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RIGIDITY AND NONEXISTENCE OF RIEMANNIAN IMMERSIONS IN SEMI-RIEMANNIAN WARPED PRODUCTS VIA PARABOLICITY

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Abstract. In this paper, we study complete Riemannian immersions into a semi-Riemannian warped product obeying suitable curvature constraints. Under appropriate differential inequalities involving higher order mean curvatures, we establish rigidity and nonexistence results concerning these immersions. Applications to the cases that the ambient space is either an Einstein manifold, a steady state type spacetime or a pseudo-hyperbolic space are given, and a particular investigation of entire graphs constructed over the fiber of the ambient space is also made. Our approach is based on a parabolicity criterion related to a linearized differential operator which is a divergence-type operator and can be regarded as a natural extension of the standard Laplacian.

1. Introduction

The aim of the present paper is to study the geometry of complete n-dimensional Riemannian immersions into a semi-Riemannian warped product of the type $\overline{M}^{n+1} = \epsilon I \times_f M^n$, where M^n is an n-dimensional connected oriented Riemannian manifold, $I \subseteq \mathbb{R}$ is an open interval, $f : I \to \mathbb{R}$ is a positive smooth function and $\epsilon = \pm 1$, being $\epsilon = 1$ when $\overline{M}^{n+1}$ is a Riemannian space and $\epsilon = -1$ when $\overline{M}^{n+1}$ is a Lorentzian space. In the Lorentzian case, $\overline{M}^{n+1}$ is called a generalized Robertson-Walker (GRW) spacetime.

This thematic has been treated by several authors along the last years, which have used a considerable amount of analytical tools in their investigations. Furthermore, we observe that in a considerable part of these it is assumed that the ambient space obeys appropriate curvature constraints. See, for instance,

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the works [3,4,6–8,11,12,16–18,21–24,29,30]. See also Chapter 7 of the excellent book of Alías, Mastrolia and Rigoli [13] and references therein.

Here, under appropriate differential inequalities involving higher order mean curvatures and assuming that the ambient space obeys suitable curvature constraints, we establish new rigidity and nonexistence results concerning these immersions. Applications to the cases that the ambient space is either an Einstein manifold, a steady state type spacetime or a pseudo-hyperbolic space are given, and a particular investigation of entire graphs construct over the fiber of the ambient space is also made. Our approach is based on a parabolicity criterion related to a linearized differential operator which is a divergence-type operator and can be regarded as a natural extension of the standard Laplacian. This criterion is obtained as a application of Theorem 2.6 in [28].

This manuscript is organized in the following way: In Section 2 we recall some basic facts related to Riemannian immersions in semi-Riemannian warped products and we quotes the auxiliaries lemmas which will be used to prove our main results. We start Section 3 establishing our parabolicity criterion for complete Riemannian immersions. Next, we present our rigidity and nonexistence results concerning complete spacelike hypersurfaces in a GRW spacetime and, afterwards, we treat the case of complete two-sided hypersurfaces in a Riemannian warped product. Finally, we close this section with the study of entire graphs constructed over the fiber of the ambient space.

2. Preliminaries

Let $\overline{M}^{n+1}$ be a connected semi-Riemannian manifold with metric $\overline{g} = \langle\ ,\ \rangle$ of index $\nu \leq 1$, and semi-Riemannian connection $\overline{\nabla}$. For a vector field $X \in \mathfrak{X}(\overline{M})$, let $\epsilon_X = \langle X, X\rangle$. We will say that X is a *unit* vector field if $\epsilon_X = \pm 1$, and *timelike* if $\epsilon_X = -1$.

Now, let M^n be a connected, n-dimensional oriented Riemannian manifold, $I \subseteq \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. A particular class of semi-Riemannian manifolds having conformal fields is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w\rangle_p = \epsilon\langle(\pi_I)_* v, (\pi_I)_* w\rangle + f(\pi_I(p))^2\langle(\pi_M)_* v, (\pi_M)_* w\rangle$$

for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$, where $\epsilon = \epsilon_{\partial_t}$ and ∂_t is the standard unit vector field tangent to I. Such a space is a particular case of a semi-Riemannian *warped product*, and, from now on, we will just write $\overline{M}^{n+1} = \epsilon I \times_f M^n$ to denote it. In the Lorentzian setting $\nu = 1$ or, equivalently, when $\epsilon = -1$, adopting the terminology established in [14], $\overline{M}^{n+1}$ is called a *generalized Robertson-Walker* (GRW) spacetime.

2.1. Riemannian immersions in semi-Riemannian warped products

In all that follows, we will consider *Riemannian immersions* $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, immersions from a connected, n-dimensional orientable differentiable manifold Σ^n into a semi-Riemannian warped product $\overline{M}^{n+1} = \epsilon I \times_f M^n$, such that the induced metric $g = \psi^*(\overline{g})$ turns Σ^n into a Riemannian manifold. In the Lorentz case $\nu = 1$, we will refer to (Σ^n, g) as a *spacelike hypersurface* of $\overline{M}^{n+1}$. For sake of simplicity, we will also denote g by $\langle\, , \rangle$ and ∇ will stand for its Levi-Civita connection, while $\overline{\nabla}$ will represent the Levi-Civita connection of the ambient space.

In this setting, we will orient Σ^n by the choice of a unit normal vector field N on it. So, we have that $\epsilon = \epsilon_{\partial_t} = \epsilon_N$. Denoting by A the Weingarten operator corresponding to N, at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$.

For $0 \leq r \leq n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p; this way one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by construction. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

Also, we define the r-th mean curvature H_r of ψ, $0 \leq r \leq n$, by

$$\binom{n}{r} H_r = \epsilon^r S_r = \sigma_r(\epsilon\lambda_1, \ldots, \epsilon\lambda_n).$$

We observe that $H_0 = 1$ and H_1 is the usual mean curvature H of Σ^n.

For $t_0 \in I$, we orient the *slice* $\Sigma^n_{t_0} = \{t_0\} \times M^n$ by using the unit normal vector field ∂_t. According to Example 5.6 of [5] and Section 2 of [12], Σ_{t_0} has constant r-th mean curvature $H_r = (-\epsilon)^r \left(\frac{f'(t_0)}{f(t_0)}\right)^r$ with respect to ∂_t.

For $0 \leq r \leq n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = \epsilon^r S_r I - \epsilon A P_{r-1}. \tag{2.1}$$

With a trivial induction, from (2.1) we verify that

$$P_r = \epsilon^r (S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r), \tag{2.2}$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ diagonalizing A at $p \in \Sigma^n$ also diagonalize all of the P_r at p. So, let $\{e_1, \ldots, e_n\}$ be an orthonormal frame on $T_p\Sigma$ which diagonalizes A_p, $A_p(e_i) = \lambda_i(p)e_i$, then from (2.2) we have that

$$(P_r)_p e_i = \epsilon^r \sum_{i_1<\cdots<i_r, i_j\neq i} \lambda_{i_1}(p)\cdots\lambda_{i_r}(p)e_i. \tag{2.3}$$

For each Newton transformation P_r, $0 \leq r \leq n$, we associate a second order linear differential operator $L_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$L_r(\xi) = \operatorname{tr}(P_r \circ \nabla^2\xi),$$

where $\nabla^2\xi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator equivalent to the Hessian operator of ξ, defined by

$$\langle \nabla^2\xi(X), Y\rangle = \langle \nabla_X \nabla\xi, Y\rangle$$

for all vector fields $X, Y \in \mathfrak{X}(\Sigma)$.

Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on Σ^n. We have that

$$\begin{aligned} \operatorname{div}(P_r(\nabla\xi)) &= \sum_{i=1}^n \langle (\nabla_{e_i} P_r)(\nabla\xi), e_i\rangle + \sum_{i=1}^n \langle P_r(\nabla_{e_i}\nabla\xi), e_i\rangle \\ &= \langle \operatorname{div} P_r, \nabla\xi\rangle + L_r(\xi), \end{aligned} \tag{2.4}$$

where the divergence of P_r on Σ^n is defined by

$$\operatorname{div} P_r := \operatorname{tr}(\nabla P_r) = \sum_{i=1}^n (\nabla_{e_i} P_r)(e_i) \quad \text{and} \quad \operatorname{div} P_0 = \operatorname{div} I = 0.$$

We close this subsection recalling a terminology introduced in [2]. We say that a Riemannian immersion $\psi : \Sigma^n \to \epsilon I \times_f M^n$ is *bounded away from the future infinity* of $\epsilon I \times_f M^n$ if there exists $\bar{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in \epsilon I \times_f M^n : t \leq \bar{t}\},$$

and we say that it is *bounded away from the past infinity* of $\epsilon I \times_f M^n$ if there exists $\underline{t} \in I$ such that

$$\psi(\Sigma) \subset \{(t, x) \in \epsilon I \times_f M^n : t \geq \underline{t}\}.$$

2.2. Some auxiliary lemmas

From (2.4), we have that the operator L_r is elliptic if and only if P_r is positive definite (for an appropriate choice of the orientation N of Σ^n). In particular, the Laplace-Beltrami operator $L_0 = \Delta$ is always elliptic. The following two lemmas establish sufficient conditions to guarantee the ellipticity of the operator L_1 and L_r when $r \geq 2$ (see, for instance, Lemmas 3.2 and 3.3 of [6]).

Lemma 2.1. *Let $\overline{M}^{n+1} = \epsilon I \times_f M^n$ be a semi-Riemannian warped product and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion. If $H_2 > 0$ on Σ^n, then L_1 is elliptic or, equivalently, P_1 is positive definite (for a appropriate choice of the Gauss map N).*

In what follows, by an *elliptic point* in a Riemannian immersion we mean a point where all principal curvatures have the same sign.

Lemma 2.2. *Let $\overline{M}^{n+1} = \epsilon I \times_f M^n$ be a semi-Riemannian warped product and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of the Gauss map N, and $H_{r+1} > 0$ on Σ^n for $2 \le r \le n-1$, then for all $1 \le j \le r$ the operator L_j is elliptic or, equivalently, P_j is positive definite* (*for a appropriate choice of the Gauss map N, if j is odd*).

Now, we will consider two particular functions naturally attached to Σ^n, namely, the (vertical) *height function* $h = (\pi_I)|_\Sigma$ and the *angle function* $\Theta = \langle N, \partial_t\rangle$.

A simple computation shows that the gradient of π_I on $\epsilon I \times_f M^n$ is given by

$$\overline{\nabla}\pi_I = \epsilon\langle\overline{\nabla}\pi_I, \partial_t\rangle\partial_t = \epsilon\partial_t. \tag{2.5}$$

So, from (2.5) we verify that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \epsilon\partial_t^\top = \epsilon\partial_t - \Theta N. \tag{2.6}$$

In particular, from (2.6) we get

$$|\nabla h|^2 = \epsilon\left(1 - \Theta^2\right), \tag{2.7}$$

where $|\ \ |$ denotes the norm of a vector field on Σ^n.

Our next lemma gives a sufficient condition to guarantee the existence of an elliptic point in a Riemannian immersion. For its proof, see Lemma 5.4 of [5] and Lemma 4 of [8].

Lemma 2.3. *Let $\overline{M}^{n+1} = \epsilon I \times_f M^n$ be a semi-Riemannian warped product and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion. If $-\epsilon f(h)$ attains a local minimum at some $p \in \Sigma^n$, such that $f'(h(p)) \neq 0$, then p is an elliptic point for Σ^n.*

From Lemma 4.1 of [6] and Proposition 6 of [12] we get the following suitable formulas:

Lemma 2.4. *Let $\psi : \Sigma^n \to \epsilon I \times_f M^n$ be a Riemannian immersion and let $g : I \to \mathbb{R}$ be any primitive of the warping function f. Then, for every $r = 0, \ldots, n-1$,*

$$L_r(g(h)) = \epsilon c_r\left(f'(h)H_r + H_{r+1}f(h)\Theta\right),$$

where $c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$.

Moreover, from Lemma 3.1 of [5] and equation (3.12) of the proof of Theorem 2 in [8], the divergence of the Newton transformation P_r is given by:

Lemma 2.5. *Let $\psi : \Sigma^n \to \epsilon I \times_f M^n$ be a Riemannian immersion. Then*

$$\langle \mathrm{div} P_1, \nabla h\rangle = -\epsilon(\mathrm{Ric}_M(N^*, N^*) + \epsilon(n-1)(\log f)''(h)|\nabla h|^2)\Theta, \tag{2.8}$$

where Ric_M *denotes the Ricci curvature of the fiber* M^n *and* $N^* = N - \epsilon\Theta\partial_t$ *is the projection of the* N *onto* M^n*. Moreover, when* M^n *has constant sectional curvature* κ*,*

$$\langle \mathrm{div} P_r, \nabla h\rangle = -\epsilon(n-r)\left(\frac{\kappa}{f^2(h)} + \epsilon(\log f)''(h)\right)\langle P_{r-1}\nabla h, \nabla h\rangle\Theta. \tag{2.9}$$

3. Main results

This section is devoted to present our rigidity and nonexistence results concerning complete Riemannian immersions in a semi-Riemannian warped product. Our approach is based on a suitable parabolicity criterion which is obtained as an application of Theorem 2.6 in [28].

3.1. A parabolicity criterion for Riemannian immersions

Considering the setting of the previous section, we define the operator $\mathcal{L}_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by

$$\mathcal{L}_r(\xi) := \mathrm{div}(P_r(\nabla\xi)). \tag{3.1}$$

According to Definition 5.4 in [11] and Definition 30 in [12], we say that a Riemannian immersion $\psi : \Sigma^n \to \epsilon I \times_f M^n$ is *$\mathcal{L}_r$-parabolic* if the only bounded above C^1 solutions of the differential inequality $\mathcal{L}_r\xi \geq 0$ are the constant ones.

The next result provides sufficient conditions which guarantee the $\mathcal{L}_r$-parabolicity of Riemannian immersions in a semi-Riemannian warped product.

Proposition 3.1. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete Riemannian immersion in $\overline{M}^{n+1} = \epsilon I \times_f M^n$. Suppose that the Newton transformation P_r is positive semi-definite and* $\sup_\Sigma H_r < +\infty$ *for some $0 \leq r \leq n$. If, for some reference point $o \in \Sigma^n$,*

$$\int_0^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)} = +\infty, \tag{3.2}$$

where B_t is the geodesic ball of radius t in Σ^n centered at the origin o, then Σ^n is $\mathcal{L}_r$-parabolic.

Proof. Let us consider on Σ^n the symmetric $(0,2)$-tensor field ξ_r given by

$$\xi_r(X, Y) := \langle P_r X, Y\rangle$$

for all $X, Y \in T\Sigma$ or, equivalently,

$$\xi_r(\nabla u, \cdot)^\sharp = P_r(\nabla u),$$

where $\sharp : T^*\Sigma \to T\Sigma$ denotes the musical isomorphism. Hence, we have

$$\mathcal{L}_r(u) = \mathrm{div}\left(\xi_r(\nabla u, \cdot)^\sharp\right).$$

Moreover, since we are assuming that P_r is positive semi-definite, $\mathrm{tr}(P_r) = c_r H_r$ (see Lemma 2.1 of [19]) and $\sup_\Sigma H_r < +\infty$, we can define a positive continuous function ξ_{r+} on $[0, +\infty)$ by:

$$\xi_{r+}(t) := c_r \sup_{\partial B_t} H_r.$$

Thus, for all $X \in T\Sigma$ with $|X| = 1$, we obtain

$$0 \leq \xi_r(X, X) \leq \xi_{r+}(t) \leq c_r \sup_\Sigma H_r < +\infty.$$

So, we get

$$\int_0^{+\infty} \frac{dt}{\xi_{r+}(t)\mathrm{vol}(\partial B_t)} \geq \left(c_r \sup_\Sigma H_r \right)^{-1} \int_0^{+\infty} \frac{dt}{\mathrm{vol}(\partial B_t)}.$$

Consequently, from hypothesis (3.2) we get

$$\int_0^{+\infty} \frac{dt}{\xi_{r+}(t)\mathrm{vol}(\partial B_t)} = +\infty.$$

Therefore, we are in position to apply Theorem 2.6 of [28] to conclude the proof. □

3.2. Rigidity and nonexistence of spacelike hypersurfaces

When the ambient space is a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$, since ∂_t is a unitary timelike vector field globally defined on $\overline{M}^{n+1}$, there exists a unique timelike unitary normal vector field N globally defined on a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ which is in the same time-orientation as ∂_t. We then say that N is *future-pointing* and, from the Cauchy-Schwarz inequality for timelike vectors, we have that $\Theta \leq -1$.

Taking into account the previous digression, we can state and prove our first rigidity result.

Theorem 3.2. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and $H > 0$ on Σ^n. If hypothesis* (3.2) *is satisfied and*

$$H \geq \frac{f'}{f}(h), \tag{3.3}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. From (3.1) and Lemma 2.4 we have

$$\mathcal{L}_0(g(h)) = -n(f'(h) + f(h)H\Theta). \tag{3.4}$$

Since we are assuming $H > 0$ on Σ^n and taking into account that $\Theta \leq -1$, from (3.4) we get

$$\mathcal{L}_0(g(h)) \geq nf(h)\left(H - \frac{f'}{f}(h)\right). \tag{3.5}$$

Thus, from inequalities (3.3) and (3.5) we obtain that $\mathcal{L}_0(g(h)) \geq 0$.

Moreover, hypothesis (3.2) guarantees that Σ^n is $\mathcal{L}_0$-parabolic. But, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we have that $g(h)$ is bounded from above. Consequently, $g(h)$ is constant on Σ^n. Therefore, we conclude that the height function h is constant and, hence, Σ^n must be a slice of $\overline{M}^{n+1}$. □

Next, we will consider a natural extension of the $(n+1)$-dimensional steady state spacetime $-\mathbb{R}\times_{e^t}\mathbb{R}^n$, the so-called *steady state-type spacetime* $\overline{M}^{n+1} = -\mathbb{R}\times_{e^t} M^n$, where M^n is a connected n-dimensional Riemannian manifold (see Section 4 of [2]). It is worth to note that when a steady state-type spacetime admits a complete spacelike hypersurface which is bounded away from the future infinity, Lemma 7 of [2] guarantees that its Riemannian fiber M^n is necessarily complete. In this setting, Theorem 3.2 reads as follows.

Corollary 3.3. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R}\times_{e^t} M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$. If $H \geq 1$ and hypothesis* (3.2) *is satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

For $r = 1$, we will suppose that the GRW spacetime obeys a suitable curvature constraint.

Theorem 3.4. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I\times_f M^n$ which obeys the following curvature constraint*

$$\mathrm{Ric}_M \leq (n-1)\inf_I(ff'' - (f')^2)\langle\,,\rangle_M, \tag{3.6}$$

where Ric_M stands for the Ricci tensor of M^n. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, $H > 0$ with $\sup_\Sigma H < +\infty$, and $H_2 > 0$. If hypothesis (3.2) *is satisfied and*

$$\frac{H_2}{H} \geq \frac{f'}{f}(h), \tag{3.7}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. From (3.1), jointly with Lemma 2.4 and equation (2.8) of Lemma 2.5, we obtain that

$$\begin{aligned}\mathcal{L}_1(g(h)) = &-f(h)((n-1)(\log f)''(h)|\nabla h|^2 - \mathrm{Ric}_M(N^*,N^*))\Theta\\ &-c_1(f'(h)H + f(h)H_2\Theta),\end{aligned} \tag{3.8}$$

where $N^* = N + \Theta\partial_t$.

On the other hand, from (2.7) we have that

$$\langle N^*, N^*\rangle_M = \frac{1}{f^2(h)}|\nabla h|^2.$$

So, using curvature constraint (3.6) and $\Theta \leq -1$, from (3.8) we get that

$$\mathcal{L}_1(g(h)) \geq c_1 f(h) H \left(\frac{H_2}{H} - \frac{f'}{f}(h)\right).$$

Thus, from (3.7) we have that $\mathcal{L}_1(g(h)) \geq 0$. Moreover, by assumptions $\sup_\Sigma H < +\infty$ and (3.2), and taking into account that Lemma 2.1 guarantees that P_1 is positive definite, it follows from Proposition 3.1 that Σ^n is $\mathcal{L}_1$-parabolic.

Consequently, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we obtain that $g(h)$ is constant on Σ^n. Therefore, we conclude that the height function h is constant, which means that Σ^n is a slice of $\overline{M}^{n+1}$. □

When the ambient spacetime is an Einstein manifold, Theorem 3.4 reads as follows:

Corollary 3.5. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a Einstein GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$. If $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and hypotheses (3.2) and (3.7) are satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. From Corollary 9.107 of [20] (see also Section 2 of [15]) we have that $\overline{M}^{n+1}$ is an Einstein manifold with Ricci tensor $\overline{\mathrm{Ric}} = \overline{c}\,\overline{g}$, $\overline{c} \in \mathbb{R}$, if and only if the fiber (M^n, g_M) has constant Ricci curvature $\mathrm{Ric}_M = c\langle\,,\rangle_M$ and the warping function f satisfies the differential equations

$$\frac{f''}{f} = \frac{\overline{c}}{n} \quad \text{and} \quad \frac{\overline{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2}. \tag{3.9}$$

Hence, from (3.9) we obtain $(n-1)(\log f)'' = \frac{c}{f^2}$. Therefore, in this case, we have that

$$\mathrm{Ric}_M = (n-1)\inf_I (ff'' - (f')^2)\langle\,,\rangle_M$$

and, consequently, the result follows by applying Theorem 3.4. □

Considering once more a steady state-type spacetime, from Theorem 3.4 we get the following consequence.

Corollary 3.6. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a steady state-type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonpositive Ricci curvature. Suppose that Σ^n is bounded away from* *the future infinity of* $\overline{M}^{n+1}$. *If* $\sup_\Sigma H < +\infty$, $H_2 \geq H > 0$ *and hypothesis* (3.2) *is satisfied, then* Σ^n *is a slice of* $\overline{M}^{n+1}$.

When $2 \leq r \leq n-1$, we will assume that the Riemannian fiber of the GRW spacetime has constant sectional curvature. In this setting, we will use Lemma 2.3 to guarantee the ellipticity of the operator $\mathcal{L}_r$.

Theorem 3.7. *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete spacelike hypersurface immersed into a GRW spacetime* $\overline{M}^{n+1} = -I \times_f M^n$ *whose fiber* M^n *has constant sectional curvature* κ *satisfying the following curvature constraint*

$$\kappa \leq \inf_I (ff'' - (f')^2). \tag{3.10}$$

Suppose that Σ^n *is bounded away from the future infinity of* $\overline{M}^{n+1}$, $H_r > 0$ *with* $\sup_\Sigma H_r < +\infty$, *and* $H_{r+1} > 0$ *for some* $2 \leq r \leq n-1$. *Assume in addition that* $f(h)$ *attains a local minimum at some point* $p \in \Sigma^n$ *such that* $f'(h(p)) \neq 0$. *If hypothesis* (3.2) *is satisfied and*

$$\frac{H_{r+1}}{H_r} \geq \frac{f'}{f}(h), \tag{3.11}$$

then Σ^n *is a slice of* $\overline{M}^{n+1}$.

Proof. Since M^n has constant sectional curvature κ, from Lemma 2.4 jointly with equation (2.9) of Lemma 2.5 and (3.1) we obtain

$$\begin{aligned}\mathcal{L}_r(g(h)) = {} & f(h)(n-r)\left(\frac{\kappa}{f^2(h)} - (\log f)''(h)\right)\langle P_{r-1}\nabla h, \nabla h\rangle\Theta \\ & - c_r(f'(h)H_r + f(h)H_{r+1}\Theta).\end{aligned} \tag{3.12}$$

On the other hand, since $f(h)$ attains a local minimum at some point $p \in \Sigma^n$ such that $f'(h(p)) \neq 0$, Lemma 2.3 guarantees that p is an elliptic point of Σ^n. So, using the assumption $H_{r+1} > 0$, from Lemma 2.2 we have that the operator L_j is elliptic or, equivalently, P_j is positive definite for all $1 \leq j \leq r$.

Thus, taking into account curvature constraint (3.10) and that $\Theta \leq -1$, from (3.12) we obtain

$$\begin{aligned}\mathcal{L}_r(g(h)) &\geq \frac{1}{f(h)}(n-r)\left((ff'' - (f')^2)(h) - \kappa\right)\langle P_{r-1}\nabla h, \nabla h\rangle \\ &\quad + c_r(f(h)H_{r+1} - f'(h)H_r) \\ &\geq c_r f(h) H_r\left(\frac{H_{r+1}}{H_r} - \frac{f'}{f}(h)\right).\end{aligned} \tag{3.13}$$

Hence, from inequalities (3.11) and (3.13) we get that $\mathcal{L}_r(g(h)) \geq 0$ on Σ^n. Moreover, from hypotheses $\sup_\Sigma H_r < +\infty$ and (3.2), we have that Σ^n is $\mathcal{L}_r$-parabolic. Therefore, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. $\square$

In the special case when the warping function is given by $f \equiv 1$, that is, $\overline{M}^{n+1} = -I \times M^n$ is a *static* GRW spacetime, we can reason as in the proof of Theorem 3.7 (but replacing the hypothesis $f' \neq 0$ by the existence of an elliptic point) to get the following nonexistence result:

Corollary 3.8. *Let $\overline{M}^{n+1} = -I \times M^n$ be a static GRW spacetime whose fiber M^n has constant sectional curvature $\kappa \leq 0$. There is no complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ bounded away from the future infinity of $\overline{M}^{n+1}$ such that, for some $2 \leq r \leq n-1$, $H_r > 0$ with* $\sup_\Sigma H_r < +\infty$*, $H_{r+1} > 0$, having an elliptic point and satisfying hypothesis* (3.2).

Proof. Let us suppose by contradiction the existence of such a spacelike hypersurface. From Lemma 2.4 jointly with equation (2.9) of Lemma 2.5 and (3.1) we obtain

$$\mathcal{L}_r(h) = ((n-r)\kappa \langle P_{r-1}\nabla h, \nabla h\rangle - c_r H_{r+1})\,\Theta. \tag{3.14}$$

Thus, since $\kappa \leq 0$, $\Theta \leq -1$ and taking into account the existence of an elliptic point jointly with $H_{r+1} > 0$, from Lemma 2.2 and (3.14) we get that $\mathcal{L}_r(h) \geq 0$ on Σ^n. Moreover, from hypotheses $\sup_\Sigma H_r < +\infty$ and (3.2), we have that Σ^n is $\mathcal{L}_r$-parabolic. Therefore, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. However, since $\overline{M}^{n+1} = -I \times M^n$ is a static GRW spacetime, each slice has identically zero higher order mean curvatures, contradicting the hypotheses that $H_r > 0$ and $H_{r+1} > 0$. $\square$

Proceeding, we will consider also the case when the spacelike hypersurface is bounded away from the past infinity of a GRW spacetime whose Riemannian fiber has constant sectional curvature obeying a curvature constraint which corresponds to the so-called *null convergence condition* (NCC). For more details concerning the NCC, see [26].

Theorem 3.9. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$ whose fiber M^n has constant sectional curvature κ satisfying the NCC*

$$\kappa \geq \sup_I (ff'' - (f')^2). \tag{3.15}$$

Suppose that Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$, $H_{r-1} > 0$ and $H_r > 0$ with $\sup_\Sigma H_r < +\infty$ *for some $2 \leq r \leq n-1$. Assume in addition that the sectional curvature of Σ^n, K_Σ, is such that*

$$K_\Sigma \leq \frac{f''}{f}(h). \tag{3.16}$$

If hypothesis (3.2) *is satisfied and*

$$\frac{H_{r+1}}{H_r} \leq -\frac{1}{\Theta}\frac{f'}{f}(h), \tag{3.17}$$

then Σ^n *is a slice of* $\overline{M}^{n+1}$.

Proof. We define a self-adjoint operator $\mathcal{P}_{r-1}:\mathfrak{X}(\Sigma^n)\to\mathfrak{X}(\Sigma^n)$ by

$$\mathcal{P}_{r-1}=H_{r-1}P_{r-1}.$$

For each $p\in\Sigma^n$, we take a local orthonormal frame $\{e_1,\ldots,e_n\}$ such that $Ae_i=\lambda_i e_i$. From (2.3) we have that

$$P_{r-1}e_i=(-1)^{r-1}\sum_{i_1<\cdots<i_{r-1},i_j\neq i}\lambda_{i_1}\cdots\lambda_{i_{r-1}}e_i.$$

Thus, for any $i\in\{1,\ldots,n\}$, we get

$$\langle\mathcal{P}_{r-1}e_i,e_i\rangle=\binom{n}{r-1}^{-1}\sum_{i_1<\cdots<i_{r-1},i_j\neq i,j_1<\cdots<j_{r-1}}(\lambda_{i_1}\lambda_{j_1})\cdots(\lambda_{i_{r-1}}\lambda_{j_{r-1}}).$$

On the other hand, from Gauss equation we have that

$$K_\Sigma(e_i,e_j)=\overline{K}(e_i,e_j)-\lambda_i\lambda_j, \tag{3.18}$$

where K_Σ and $\overline{K}$ are the sectional curvatures of Σ^n and $\overline{M}^{n+1}$, respectively.

With a straightforward computation using a general relationship between the curvature tensor of a warped product and the curvature tensor of its base and its fiber (cf. Proposition 7.42 of [27]; see also equation (6.6) of [6]) we obtain that

$$\begin{aligned}\overline{R}(U,V)W&=R_{M^n}(U^*,V^*)W^*+((\log f)'(h))^2(\langle U,W\rangle V-\langle V,W\rangle U)\\&\quad-(\log f)''(h)\langle W,\partial_t\rangle(\langle U,\partial_t\rangle V-\langle V,\partial_t\rangle U)\\&\quad-(\log f)''(h)(\langle U,W\rangle\langle V,\partial_t\rangle-\langle U,\partial_t\rangle\langle V,W\rangle)\partial_t\end{aligned} \tag{3.19}$$

for arbitrary vector fields U,V,W in $\overline{M}^{n+1}$, where $U^*=(\pi_{M^n})_*U=U+\langle U,\partial_t\rangle\partial_t$. Then, for an orthonormal basis $\{X,Y\}$ of an arbitrary 2-plane tangent to Σ^n, equation (3.19) gives

$$\begin{aligned}\overline{K}(X,Y)&=\frac{1}{f^2(h)}K_M(X^*,Y^*)|X^*\wedge Y^*|^2\\&\quad+((\log f)'(h))^2(\langle X,X\rangle\langle Y,Y\rangle-\langle Y,X\rangle\langle X,Y\rangle)\\&\quad-(\log f)''(h)\langle X,\partial_t\rangle(\langle X,\partial_t\rangle\langle Y,Y\rangle-\langle Y,\partial_t\rangle\langle X,Y\rangle)\\&\quad-(\log f)''(h)(\langle X,X\rangle\langle Y,\partial_t\rangle-\langle X,\partial_t\rangle\langle Y,X\rangle)\langle\partial_t,Y\rangle\\&=\frac{1}{f^2(h)}K_M(X^*,Y^*)|X^*\wedge Y^*|^2+((\log f)'(h))^2\\&\quad-(\log f)''(h)(\langle X,\partial_t\rangle^2+\langle Y,\partial_t\rangle^2).\end{aligned} \tag{3.20}$$

Since $\nabla h=-\partial_t^\top=-\partial_t-\Theta N$, we have that

$$\langle X,\partial_t\rangle^2=\langle X,-\nabla h-\Theta N\rangle^2=\langle X,\nabla h\rangle^2. \tag{3.21}$$

Moreover,

$$
\begin{aligned}
|X^* \wedge Y^*|^2 &= |X^*|^2|Y^*|^2 - \langle X^*, Y^*\rangle^2 \\
&= \langle X^*, X^*\rangle\langle Y^*, Y^*\rangle - \langle X^*, Y^*\rangle^2 \\
&= (1+\langle X, \partial_t\rangle^2)(1+\langle Y, \partial_t\rangle^2) - \langle X, \partial_t\rangle^2\langle Y, \partial_t\rangle^2 \\
&= 1+\langle X, \partial_t\rangle^2 + \langle Y, \partial_t\rangle^2 \\
&= 1+\langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2. \qquad (3.22)
\end{aligned}
$$

Consequently, inserting (3.21) and (3.22) into (3.20) we get

$$
\begin{aligned}
\overline{K}(X,Y) &= \frac{1}{f^2(h)}K_M(X^*,Y^*)(1+\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2) + ((\log f)'(h))^2 \\
&\quad - (\log f)''(h)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2) \\
&= \frac{1}{f^2(h)}K_M(X^*,Y^*) + ((\log f)'(h))^2 \\
&\quad + \left(\frac{1}{f^2(h)}K_M(X^*,Y^*) - (\log f)''(h)\right)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2) \\
&= \frac{1}{f^2(h)}K_M(X^*,Y^*) + \left(\frac{f'}{f}(h)\right)^2 \\
&\quad + \frac{1}{f^2(h)}(K_M(X^*,Y^*) - f(h)f''(h) + (f'(h))^2)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2). \qquad (3.23)
\end{aligned}
$$

Since we are assuming the NCC (3.15), from (3.23) we deduce the following inequality

$$\overline{K}(X,Y) \geq \frac{f''}{f}(h). \qquad (3.24)$$

Thus, from (3.18) and (3.24) we obtain

$$\lambda_i\lambda_j = \overline{K}(e_i,e_j) - K_\Sigma(e_i,e_j) \geq \frac{f''}{f}(h) - K_\Sigma(e_i,e_j). \qquad (3.25)$$

Consequently, from (3.16) and (3.25) we have $\lambda_i\lambda_j \geq 0$ for all $i,j \in \{1,\dots,n\}$, with $i \neq j$. Hence,

$$\langle \mathcal{P}_{r-1}e_i, e_i\rangle = \binom{n}{r-1}^{-1}\sum(\lambda_{j_1}\lambda_{i_1})\cdots(\lambda_{j_{r-1}}\lambda_{i_{r-1}}) \geq 0. \qquad (3.26)$$

So, from (3.26) we conclude that $\mathcal{P}_{r-1}$ is positive semi-definite. Thus, since H_{r-1} and H_r are positive, $\Theta \leq -1$ and taking into account that (3.15) is satisfied, from (3.12) we obtain

$$
\begin{aligned}
\mathcal{L}_r(g(h)) &= f(h)(n-r)\left(\frac{\kappa}{f^2(h)} - (\log f)''(h)\right)\frac{1}{H_{r-1}}\langle \mathcal{P}_{r-1}\nabla h, \nabla h\rangle\Theta \\
&\quad - c_r(f'(h)H_r + f(h)H_{r+1}\Theta)
\end{aligned}
$$

$$\leq -c_r f(h) H_r \Theta \left(\frac{H_{r+1}}{H_r} + \frac{1}{\Theta} \frac{f'}{f}(h) \right). \tag{3.27}$$

Hence, considering inequality (3.17) into (3.27) we get that $\mathcal{L}_r(g(h)) \leq 0$ on Σ^n. Moreover, hypotheses $\sup_\Sigma H_r < +\infty$ and (3.2) assure that Σ^n is $\mathcal{L}_r$-parabolic. Therefore, since Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. □

Remark 3.10. Concerning Theorem 3.9, we observe that when Σ^n has an elliptic point, hypothesis (3.16) can be dropped. Furthermore, we point out that inequality (3.17) was already used in Theorem 4 of [16] to obtain an extension of Theorem 3.7 in [7] and Theorem 4.1 in [29].

3.3. Rigidity and nonexistence of two-sided hypersurfaces

Similarly to the case of spacelike hypersurfaces in GRW spacetimes, in this subsection we will establish rigidity and nonexistence results concerning complete two-sided hypersurfaces immersed in a Riemannian warped product. We recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, there is on it a globally defined unit normal vector field N.

Theorem 3.11. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_f M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis* (3.2) *is satisfied and*

$$0 < H \leq \frac{f'}{f}(h), \tag{3.28}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Taking into account that $H > 0$ and $-1 \leq \Theta \leq 0$, from Lemma 2.4 jointly with (3.28) we obtain that

$$\mathcal{L}_0(g(h)) = n(f'(h) + f(h)H\Theta) \geq nf(h)\left(\frac{f'}{f}(h) - H \right). \tag{3.29}$$

Hence, using inequality (3.28) in (3.29), we get that $\mathcal{L}_0(g(h)) \geq 0$. Moreover, by hypothesis (3.2) we have that Σ^n is $\mathcal{L}_0$-parabolic. So, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we obtain that $g(h)$ is constant on Σ^n and, therefore, Σ^n is a slice of $\overline{M}^{n+1}$. □

When the warping function f is either exponential or hyperbolic cosine, following the terminology introduced by [31], the corresponding warped product $\mathbb{R} \times_{e^t} M^n$ or $\mathbb{R} \times_{\cosh t} M^n$ has been referred to as a *pseudo-hyperbolic space*. Tashiro's terminology is due to the fact that with suitable choices of the fiber M^n we obtain warped products which are isometric to the hyperbolic space. For more details about these spaces see, for instance, [9, 10, 24, 25]. In this context, we get the following applications of Theorem 3.11.

Corollary 3.12. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis* (3.2) *is satisfied and $0 < H \leq 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 3.13. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis* (3.2) *is satisfied and $0 < H \leq \tanh(h)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In our next result, we will suppose that the ambient space obeys a suitable curvature constraint which is the opposite of that assumed in the results of [25].

Theorem 3.14. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_f M^n$, which obeys the following curvature constraint*

$$\mathrm{Ric}_M \geq (n-1) \sup_I ((f')^2 - f f'') \langle \, , \rangle_M, \tag{3.30}$$

where Ric_M stands for the Ricci tensor of M^n. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis (3.2) *is satisfied, $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and*

$$\frac{H_2}{H} \leq \frac{f'}{f}(h), \tag{3.31}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. From Lemma 2.4 and equation (2.8) of Lemma 2.5 we obtain that

$$\begin{aligned} \mathcal{L}_1(g(h)) = f(h)(&-\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)|\nabla h|^2)\Theta \\ &+ c_1(f'(h)H + f(h)H_2\Theta), \end{aligned} \tag{3.32}$$

where $N^* = N - \Theta \partial_t$.

Taking into account that $|N^*|_M^2 = \frac{1}{f^2(h)}|\nabla h|^2$, using curvature constraint (3.30) we obtain

$$(n-1)\left(\frac{(f')^2 - f f''}{f^2}\right)(h)|\nabla h|^2 - \mathrm{Ric}_M(N^*, N^*) \leq 0. \tag{3.33}$$

Thus, since we are assuming $-1 \leq \Theta \leq 0$, considering (3.33) into (3.32) we get

$$\mathcal{L}_1(g(h)) \geq c_1 f(h) H \left(\frac{f'}{f}(h) - \frac{H_2}{H}\right).$$

Hence, using hypothesis (3.31) we reach at $\mathcal{L}_1(g(h)) \geq 0$. Moreover, since Lemma 2.1 gives that P_1 is positive definite, we can apply Proposition 3.1 to guarantee that Σ^n is $\mathcal{L}_1$-parabolic. So, since Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we get that the function $g(h)$ is constant. Therefore, we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. □

We can reason as in the proof of Corollary 3.5, obtaining the following consequence of Theorem 3.14:

Corollary 3.15. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into an Einstein warped product $\overline{M}^{n+1} = I \times_f M^n$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and hypotheses* (3.2) *and* (3.31) *are satisfied, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient is a pseudo-hyperbolic space, Theorem 3.14 leads us to the following applications:

Corollary 3.16. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{e^t} M^n$ whose fiber M^n has nonnegative Ricci curvature. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis* (3.2) *is satisfied, $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and $\frac{H_2}{H} \leq 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Corollary 3.17. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a pseudo-hyperbolic space $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} M^n$ whose Ricci tensor of the fiber M^n is such that $\mathrm{Ric}_M \geq -(n-1)\langle\, ,\rangle_M$. Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$ and that $-1 \leq \Theta \leq 0$. If hypothesis* (3.2) *is satisfied, $H > 0$ with $\sup_\Sigma H < +\infty$, $H_2 > 0$ and $\frac{H_2}{H} \leq \tanh(h)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In our next results, we deal with higher order mean curvatures.

Theorem 3.18. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1} = I \times_f M^n$ whose fiber M^n has constant sectional curvature κ and it obeys the curvature constraint*

$$\kappa \geq \sup_I((f')^2 - ff''). \tag{3.34}$$

Suppose that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, $-1 \leq \Theta \leq 0$ and that the sectional curvature of Σ^n, K_Σ, is such that

$$K_\Sigma \geq \frac{1}{f^2(h)}(\kappa - (f'(h))^2). \tag{3.35}$$

If hypothesis (3.2) *is satisfied, $H_{r-1} > 0$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$, and*

$$\frac{H_{r+1}}{H_r} \leq \frac{f'}{f}(h) \tag{3.36}$$

for some $2 \leq r \leq n-1$, then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since the fiber M^n has constant sectional curvature κ, from (3.1) jointly with Lemma 2.4 and equation (2.9) of Lemma 2.5 we obtain

$$\begin{aligned}\mathcal{L}_r(g(h)) = &-(n-r)f(h)\left(\frac{\kappa}{f^2(h)} + (\log f)''(h)\right)\langle P_{r-1}\nabla h, \nabla h\rangle\Theta \\ &+ c_r(f'(h)H_r + f(h)H_{r+1}\Theta).\end{aligned} \tag{3.37}$$

On the other hand, as in the proof of Theorem 3.9, we will consider the self-adjoint operator $\mathcal{P}_{r-1} : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ defined by

$$\mathcal{P}_{r-1} = H_{r-1}P_{r-1}.$$

For each $p \in \Sigma^n$, we take a local orthonormal frame $\{e_1, \dots, e_n\}$ such that $Ae_i = \lambda_i e_i$. From (2.3) we have that $P_{r-1}e_i = \sum_{i_1<\cdots<i_{r-1}, i_j\neq i} \lambda_{i_1}\cdots\lambda_{i_{r-1}}$. Thus, for any $i \in \{1, \dots, n\}$, we get

$$\langle \mathcal{P}_{r-1}e_i, e_i\rangle = \binom{n}{r-1}^{-1} \sum_{i_1<\cdots<i_{r-1}, i_j\neq i, j_1<\cdots<j_{r-1}} (\lambda_{i_1}\lambda_{j_1})\cdots(\lambda_{i_{r-1}}\lambda_{j_{r-1}}).$$

From Gauss equation, we have that

$$K_\Sigma(e_i, e_j) = \overline{K}(e_i, e_j) + \lambda_i\lambda_j.$$

Moreover, using once more Proposition 7.42 of [27] we get

$$\begin{aligned}\overline{R}(U,V)W = &R_{M^n}(U^*, V^*)W^* - ((\log f)'(h))^2(\langle U, W\rangle V - \langle V, W\rangle U) \\ &- (\log f)''(h)\langle W, \partial_t\rangle(\langle U, \partial_t\rangle V - \langle V, \partial_t\rangle U) \\ &- (\log f)''(h)(\langle U, W\rangle\langle V, \partial_t\rangle - \langle U, \partial_t\rangle\langle V, W\rangle)\partial_t\end{aligned}$$

for arbitrary vector fields U, V, W in $\overline{M}^{n+1}$, where $U^* = (\pi_{M^n})_*U = U - \langle U, \partial_t\rangle\partial_t$.

Thus, for an orthonormal basis $\{X, Y\}$ we find that

$$\begin{aligned}\overline{K}(X,Y) = &\frac{1}{f^2(h)}K_M(X^*, Y^*)|X^*\wedge Y^*|^2 \\ &- ((\log f)'(h))^2 - (\log f)''(h)(\langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2).\end{aligned}$$

Since that $|X^*\wedge Y^*|^2 = 1 - \langle X, \nabla h\rangle^2 - \langle Y, \nabla h\rangle^2$, we get

$$\begin{aligned}\overline{K}(X,Y) &= \frac{1}{f^2(h)}K_M(X^*, Y^*)(1 - (\langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2)) \\ &\quad - ((\log f)'(h))^2 - (\log f)''(h)(\langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2) \\ &= \frac{1}{f^2(h)}K_M(X^*, Y^*) - ((\log f)'(h))^2 \\ &\quad - \left(\frac{1}{f^2(h)}K_M(X^*, Y^*) + (\log f)''(h)\right)(\langle X, \nabla h\rangle^2 + \langle Y, \nabla h\rangle^2) \\ &= \frac{1}{f^2(h)}K_M(X^*, Y^*) - \left(\frac{f'}{f}(h)\right)^2\end{aligned}$$

$$-\left(\frac{1}{f^2(h)}K_M(X^*,Y^*)+\frac{ff''-(f')^2}{f^2}(h)\right)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2).$$

We also note that

$$\begin{aligned}\lambda_i\lambda_j &= K_\Sigma(e_i,e_j)-\overline{K}(e_i,e_j)\\ &= K_\Sigma(e_i,e_j)+\left(\frac{f'}{f}(h)\right)^2-\frac{1}{f^2(h)}K_M(X^*,Y^*)\\ &\quad+\frac{1}{f^2(h)}(K_M(X^*,Y^*)+(ff''-(f')^2)(h))(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2).\end{aligned}$$

Then, taking into account the curvature constraint (3.34) and inequality (3.35), we get $\lambda_i\lambda_j\geq 0$ for all $i,j\in\{1,\dots,n\}$, with $i\neq j$. So, $\langle\mathcal{P}_{r-1}e_i,e_i\rangle\geq 0$, and $\mathcal{P}_{r-1}$ is positive semi-definite. Thus, taking into account that H_{r-1} and H_r are positive and $-1\leq\Theta\leq 0$, from (3.34) and (3.37) we get

$$\begin{aligned}\mathcal{L}_r(g(h)) = &-(n-r)f(h)\left(\frac{\kappa}{f^2(h)}+(\log f)''(h)\right)\frac{1}{H_{r-1}}\langle\mathcal{P}_{r-1}\nabla h,\nabla h\rangle\Theta\\ &+c_r(f'(h)H_r+f(h)H_{r+1}\Theta)\\ \geq\ & c_rf(h)H_r\left(\frac{f'}{f}(h)-\frac{H_{r+1}}{H_r}\right).\end{aligned}\tag{3.38}$$

Hence, considering (3.36) into (3.38) we conclude that $\mathcal{L}_r(g(h))\geq 0$ on Σ^n. Consequently, since we are assuming that Σ^n is bounded away from the future infinity of $\overline{M}^{n+1}$, we can apply Proposition 3.1 to obtain that h is constant on Σ^n. Therefore, Σ^n must be a slice of $\overline{M}^{n+1}$. □

From Theorem 3.18 we get the following nonexistence result:

Corollary 3.19. *Let $\overline{M}^{n+1}=I\times M^n$ be a Riemannian warped product whose fiber M^n has constant nonnegative sectional curvature κ. There is no complete two-sided hypersurface $\psi:\Sigma^n\to\overline{M}^{n+1}$ bounded away from the future infinity of $\overline{M}^{n+1}$, with $-1\leq\Theta\leq 0$, satisfying hypothesis* (3.2) *and such that $K_\Sigma\geq\kappa$, $H_{r-1}>0$, $H_r>0$ with $\sup_\Sigma H_r<+\infty$ and $H_{r+1}\leq 0$ for some $2\leq r\leq n-1$.*

Related to the higher order mean curvatures, we also establish the following result:

Theorem 3.20. *Let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a complete two-sided hypersurface immersed into a Riemannian warped product $\overline{M}^{n+1}=I\times_f M^n$ whose fiber M^n has constant sectional curvature κ satisfying*

$$\kappa\leq\inf_I((f')^2-ff''). \tag{3.39}$$

Suppose that Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$ and that $-1\leq\Theta<0$. Assume in addition that $f(h)$ attains a local maximum at some *point $p \in \Sigma^n$ such that $f'(h(p)) \neq 0$. If hypothesis* (3.2) *is satisfied,* $H_r > 0$ *with* $\sup_\Sigma H_r < +\infty$, $H_{r+1} > 0$ *and*

$$\frac{H_{r+1}}{H_r} \geq -\frac{1}{\Theta}\frac{f'}{f}(h) \tag{3.40}$$

for some $2 \leq r \leq n-1$*, then* Σ^n *is a slice of* $\overline{M}^{n+1}$*.*

Proof. Since we are assuming that $f(h)$ attains a local maximum at some point $p \in \Sigma^n$ such that $f'(h(p)) \neq 0$, Lemma 2.3 guarantees that p is an elliptic point of Σ^n. So, using the assumption $H_{r+1} > 0$, from Lemma 2.2 the operator L_j is elliptic or, equivalently, P_j is positive definite for all $1 \leq j \leq r$. Thus, taking into account curvature constraint (3.39) and since $-1 \leq \Theta < 0$, from (3.37) we obtain

$$\mathcal{L}_r(g(h)) \leq c_r f(h) H_r \Theta \left(\frac{H_{r+1}}{H_r} + \frac{1}{\Theta}\frac{f'}{f}(h) \right).$$

Hence, from hypothesis (3.40) we have that $\mathcal{L}_r(g(h)) \leq 0$ on Σ^n. Therefore, since Σ^n is bounded away from the past infinity of $\overline{M}^{n+1}$, we can apply once more Proposition 3.1 to conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. □

3.4. Applications to entire graphs

Let $\Omega \subseteq M^n$ be a connected domain of M^n. A (vertical) graph over Ω is determined by a smooth function $u \in C^\infty(\Omega)$ and it is given by

$$\Sigma(u) = \{(u(x), x) : x \in \Omega\} \subset \epsilon I \times_f M^n.$$

The metric induced on Ω from the metric on the ambient space via $\Sigma^n(u)$ is

$$\langle\, , \rangle = \epsilon du^2 + f^2(u)\langle\, , \rangle_M. \tag{3.41}$$

We observe that for a graph $\Sigma(u)$, its height function h is nothing but the function u seen as a function on $\Sigma(u)$. Therefore, in what follows, Du stands for the gradient of u, as a function on M^n, while $\nabla u = \nabla h$ stands for the gradient of the height function, as a function on $\Sigma(u)$.

The graph is said to be *entire* if $\Omega = M^n$. It can be easily seen that in the case $\epsilon = 1$, when the function $f(u)$ is bounded on M^n, the entire graph $\Sigma(u)$ is complete. In particular, this occurs when $\Sigma(u)$ lies between two slices of $I \times_f M^n$. In the case $\epsilon = -1$, a graph $\Sigma(u)$ is a spacelike hypersurface if and only if $|Du|^2_{M^n} < f^2(u)$, where $|Du|_{M^n}$ stands for the norm of Du with respect to the metric $\langle\, , \rangle_M$ in Ω. From Lemma 3.1 of [14], in the case where M^n is a simply connected manifold, every complete spacelike hypersurface Σ^n in $-I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph in such space. In particular, this happens for complete spacelike hypersurfaces contained in a timelike bounded region. However, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph in a complete GRW spacetime is not necessarily complete, in the sense that the induced Riemannian metric (3.41) is not necessarily complete on M^n.

For instance, Albujer constructed explicit examples of noncomplete entire maximal spacelike graphs (that is, whose mean curvature is identically zero) in the Lorentzian product space $-\mathbb{R}\times\mathbb{H}^2$ (see Section 3 of [1]).

Being $\Sigma(u)\subset\epsilon I\times_f M^n$ an entire graph, its orientation N which corresponds to the choices made in Subsections 3.2 and 3.3 is given by

$$N=\frac{f(u)}{W(u)}\left(-\epsilon\partial_t+\frac{1}{f^2(u)}Du\right), \tag{3.42}$$

where $W(u):=\sqrt{f^2(u)+\epsilon|Du|^2_{M^n}}$.

When $\overline{M}^{n+1}=-I\times_f M^n$ is a GRW spacetime, we can restate Theorem 3.7 in the context of entire graphs as follows:

Corollary 3.21. *Let $\overline{M}^{n+1}=-I\times_f M^n$ be a GRW spacetime whose fiber M^n has constant sectional curvature κ satisfying curvature constraint* (3.10) *and let $\Sigma(u)$ be an entire graph determined by a bounded function $u\in C^\infty(M)$ such that, for some $2\leq r\leq n-1$, $H_r>0$ with $\sup_M H_r<+\infty$ and $H_{r+1}>0$. Suppose that $f(u)$ attains a local minimum at some point $x\in M^n$ such that $f'(u(x))\neq 0$ and that $|Du|^2_M\leq\alpha f^2(u)$ for some constant $0<\alpha<1$. If hypothesis* (3.2) *is satisfied by $\Sigma(u)$ and*

$$\frac{H_{r+1}}{H_r}\geq\frac{f'}{f}(u), \tag{3.43}$$

then $u\equiv t_0$ for some $t_0\in I$.

Proof. As in the beginning of the proof of Corollary 5.1 in [7], our constraint on $|Du|_M$ guarantees that $\Sigma(u)$ is complete. Consequently, since we are also assuming that hypotheses (3.2) and (3.43) are satisfied, we can apply Theorem 3.7 to conclude the result. □

Taking into account (3.42), it is not difficult to see that we can also reformulate Theorem 3.9 in the context of entire graphs as follows:

Corollary 3.22. *Let $\overline{M}^{n+1}=-I\times_f M^n$ be a GRW spacetime whose fiber M^n has constant sectional curvature κ satisfying the NCC* (3.15) *and let $\Sigma(u)$ be an entire graph determined by a bounded function $u\in C^\infty(M)$ such that, for some $2\leq r\leq n-1$, $H_{r-1}>0$ and $H_r>0$ with $\sup_\Sigma H_r<+\infty$. Suppose that the sectional curvature of $\Sigma(u)$ satisfies* (3.16) *and that $|Du|^2_M\leq\alpha f^2(u)$ for some constant $0<\alpha<1$. If hypothesis* (3.2) *is satisfied by $\Sigma(u)$ and*

$$\frac{H_{r+1}}{H_r}\leq\frac{f'}{f^2}(u)W(u),$$

then $u\equiv t_0$ for some $t_0\in I$.

When the ambient space is a Riemannian warped product, it is not difficult to verify that all results in Subsection 3.3 can be also rewritten for the context of entire graphs. In particular, we quote the following nonparametric versions of Theorems 3.18 and 3.20:

Corollary 3.23. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product whose fiber M^n has constant sectional curvature κ obeying the curvature constraint* (3.34) *and let be $\Sigma(u)$ be an entire graph determined by a bounded function $u \in C^\infty(M)$ such that, for some $2 \le r \le n-1$, $H_{r-1} > 0$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$. Suppose that the sectional curvature of $\Sigma(u)$ satisfies* (3.35) *and that $|Du|_M < +\infty$. If hypothesis* (3.2) *is satisfied by $\Sigma(u)$ and*

$$\frac{H_{r+1}}{H_r} \le \frac{f'}{f}(u),$$

then $u \equiv t_0$ for some $t_0 \in I$.

Corollary 3.24. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a Riemannian warped product whose fiber M^n has constant sectional curvature κ obeying the curvature constraint* (3.39) *and let be $\Sigma(u)$ be an entire graph determined by a bounded function $u \in C^\infty(M)$ such that, for some $2 \le r \le n-1$, $H_r > 0$ with $\sup_\Sigma H_r < +\infty$ and $H_{r+1} > 0$. Suppose that $f(u)$ attains a local maximum at some point $x \in M^n$ such that $f'(u(x)) \neq 0$ and that $|Du|_M < +\infty$. If hypothesis* (3.2) *is satisfied by $\Sigma(u)$ and*

$$\frac{H_{r+1}}{H_r} \ge \frac{f'}{f^2}(u)W(u),$$

then $u \equiv t_0$ for some $t_0 \in I$.

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