ANTI-FLIPS OF THE BLOW-UPS OF THE PROJECTIVE SPACES AT TORUS INVARIANT POINTS

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Abstract. We explicitly construct the smooth toric Fano variety which is isomorphic to the blow-up of the projective space at torus invariant points in codimension one by anti-flips.

1. Introduction

The blow-up of the projective plane $\mathbb{P}^2$ at 1, 2 or 3 torus invariant points is isomorphic to the Hirzebruch surface F_1 of degree 1, the del Pezzo surface S_7 of degree 7 or the del Pezzo surface S_6 of degree 6, respectively. As is well known, they are *Fano* varieties, that is, their anti-canonical divisors are ample. For $d \geq 3$, let B_n^d be the blow-up of $\mathbb{P}^d$ at n torus invariant points. Then, B_1^d is a Fano variety, while B_n^d is not a Fano variety for $n \geq 2$ (see e.g. [3]).

In this paper, we construct the smooth Fano variety $\widetilde{B}_n^d$ which is birationally equivalent to B_n^d by a finite succession of *anti-flips*. For this construction, we investigate the *primitive relations* for toric anti-flips. The construction of $\widetilde{B}_n^d$ is a generalization for the theory of pseudo-symmetric and symmetric toric varieties by Ewald [5] and Voskresenskij-Klyachko [12]. The following is the main theorem of this paper.

Theorem 1.1 (Theorem 3.3). *Let B_n^d be the blow-up of $\mathbb{P}^d$ at n torus invariant points. Suppose that $d \geq 3$ and $n \geq 2$.*

If $2n - 1 < d$ or d is even, then there exists a finite succession $B_n^d \dashrightarrow \widetilde{B}_n^d$ of anti-flips such that $\widetilde{B}_n^d$ is a smooth toric Fano variety. More precisely, if $2n - 1 < d$, then $\widetilde{B}_n^d$ has a $(\mathbb{P}^1)^n$-bundle structure over $\mathbb{P}^{d-n}$, while otherwise every extremal ray of the Kleiman-Mori cone of $\widetilde{B}_n^d$ is of small type.

If $2n - 1 \geq d$ and d is odd, then there does not exist a smooth Fano variety which is isomorphic to B_n^d in codimension one.

This paper is organized as follows: Section 2 is devoted to the calculation of anti-flips by using the notion of primitive relations. This will be useful for the birational geometry of toric varieties. In Section 3, we prove Theorem 1.1. In each case, we can explicitly describe the number of anti-flips to obtain the smooth toric Fano variety (see Theorem 3.3).

2. Preliminary

In this section, we quickly review the notion of primitive collections and relations for toric varieties introduced by Batyrev [1] (see also [2, 11]). They are convenient to describe the fan associated to a smooth complete toric variety. By using them, we can explicitly calculate some important operations in the birational geometry like blow-ups, blow-downs and anti-flips. For the basic theory of the toric geometry, see [4, 7, 9]. Moreover, for the toric Mori theory, see [6, 8, 10]. We will work over an algebraically closed field $K = \overline{K}$.

Let $X = X_\Sigma$ be the smooth projective toric d-fold associated to a fan Σ in $N := \mathbb{Z}^d$. Put

$$\mathrm{G}(\Sigma) := \{\text{the primitive generators for 1-dimensional cones in } \Sigma\} \subset N.$$

There is a one-to-one correspondence between $\mathrm{G}(\Sigma)$ and the set of torus invariant prime divisors on X. In particular, for another smooth projective toric d-fold $X' = X_{\Sigma'}$, if $\mathrm{G}(\Sigma) = \mathrm{G}(\Sigma')$, then X and X' are isomorphic in codimension one.

The following notion is very important for our theory.

Definition 2.1. A non-empty subset $P \subset \mathrm{G}(\Sigma)$ is a *primitive collction* of Σ (or X) if

   (1) P does not generate a cone in Σ, while
   (2) $P \setminus \{u\}$ generates a cone in Σ for any $u \in \mathrm{G}(\Sigma)$.

For a primitive collection $P = \{u_1, \ldots, u_l\}$, there exists a unique cone $\sigma(P) \in \Sigma$ which contains $u_1 + \cdots + u_l$ in its relative interior. Let $\{v_1, \ldots, v_m\} \subset \mathrm{G}(\Sigma)$ be the generators for $\sigma(P)$ (in particular, m is the dimension of $\sigma(P)$). Then, we have a linear relation

$$u_1 + \cdots + u_l = a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m$ are positive integers. We call this relation the *primitive relation* for $P = \{u_1, \ldots, u_l\}$.

It is well known that for any primitive collection of $X = X_\Sigma$, we can associate a numerical 1-cycle on X by using its primitive relation (see e.g. [1]). In particular, the numerical 1-cycles associated to the primitive collections of X generate the Kleiman-Mori cone $\mathrm{NE}(X)$, which is always polyhedral when X is

a projective toric variety. So, we say that a primitive collection (or a primitive relation) is *extremal* if the associated numerical 1-cycle generates an extremal ray $R \subset \mathrm{NE}(X)$. Thus, we obtain the extremal contraction $\varphi_R : X \to \overline{X}$ associated to an extremal primitive relation $u_1 + \cdots + u_l = a_1 v_1 + \cdots + a_m v_m$. For the type of φ_R, the following hold:

- If $m = 0$, then $\varphi_R : X \to \overline{X}$ is a *Mori fiber space*. In this case, φ_R is nothing but a $\mathbb{P}^{l-1}$-bundle structure over $\overline{X}$.
- If $m = 1$, then $\varphi_R : X \to \overline{X}$ is a *divisorial contraction*. Moreover, if $a_1 = 1$, then φ_R is a blow-up of $\overline{X}$ along a $(d-l)$-dimensional torus invariant subvariety.
- If $m \geq 2$, then $\varphi_R : X \to \overline{X}$ is a *small contraction*. Moreover, if $l - (a_1 + \cdots + a_m) > 0$ (resp. < 0, $= 0$), then φ_R is a *flipping* (resp. *anti-flipping*, *flopping*) contraction. For a flipping (resp. anti-flipping, flopping) contraction $\varphi_R : X \to \overline{X}$, we can construct a *flip* (resp. *anti-flip*, *flop*)

$$X \dashrightarrow X^+$$

$$\varphi_R \searrow \quad \swarrow \varphi_R^+$$

$$\overline{X}$$
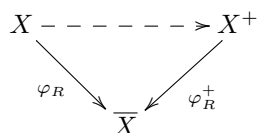
by the toric Mori theory. An anti-flip is the inverse operation of a flip, that is, if $X \dashrightarrow X^+$ is a flip, then its inverse rational map $X^+ \dashrightarrow X$ is an anti-flip.

We should remark that Σ can be recovered by all the primitive relations of Σ. Namely, we can describe a fan by giving all the primitive relations of it.

For blow-ups, the primitive collections can be calculated as follows:

Proposition 2.2 ([11], Theorem 4.3). *Let $X = X_\Sigma$ be a smooth projective toric variety and $X' \to X$ be the blow-up with respect to an l-dimensional cone $\langle u_1, \ldots, u_l \rangle$ in Σ, where $\{u_1, \ldots, u_l\} \subset \mathrm{G}(\Sigma)$ (remark that l is the codimension of the center of the blow-up). Put $v := u_1 + \cdots + u_l$. Then, the primitive collections of X' are*

(1) $\{u_1, \ldots, u_l\}$ *(whose primitive relation is $u_1 + \cdots + u_l = v$),*
(2) *any primitive collection P in Σ such that $\{u_1, \ldots, u_l\} \not\subset P$ and*
(3) $(P \setminus \{u_1, \ldots, u_l\}) \cup \{v\}$ *for any primitive collection P of Σ such that $P \setminus \{u_1, \ldots, u_l\}$ is a minimal element in*

$$\{Q \setminus \{u_1, \ldots, u_l\} \mid Q \text{ is a primitive collection of } \Sigma, \ Q \cap \{u_1, \ldots, u_l\} \neq \emptyset \}.$$

Conversely, we can calculate the primitive collections of a blow-down of a smooth projective toric variety.

Proposition 2.3 ([11], Corollary 4.9). *Let $X = X_\Sigma$ be a smooth projective toric variety and $X \to \overline{X}$ the blow-down with respect to an extremal primitive relation*

$$u_1 + \cdots + u_l = v$$

of X. Then, the primitive collections of $\overline{X}$ are

    (1) *any primitive collection P of Σ such that $P \neq \{u_1, \ldots, u_l\}$ and $v \notin P$, and*

    (2) *$(P \setminus \{v\}) \cup \{u_1, \ldots, u_l\}$ for any primitive collection P of Σ such that $v \in P$ and $(P \setminus \{v\}) \cup S$ is not a primitive collection of Σ for any proper subset $S \subset \{u_1, \ldots, u_l\}$.*

By combining Propositions 2.2 and 2.3, we obtain the following. This theorem is essential for the calculations in Section 3.

Theorem 2.4. *Let $X = X_\Sigma$ be a smooth projective toric variety and $X \dashrightarrow X^+ = X_{\Sigma^+}$ the anti-flip with respect to an extremal primitive relation*

$$u_1 + \cdots + u_l = v_1 + \cdots + v_m \ (l < m)$$

of Σ, where $\{u_1, \ldots, u_l, v_1, \ldots, v_m\} \subset \mathrm{G}(\Sigma)$. Then, X^+ is also a smooth projective toric variety and the primitive collections of Σ^+ are

    (1) *$\{v_1, \ldots, v_m\}$ whose primitive relation is*

$$v_1 + \cdots + v_m = u_1 + \cdots + u_l,$$

    (2) *any primitive collection P of Σ such that $\{v_1, \ldots, v_m\} \not\subset P$ and $P \neq \{u_1, \ldots, u_l\}$, and*

    (3) *$(P \setminus \{v_1, \ldots, v_m\}) \cup \{u_1, \ldots, u_l\}$ for any primitive collection P of Σ such that $P \setminus \{v_1, \ldots, v_m\}$ is a minimal element in*

$$\{P \setminus \{v_1, \ldots, v_m\} \mid P \text{ is a primitive collection of } \Sigma, \ P \cap \{v_1, \ldots, v_m\} \neq \emptyset\}$$

    and $(P \setminus \{v_1, \ldots, v_m\}) \cup S$ does not contain a primitive collection for any proper subset $S \subset \{u_1, \ldots, u_l\}$.

Proof. X^+ is obtained by blowing-up X along the torus invariant subvariety associated to the cone $\langle v_1, \ldots, v_m \rangle$ and by blowing-down with respect to the extremal primitive relation

$$u_1 + \cdots + u_l = v,$$

where $v := v_1 + \cdots + v_m$. Therefore, we can apply Propositions 2.2 and 2.3. $\quad\square$

Remark 2.5. Obviously, Theorem 2.4 is valid for smooth flips (the case where $l > m$) and smooth flops (the case where $l = m$) as well.

Low-dimensional examples for this calculation are given in Section 3 (see Examples 3.6 and 3.7).

We end this section by giving a characterization of *Fano* varieties using the notion of primitive relations for the reader's convenience:

Proposition 2.6 (see e.g. [2])**.** *Let $X = X_\Sigma$ be a smooth projective toric variety. Then X is a Fano variety (resp. weak Fano variety) if and only if for any primitive relation*

$$u_1 + \cdots + u_l = a_1 v_1 + \cdots + a_m v_m$$

of Σ, $l - (a_1 + \cdots + a_m) > 0$ *(resp.* ≥ 0*) holds. We call* $l - (a_1 + \cdots + a_m)$ *the degree of the primitive collection (or relation).*

3. Blow-ups and anti-flips

First, we give the description of the fans associated to the projective spaces and their blow-ups at torus invariant points. We will use this notation throughout this section.

For any natural number d, the d-dimensional projective space $\mathbb{P}^d$ is the simplest complete toric d-fold whose fan Σ_0^d is described as follows: Let $\{e_1, \ldots, e_d\}$ be the standard basis for $N := \mathbb{Z}^d$, and put $x_1 := e_1, \ldots, x_d := e_d, x_{d+1} := -(e_1 + \cdots + e_d)$. Then, Σ_0^d has the unique primitive collection with the following primitive relation

$$x_1 + \cdots + x_{d+1} = 0,$$

where $G(\Sigma_0^d) = \{x_1, \ldots, x_{d+1}\}$. Namely, $\mathbb{P}^d$ can be expressed by this only one simple equality.

Suppose that $d \geq 2$, and let $\pi : B_n^d \to \mathbb{P}^d$ be the blow-up of $\mathbb{P}^d$ at n torus invariant points for $1 \leq n \leq d + 1$ and Σ_n^d the fan associated to B_n^d. Here, we should remark that $\mathbb{P}^d$ has exactly $d + 1$ torus invariant points. We may assume $n \geq 2$, since B_1^d itself is a Fano manifold. By using Proposition 2.2 n times, we obtain the following.

Proposition 3.1. *The primitive relations of* Σ_n^d *are*

$$x_i + y_i = 0 \ (1 \leq i \leq n), \ x_1 + \cdots + \check{x}_i + \cdots + x_{d+1} = y_i \ (1 \leq i \leq n) \ and$$

$$y_i + y_j = x_1 + \cdots + \check{x}_i + \cdots + \check{x}_j + \cdots + x_{d+1} \ (1 \leq i < j \leq n),$$

where $G(\Sigma_n^d) = \{x_1, \ldots, x_{d+1}, y_1, \ldots, y_n\}$. *In particular,* Σ_n^d *has exactly* $\frac{n(n+3)}{2}$ *primitive collections.*

The main purpose of this paper is to construct the smooth toric Fano variety $\widetilde{B}_n^d$ associated to the fan $\widetilde{\Sigma}_n^d$ such that $G(\widetilde{\Sigma}_n^d) = G(\Sigma_n^d)$, that is, $\widetilde{B}_n^d$ and B_n^d are isomorphic in codimension one. B_2^2 and B_3^2 themselves are del Pezzo surfaces, so we assume $d \geq 3$. Proposition 2.6 tells us that B_n^d is not a Fano variety for $d \geq 3$. We use the notation $C_{n,r} = \frac{n!}{r!(n-r)!}$ for $1 \leq r \leq n$.

Lemma 3.2. *Let* $1 \leq r \leq n - 1$. *Suppose that* $2r + 1 < d$ *and there exists a smooth projective toric* d-*fold* $B_{n,r}^d$ *associated to the fan* $\Sigma_{n,r}^d$ *such that the primitive relations of* $\Sigma_{n,r}^d$ *are*

(I) $x_i + y_i = 0 \ (1 \leq i \leq n)$,

(II) $\sum_{x \in \{x_1, \ldots, x_{d+1}\} \setminus \{x_{i_1}, \ldots, x_{i_r}\}} x = y_{i_1} + \cdots + y_{i_r} \ (1 \leq i_1 < \cdots < i_r \leq n)$ *and*

(III) $y_{j_1} + \cdots + y_{j_{r+1}} = \sum_{x \in \{x_1, \ldots, x_{d+1}\} \setminus \{x_{j_1}, \ldots, x_{j_{r+1}}\}} x \ (1 \leq j_1 < \cdots < j_{r+1} \leq n)$, *where* $G(\Sigma_{n,r}^d) = G(\Sigma_n^d) = \{x_1, \ldots, x_{d+1}, y_1, \ldots, y_n\}$.

Then, there exists a sequence of smooth anti-flips

$$B_{n,r}^d =: B_{n,r}^d(0) \dashrightarrow B_{n,r}^d(1) \dashrightarrow \cdots \dashrightarrow B_{n,r}^d(\mathrm{C}_{n,r+1}) =: B_{n,r+1}^d$$

such that the primitive relations of the fan $\Sigma_{n,r+1}^d$ *associated to* $B_{n,r+1}^d$ *are*

$$x_i + y_i = 0 \ (1 \le i \le n),$$

$$\sum_{x \in \{x_1,\dots,x_{d+1}\}\setminus\{x_{j_1},\dots,x_{j_{r+1}}\}} x = y_{j_1} + \cdots + y_{j_{r+1}} \ (1 \le j_1 < \cdots < j_{r+1} \le n) \ and$$

$$y_{k_1} + \cdots + y_{k_{r+2}} = \sum_{x \in \{x_1,\dots,x_{d+1}\}\setminus\{x_{k_1},\dots,x_{k_{r+2}}\}} x \ (1 \le k_1 < \cdots < k_{r+2} \le n),$$

where $\mathrm{G}(\Sigma_{n,r+1}^d) = \mathrm{G}(\Sigma_n) = \{x_1,\dots,x_{d+1},y_1,\dots,y_n\}.$

Proof. First, we remark that $2r+1 < d$ means that the degrees of the primitive relations in (III) are negative. Moreover, they are extremal by the symmetry of the fan $\Sigma_{n,r}^d$, and the associated extremal contractions are anti-flipping contractions. In particular, $B_{n,r}^d$ is not a Fano variety.

Take a primitive relation

$$(\star) \qquad y_{s_1} + \cdots + y_{s_{r+1}} = \sum_{x \in \{x_1,\dots,x_{d+1}\}\setminus\{x_{s_1},\dots,x_{s_{r+1}}\}} x$$

in (III), and let

$$B_{n,r}^d(0) := B_{n,r}^d \dashrightarrow B_{n,r}^d(1)$$

be the associated anti-flip. Put $\Sigma_{n,r}^d(1)$ be the fan associated to $B_{n,r}^d(1)$. By (2) in Theorem 2.4, the primitive relations of $\Sigma_{n,r}^d$ in (I) are also primitive relations of $\Sigma_{n,r}^d(1)$. Also, (2) in Theorem 2.4 says that the primitive relations in (III) other than the above primitive relation $(\star)$ are also primitive relations of $\Sigma_{n,r}^d(1)$, while $\{x_1,\dots,x_{d+1}\} \setminus \{x_{i_1},\dots,x_{i_r}\}$ (which is a primitive collection of $\Sigma_{n,r}^d$ in (II)) is a primitive collection of $\Sigma_{n,r}^d(1)$ if and only if it does not contain $\{x_1,\dots,x_{d+1}\} \setminus \{x_{s_1},\dots,x_{s_{r+1}}\}$, that is, $\{x_{i_1},\dots,x_{i_r}\} \not\subset \{x_{s_1},\dots,x_{s_{r+1}}\}$. On the other hand,

$$\sum_{x \in \{x_1,\dots,x_{d+1}\}\setminus\{x_{s_1},\dots,x_{s_{r+1}}\}} x = y_{s_1} + \cdots + y_{s_{r+1}}$$

is of course a new primitive relation of $\Sigma_{n,r}^d(1)$. Moreover, (3) in Theorem 2.4 tells us that we have another new primitive collection $\{y_i, y_{s_1},\dots,y_{s_{r+1}}\}$ of $\Sigma_{n,r}^d(1)$ if $y_i \notin \{y_{s_1},\dots,y_{s_{r+1}}\}$ and $\{y_i, y_{s_1},\dots,y_{s_{r+1}}\}$ contains no primitive collection of $\Sigma_{n,r}^d$ other than $\{y_{s_1},\dots,y_{s_{r+1}}\}$ since $\{x_i, y_i\}$ is a primitive collection of $\Sigma_{n,r}^d$. We should remark that a primitive collection in (II) has the non-empty intersection with $\{x_1,\dots,x_{d+1}\} \setminus \{x_{s_1},\dots,x_{s_{r+1}}\}$, however it does not fulfill the condition in (3) in Theorem 2.4. Thus, we obtain the primitive relations

$$x_i + y_i = 0 \ (1 \le i \le n),$$

$$\sum_{x \in \{x_1, \ldots, x_{d+1}\} \backslash \{x_{i_1}, \ldots, x_{i_r}\}} x = y_{i_1} + \cdots + y_{i_r} \ (1 \le i_1 < \cdots < i_r \le n),$$

$$y_{j_1} + \cdots + y_{j_{r+1}} = \sum_{x \in \{x_1, \ldots, x_{d+1}\} \backslash \{x_{j_1}, \ldots, x_{j_{r+1}}\}} x \ ((j_1, \ldots, j_{r+1}) \ne (s_1, \ldots, s_{r+1}))$$

and

$$\sum_{x \in \{x_1, \ldots, x_{d+1}\} \backslash \{x_{s_1}, \ldots, x_{s_{r+1}}\}} x = y_{s_1} + \cdots + y_{s_{r+1}}$$

of $\Sigma_{n,r}^d(1)$. We remark that in this first case, any primitive relation of $\Sigma_{n,r}^d$ in (II) does not vanish, while the number of new primitive relations is only one. Continuously, by doing the anti-flip with respect to a primitive relation in (III) one by one, we obtain a sequence

$$B_{n,r}^d(0) \dashrightarrow B_{n,r}^d(1) \dashrightarrow \cdots \dashrightarrow B_{n,r}^d(\mathrm{C}_{n,r+1})$$

of anti-flips. In each step, we can calculate the primitive relations of the anti-flip with the same rules as the first case $B_{n,r}^d(0) \dashrightarrow B_{n,r}^d(1)$. Eventually, all the primitive relations in (II) vanish, and $\{k_1, \ldots, k_{r+2}\}$ becomes a primitive collection for any $1 \le k_1 < \cdots < k_{r+2} \le n$. This shows that $B_{n,r+1}^d := B_{n,r}^d(\mathrm{C}_{n,r+1})$ has the desired primitive relations. $\square$

By Proposition 3.1, we can put $\Sigma_{n,1}^d := \Sigma_n^d$. So, we can construct $B_{n,1}^d$, $B_{n,2}^d$, $B_{n,3}^d, \ldots$ inductively by Lemma 3.2 unless $2r + 1 \ge d$ or $r = n$. If $2r + 1 = d$, then $B_{n,r}^d$ has a flopping contraction, and we cannot obtain a smooth Famo variety. If $2r + 1 > d$, then $\widetilde{B}_n^d := B_{n,r}^d$ is the desired smooth Fano variety.

Thus, we obtain the following main theorem in this paper.

Theorem 3.3. *The following hold:*

(1) *If $2n - 1 < d$, then $\widetilde{B}_n^d := B_{n,n}^d$ is a smooth toric Fano variety whose primitive relations are*

$x_i + y_i = 0 \ (1 \le i \le n)$ *and* $x_{n+1} + \cdots + x_{d+1} = y_1 + \cdots + y_n$.

$\widetilde{B}_n^d$ has a $(\mathbb{P}^1)^n$-bundle structure over $\mathbb{P}^{d-n}$. Moreover, $B_n^d \dashrightarrow \widetilde{B}_n^d$ is the composition of $2^n - n - 1$ anti-flips.

(2) *If $2n - 1 \ge d$ and d is odd, then there does not exist a smooth Fano variety which is isomorphic to B_n^d in codimension one.*

(3) *If $2n - 1 \ge d$ and d is even, then put $c := \frac{d}{2}$. In this case, $\widetilde{B}_n^d := B_{n,c}^d$ is the desired smooth toric Fano variety. $\widetilde{B}_n^d$ has no bundle structure, and every extremal ray of the Kleiman-Mori cone of $\widetilde{B}_n^d$ is of small type. Moreover, $B_n^d \dashrightarrow \widetilde{B}_n^d$ is the composition of $\sum_{r=2}^c \mathrm{C}_{n,r}$ anti-flips.*

Proof. $B_{n,r}^d \dashrightarrow B_{n,r+1}^d$ is the composition of $\mathrm{C}_{n,r+1}$ anti-flips. Therefore, $B_n^d \dashrightarrow \widetilde{B}_n^d$ is the composition of $\sum_{r=2}^n \mathrm{C}_{n,r}$ anti-flips (resp. $\sum_{r=2}^c \mathrm{C}_{n,r}$) for the case (1) (resp. the case (3)).

The case (2) means $2r + 1 = d$ for some $1 \le r < n$. So, we have a flopping contraction in the middle of the operation. $\square$

Remark 3.4. In the cases (1) and (3) in Theorem 3.3, the rational map $B_n^d \dashrightarrow \widetilde{B}_n^d$ is a process of the so-called $-K_{B_n^d}$-*minimal model program* which consists of only $-K_{B_n^d}$-flips (that is, anti-flips), and $\widetilde{B}_n^d$ is the *unique* $-K_{B_n^d}$-minimal model.

The case (2) is similar. However, in this case, any $-K_{B_n^d}$-minimal model is not a Fano manifold (in particular, not unique).

Remark 3.5. The conditions $2n - 1 \geq d$ in (3) in Theorem 3.3 and $n \leq d+1$ become $c + 1 \leq n \leq 2c + 1$. Thus, we obtain exactly $c + 1$ smooth toric Fano varieties $\widetilde{B}_{c+1}^d, \widetilde{B}_{c+2}^d, \ldots, \widetilde{B}_{2c+1}^d$ in this case.

Example 3.6. We explicitly describe the 4-dimensional operations $B_2^4 \dashrightarrow \widetilde{B}_2^4$ and $B_3^4 \dashrightarrow \widetilde{B}_3^4$.

(1) The primitive relations of Σ_2^4 are

(i) $x_1 + y_1 = 0$, (ii) $x_2 + y_2 = 0$, (iii) $x_2 + x_3 + x_4 + x_5 = y_1$,

(iv) $x_1 + x_3 + x_4 + x_5 = y_2$ and (v) $y_1 + y_2 = x_3 + x_4 + x_5$,

where $\mathrm{G}(\Sigma_2^4) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2\}$. We do the anti-flip with respect to (v). Theorem 2.4 tells us that the primitive relations (iii) and (iv) are eliminated, since $\{x_3, x_4, x_5\} \subset \{x_2, x_3, x_4, x_5\}$ and $\{x_3, x_4, x_5\} \subset \{x_1, x_3, x_4, x_5\}$. So the primitive relations of the desired toric manifold $B_{2,1}^4(1) = B_{2,2}^4 = \widetilde{B}_2^4$ are

(i) $x_1 + y_1 = 0$, (ii) $x_2 + y_2 = 0$ and (v)$^+$ $x_3 + x_4 + x_5 = y_1 + y_2$.

By Proposition 2.6, $\widetilde{B}_2^4$ is a Fano variety. This case corresponds to (1) in Theorem 3.3.

(2) The primitive relations of Σ_3^4 are

(i) $x_1 + y_1 = 0$, (ii) $x_2 + y_2 = 0$, (iii) $x_3 + y_3 = 0$,

(iv) $x_2 + x_3 + x_4 + x_5 = y_1$, (v) $x_1 + x_3 + x_4 + x_5 = y_2$,

(vi) $x_1 + x_2 + x_4 + x_5 = y_3$, (vii) $y_1 + y_2 = x_3 + x_4 + x_5$,

(viii) $y_1 + y_3 = x_2 + x_4 + x_5$ and (ix) $y_2 + y_3 = x_1 + x_4 + x_5$,

where $\mathrm{G}(\Sigma_2^4) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3\}$. We do the 3-times anti-flips

$$B_3^4 = B_{3,1}^4(0) \dashrightarrow B_{3,1}^4(1) \dashrightarrow B_{3,1}^4(2) \dashrightarrow B_{3,1}^4(3)$$

with respect to (vii), (viii) and (ix). The primitive relations of $B_{3,1}^4(1)$ are (i), (ii), (iii), (vi), (viii), (ix) and

$$\text{(vii)}^+ \quad x_3 + x_4 + x_5 = y_1 + y_2,$$

since $\{x_3, x_4, x_5\} \subset \{x_2, x_3, x_4, x_5\}$ and $\{x_3, x_4, x_5\} \subset \{x_1, x_3, x_4, x_5\}$. The primitive relations of $B_{3,1}^4(2)$ are (i), (ii), (iii), (vii)$^+$, (ix) and

$$\text{(viii)}^+ \quad x_2 + x_4 + x_5 = y_1 + y_3,$$

since $\{x_2, x_4, x_5\} \subset \{x_1, x_2, x_4, x_5\}$. Finally, the primitive relations of $B_{3,1}^4(3)$ are (i), (ii), (iii), (vii)$^+$, (viii)$^+$,

(ix)$^+$ $x_1 + x_4 + x_5 = y_2 + y_3$ and (x) $y_1 + y_2 + y_3 = x_4 + x_5$.

We should remark that $\{y_1, y_2, y_3\}$ is a new primitive collection (see (3) in Theorem 2.4). By Proposition 2.6, $B_{3,1}^4(3) = B_{3,2}^4 = \widetilde{B}_3^4$ is a Fano variety, and this case corresponds to (3) in Theorem 3.3.

$\widetilde{B}_2^4$ is the smooth toric Fano 4-fold of type D_9, while $\widetilde{B}_3^4$ is the smooth toric Fano 4-fold of type M_1 (see Batyrev's list [2]).

Example 3.7. We consider the 3-dimensional case B_2^3, that is, the case (2) in Theorem 3.3. The primitive relations of Σ_2^3 are

$x_1 + y_1 = 0, x_2 + y_2 = 0, x_2 + x_3 + x_4 = y_1, x_1 + x_3 + x_4 = y_2$ and $y_1 + y_2 = x_3 + x_4$,

where $G(\Sigma_2^3) = \{x_1, x_2, x_3, x_4, y_1, y_2\}$. Let $B_2^3 \dashrightarrow B^+$ be the flop with respect to $y_1 + y_2 = x_3 + x_4$. Then, the primitive relations of B^+ are

$$x_1 + y_1 = 0, \ x_2 + y_2 = 0 \text{ and } x_3 + x_4 = y_1 + y_2$$

by Theorem 2.4 (see Remark 2.5, too). Both B_2^3 and B^+ are not Fano manifolds but weak Fano manifolds by Proposition 2.6. Namely, they are $-K_{B_2^3}$-minimal models for B_2^3. However, B_2^3 and B^+ are not isomorphic (see Remark 3.4).

Remark 3.8. For an even number d, $\widetilde{B}_d^d$ is the pseudo-symmetric toric Fano variety $\widetilde{V}^d$ in [5], while $\widetilde{B}_{d+1}^d$ is the symmetric toric Fano variety V^d in [12].

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