\hat{r}^d(t)}. \tag{2.8}$$

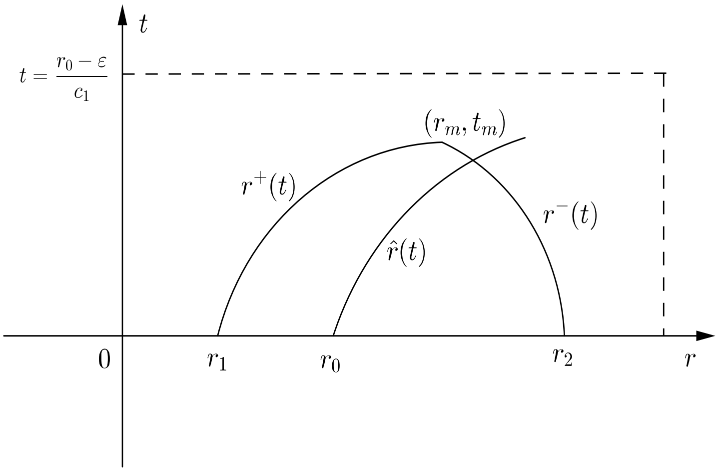

FIGURE 1. The region bounded by two characteristics

If we integrate (2.8) along the characteristic from 0 to t with $t < \frac{r_0-\varepsilon}{c_1}$ and use (2.7), then it follows that

$$\begin{aligned}\big|u(t,\hat{r}(t)) - u(0,r_0)\big| &\leq \int_0^t \frac{|U(v,r)|}{\hat{r}^d(v)}dv \leq \sqrt{t}\Big(\int_0^t \Big(\frac{U(v,\hat{r}(v))}{\hat{r}^d(v)}\Big)^2 dv\Big)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{r_0-\varepsilon}}{r_0^d\sqrt{c_1c_0}}\left(\int_{r_0}^{r} U^2(\hat{t}(s),s)ds\right)^{\frac{1}{2}} \leq \sqrt{\frac{M(r_0-\varepsilon)}{c_1c_0}}\sqrt{\varepsilon}.\end{aligned}$$

From the smoothness of $c(\cdot)$, there exists ε_0 small enough such that for any $\varepsilon \in (0,\varepsilon_0)$ it follows that on the curve $r = \hat{r}(t)$

$$0 < c_2 = \frac{c'(k_0)}{4} \leq \frac{c'\big(u(0,r(0))\big)}{2} \leq c'\big(u(t,\hat{r}(t))\big). \tag{2.9}$$

We now prove that the solutions of (2.2) can blow up for the given initial condition (2.4), namely, $S(t,\hat{r}(t))$ becomes infinite at finite time before $t = \frac{r_0-\varepsilon}{c_1}$. First, let us take small enough $r_0 > 0$ so that

$$\Big(\frac{c_1M}{4c_0^2} + \frac{3^d}{2c_1}\omega^2\Big)r_0^{1+d} < \frac{1}{3}, \quad r_0^{2d} \leq \frac{c_2c_0^2}{16c_1 3^d(c_1^2M + 2c_0^2 3^d\omega^2)}. \tag{2.10}$$

Furthermore, let us choose

$$\varepsilon < \min\Big\{\varepsilon_0, c_0, \frac{3}{4}r_0, \frac{r_0^{\frac{1}{2}-d}}{4d}\sqrt{\frac{c_0}{c_1M}}, \Big(\frac{r_0}{c_0}\Big)^{\frac{3}{2}}\frac{\sqrt{c_1M}}{4d}\Big\} \tag{2.11}$$

and

$$
\begin{aligned}
S(0,r_0) &= \big(-2c\big(u(0,r_0)\big)+\varepsilon\big)r_0^d\psi'(0)\\
&> 2c_0r_0^d\max\Big\{\frac{3}{2c_0r_0^d},\ \frac{16c_1^2 3^d}{r_0c_0c_2}\Big\}\\
&> \max\Big\{3,\ \frac{32c_1^2(3r_0)^d}{(r_0-\varepsilon)c_2}\Big\}. \qquad (2.12)
\end{aligned}
$$

Lemma 2.1. *Let $S(0,r_0)$ satisfy* (2.12)*. Then $S\big(t,\hat r(t)\big)$ goes to infinity as t approaches some t^*, which is less than $\frac{r_0-\varepsilon}{c_1}$, for sufficiently small ε in* (2.11)*.*

Proof. First, let us show that $S\big(t,\hat r(t)\big)>1$ when $S\big(t,\hat r(t)\big)$ is smooth in $[0,\frac{r_0-\varepsilon}{c_1})$. Let us assume on the contrary. That is, there exists $\tau\in(0,\ \frac{r_0-\varepsilon}{c_1})$ such that $S\big(t,\hat r(t)\big)$ belongs to $C^1([0,\tau])$, $S\big(t,\hat r(t)\big)>1$ for all $t\in\big[0,\tau\big)$, and $S\big(\tau,\hat r(\tau)\big)=1$. From (1.4), (2.1) and (2.9), one has

$$
\begin{aligned}
\partial_1\Big(\frac{1}{S}\Big) &= -\frac{1}{S^2}\Big[\frac{c'}{4cr^d}(S^2-U^2)+\frac{dc}{r}U-\frac{r^d}{2}\omega^2\sin(2u)\Big]\\
&\le -\frac{c'}{4cr^d}+\frac{1}{S^2}\Big(\frac{c'}{4cr^d}U^2+\frac{dc}{r}|U|+\frac{r^d}{2}\omega^2\sin(2u)\Big)\\
&\le -\frac{c_2}{4c_1(3r_0)^d}+\frac{1}{S^2}\Big(\frac{c_2}{4c_0r_0^d}U^2+\frac{dc_1}{r_0}|U|+\frac{(3r_0)^d\omega^2}{2}\Big). \qquad (2.13)
\end{aligned}
$$

If we integrate (2.13) from 0 to τ along the positive characteristic $r=\hat r(t)$ with $\tau<\frac{r_0-\varepsilon}{c_1}$, then it follows that

$$
\begin{aligned}
\frac{1}{S\big(\tau,\hat r(\tau)\big)} &\le \frac{1}{S(0,r_0)}+\int_0^\tau\frac{1}{S^2}\Big(\frac{c_2}{4c_0r_0^d}U^2+\frac{dc_1}{r_0}|U|+\frac{(3r_0)^d\omega^2}{2}\Big)dt\\
&\le \frac{1}{S(0,r_0)}+\frac{c_1}{4c_0^2r_0^d}\int_{r_0}^{\hat r}U^2dr+\sqrt{\frac{c_1d^2}{c_0r_0}}\Big(\int_{r_0}^{\hat r}U^2dr\Big)^{\frac12}+\frac{(3r_0)^d\omega^2}{2}\cdot\frac{r_0}{c_1}\\
&\le \frac{1}{S(0,r_0)}+\frac{c_1}{4c_0^2}Mr_0^d\varepsilon+d\sqrt{\frac{c_1}{c_0r_0}}\sqrt{M\varepsilon}r_0^d+\frac{3^d}{2c_1}\omega^2r_0^{1+d}\\
&\le \frac{1}{S(0,r_0)}+\frac{c_1}{4c_0^2}Mr_0^{1+d}+\sqrt{\frac{c_1M}{c_0r_0}}dr_0^d\varepsilon+\frac{3^d}{2c_1}\omega^2r_0^{1+d}\\
&\le \frac{2}{3}+\sqrt{\frac{c_1M}{c_0r_0}}dr_0^d\varepsilon<1,
\end{aligned}
$$

which contradicts to $S\big(\tau,\hat r(\tau)\big)=1$. In fact, the last inequality comes from the choice of r_0 in (2.10) and ε in (2.11). Therefore, $S\big(t,\hat r(t)\big)>1$ holds for $0\le t<\frac{r_0-\varepsilon}{c_1}$.

Finally, let us prove that $S(t,\hat r(t))$ becomes infinite at finite time before $t=\frac{r_0-\varepsilon}{c_1}$. Integration of (2.13) from 0 to t with $t<\frac{r_0-\varepsilon}{c_1}$, which together with the choice of $S(0,r_0)$ in (2.12) yields

$$\begin{aligned}\frac{1}{S\big(t,\hat{r}(t)\big)} &\leq \frac{1}{S(0,r_0)} - \int_0^t \frac{c_2}{4c_1(3r_0)^d}dt \\ &\quad + \int_0^t \frac{1}{S^2}\Big(\frac{c_2}{4c_0 r_0^d}U^2 + \frac{dc_1}{r_0}|U| + \frac{(3r_0)^d\omega^2}{2}\Big)dt \\ &\leq \frac{1}{S(0,r_0)} - \frac{c_2}{4c_1(3r_0)^d}t + \Big(\frac{c_1 M}{4c_0^2} + \frac{3^d\omega^2}{2c_1}\Big)r_0^{1+d} + dr_0^{d-\frac{1}{2}}\sqrt{\frac{c_1 M}{c_0}}\varepsilon \\ &\leq \frac{c_2}{4c_1(3r_0)^d}\Big(\frac{r_0-\varepsilon}{2c_1} - t + \frac{r_0}{8c_1}\Big). \qquad (2.14)\end{aligned}$$

We find that there exists t^* satisfying

$$t^* = \frac{r_0-\varepsilon}{2c_1} + \frac{r_0}{8c_1} = \frac{5r_0-4\varepsilon}{8c_1} < \frac{r_0-\varepsilon}{c_1}$$

and

$$\frac{1}{S\big(t,\hat{r}(t)\big)} \to 0$$

as $t \to t^*$ by taking $r_0 > 4\varepsilon/3$. Hence $S(t,\hat{r}(t)) \to \infty$ as $t \to t^*$. □

According to the above argument, we show that the singularity of the smooth solution u occurs at some point, that is $\nabla u \to \infty$ as $t \to t^*$ for some $t^* < \frac{r_0-\varepsilon}{c_1}$. Therefore we complete the proof of the Main theorem.

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