East Asian Math. J.
Vol. 40 (2024), No. 1, pp. 001–023
http://dx.doi.org/10.7858/eamj.2024.001

SEMI-INVARIANT SUBMANIFOLDS OF CODIMENSION 3 IN A COMPLEX SPACE FORM WITH ξ-PARALLEL STRUCTURE JACOBI OPERATOR

U - Hang KI and Hyunjung SONG*

Abstract. Let M be a semi-invariant submanifold of codimension 3 with almost contact metric structure (ϕ, ξ, η, g) in a complex space form $M_{n+1}(c)$. We denote by A, K and L the second fundamental forms with respect to the unit normal vector C, D and E respectively, where C is the distinguished normal vector, and by $R_\xi = R(\xi, \cdot)\xi$ the structure Jacobi operator. Suppose that the third fundamental form t satisfies $dt(X,Y) = 2\theta g(\phi X, Y)$ for a scalar $\theta(\neq 2c)$ and any vector fields X and Y, and at the same time $R_\xi K = KR_\xi$ and $\nabla_{\phi\nabla_\xi \xi} R_\xi = 0$. In this paper, we prove that if it satisfies $\nabla_\xi R_\xi = 0$ on M, then M is a real hypersurface of type (A) in $M_n(c)$ provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$.

1. introduction

Let $\tilde{M}$ a Kaehlerian manifoldn manifold with complex structure J. A submanifold M of $\tilde{M}$ is called a $CR\ submanifold$ if there exists a differentiable distribution $\triangle : p \to \triangle_p \subset T_pM$ on M such that $\triangle$ is J-invariant and the complementary orthogonal distribution $\triangle^\perp$ is totally real, where T_pM denote by the tangent space at each point p in M ([1], [33]). In particular, M is said to be a *semi-invariant submanifold* if dim $\triangle^\perp = 1$. In this case, M admits an almost contact metric structure (ϕ, ξ, η, g). A typical example of a semi-invariant submanifold is real hypersurfaces in $\tilde{M}$. Furthermore, nontrivial examples of semi-invariant submanifold in a complex projective space are constructed in [18] and [28]. Thus, we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

As is well known, complete and simply connected nonflat complex space form $M_n(c)$ are isometric to a complex projective space $P_n\mathbb{C}$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$ or $c < 0$.

Received September 25, 2023; Accepted November 29, 2023.
2010 *Mathematics Subject Classification.* 53B25, 53C40, 53C42.
Key words and phrases. semi-invariant submanifold, distinguished normal, complex space form, structure Jacobi operator, scalar curvature.
* Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

In the study of real hypersurfaces in a complex projective space $P_n\mathbb{C}$, Takagi ([29], [30]) classified all homogeneous Hopf hypersurfaces, and Cecil-Ryan ([5]) and Kimura ([21]) showed that they can be regarded as the tubes of constant radius over Kaehlerian submanifolds. Such tubes can be divided into six type : A_1, A_2, B, C, D and E.

In the case of real hypersurfaces in a complex hyperbolic space $H_n\mathbb{C}$, the classification of homogenous raal hypersurfaces in $H_n\mathbb{C}$ was obtained by Berndt-Tamaru([3]). Berndt ([2]) showed that all real hypersurfaces with constant principal curvatures are realized as the tubes over certain submanifolds. Such tubes are said to be real hypersurfaces of type A_0, A_1, A_2 and B.

Among the several types of real hypersurfaces appearing in Takagi′s list or Berndt′s list, several pieces are tubes over totally geodesic $P_n\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$. These and a horosphere in $H_n\mathbb{C}$ are together said to be of *type* (A).

Characterization problems for a real hypersurface of type (A) in a complex space form $M_n(c)$ were started by many authors ([7], [13], [14], [19], [22], [23], [24], etc). Two of them, we introduce the following theorem without proof due to Okumura ([24]) for $c > 0$ and Montiel-Romero ([22]) for $c < 0$ respectively.

Theorem O-MR. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. Then $A\phi = \phi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely :*

(I) *in case of $P_n\mathbb{C}$,*
  (A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \pi/2$,*
  (A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$*
(II) *in case of $H_n\mathbb{C}$,*
  (A_0) *a horosphere $H_n\mathbb{C}$, i,e, a Montiel tube,*
  (A_1) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*
  (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ $(1 \leq k \leq n-2)$.*

We define the Jacobi operator $R_\xi(X) = R(X, \xi)\xi$ with respect to the structure vector ξ for the curvature tensor R and any vector field X on M. Then R_ξ is a self-adjoint endomorphism on the tangent space of a CR submanifold M. But, it is known that there no real hypersurfaces in a complex space form $M_n(c)$ with parallel structure Jacobi operator ([27]). Using several conditions on the structure Jacobi operator R_ξ, characterization problems for a real hypersurface of type (A) have recently studied. In the previous paper ([13]), Kurihara and the present author gave, using the structure Jacobi operator, another characterization of a real hypersurface of type (A) in a complex space form. Namely they proved the following :

Theorem K([13]). *Let M be a connected real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. If it satisfies $\nabla_\xi R_\xi = 0$ and $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$, then M is a real hypersurface of type (A).*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex space form $M_{n+1}(c)$ have been studied in [18] by using properties of induced almost contact structure and those of the third fundamental form of the submanifold. Furthermore, using several conditions for the structure Jacobi operator R_ξ, semi-invariant submanifolds of codimension 3 in a complex space form were studied ([9], [12], [15], [16], [17], etc.).

In the present paper, we discuss a semi-invariant submanifold version of the Theorem K, that is, we consider a semi-invariant submanifold M of codimension 3 in a nonflat complex space form $M_{n+1}(c)$ which satisfies $R_\xi K = KR_\xi$ and at the same time the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, where $\omega(X, Y) = g(\phi X, Y)$ for any vector fields X and Y on M. Then we prove that if it satisfies $\nabla_\xi R_\xi = 0$, then M is a real hypersurface in $M_n(c)$ provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$. Further, we also prove that M satisfies $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$, then M is a real hypersurface of type (A). Our main theorem appears in section 4.

A all manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant submanifold supposed to be orientable.

2. Structure equations of semi-invariant submanifolds

In this section, elemental facts of semi-invariant submanifolds are re-called.

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature 4c with parallel almost complex structure J and a Riemannian metric tensor G, which is called a *complex space form* and denoted by $M_{n+1}(c)$. Let M be a real $(2n-1)$-dimensional Riemannian manifold immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. In the sequel, we identify $i(M)$ with M itself. We denote by g the Riemannian metric tensor on M from that of $\tilde{M}$.

If we denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M, then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C + g(KX, Y)D + g(LX, Y)E, \tag{2.1}$$

$$\begin{aligned} \tilde{\nabla}_X C &= -AX + l(X)D + m(X)E, \\ \tilde{\nabla}_X D &= -KX - l(X)C + t(X)E, \\ \tilde{\nabla}_X E &= -LX - m(X)C - t(X)D \end{aligned} \tag{2.2}$$

for any vector fields tangent to X and Y on M and any unit vector field C, D and E normal to M because we take C, D and E are mutually orthogonal, where A, K, L are called the *second fundamental forms* and l, m and t the *third fundamental forms.*

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a *CR submanifold* ([1], [33]) if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution (Δ, Δ^{-1}) such that for any point p in M we have $J\Delta_p = T_pM$, $JT_p^{\perp} \subset T_p^{\perp}M$, where $T_p^{\perp}M$ denote the normal space of M at p. In particular, M is said to be *semi-invariant submanifold*([4], [31]) provided that $dim\Delta^{\perp} = 1$ or to be a *CR submanifold* with *CR dimension* $n-1$([25]).

In this case the unit normal vector field in $J\Delta^{\perp}$ is called a *distinguished normal* to the semi-invariant submanifold and denote this by C ([31], [32]).

From now on we discuss that M is a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a Kaehlerian manifold $\tilde{M}$ of real $2(n+1)$-dimension. Then we can choose a local orthonormal frame field $\{e_1, \cdots, e_{n-1}, Je_1, \cdots, J_{e_{n-1}}, e_0 = \xi, C = J\xi, D = JE, E\}$ on the tangent space $T_p\tilde{M}$ of $\tilde{M}$ for any point p in M such that $e_1, \cdots, e_{n-1}, Je_1, \cdots, J_{e_{n-1}}, \xi \in T_pM$, and C, D, $E \in T_p^{\perp}M$.

Now, let ϕ be the restriction of J on M, then we have

$$JX = \phi X + \eta(X)C, \quad \eta(X) = g(\xi, X), \quad JC = -\xi \tag{2.3}$$

for any vector field X on M ([32]). From this it is, using Hermitian property of J, verified that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have

$$\phi^2X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X),$$
$$\phi\xi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y.

In the sequel, we denote the normal components of $\tilde{\nabla}_X C$ by $\nabla^{\perp}C$. The distinguished normal C is said to be *parallel* in the normal bundle if we have $\nabla^{\perp}C = 0$, that is, l and m vanish identically.

Using the Kaehler condition $\tilde{\nabla}J = 0$ and the Gauss and Weingarten formulas,we obtain from (2.3)

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.4}$$

$$\nabla_X\xi = \phi AX, \tag{2.5}$$

$$KX = \phi LX - m(X)\xi, \tag{2.6}$$

$$LX = -\phi KX + l(X)\xi \tag{2.7}$$

for any vectors X and Y on M. From the last two equations, we have

$$g(K\xi, X) = -m(X), \tag{2.8}$$

$$g(L\xi, X) = l(X). \tag{2.9}$$

Using the frame field $\{e_0 = \xi, e_1, \cdots, e_{n-1}, \phi e_1, \cdots, \phi e_{n-1}\}$ on M it follows from (2.6) $\sim$ (2.9) that

$$T_rK = \eta(K\xi) = -m(\xi), \quad T_rL = \eta(L\xi) = l(\xi). \tag{2.10}$$

Now, we retake D and E, there is no loss of generality such that we may assume $T_rL = 0$(cf. [18]). So we have

$$l(\xi) = 0. \tag{2.11}$$

In what follows, to write our formulas in a convention form, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $T_rA = h$, $T_rK = k$, $T_r({}^tAA) = h_{(2)}$ and for a function f we denote by ∇f the gradient vector field of f.

From (2.10) we also have

$$m(\xi) = -k. \tag{2.12}$$

From (2.6) and (2.7) we also get

$$\eta(X)l(\phi Y) - \eta(Y)l(\phi X) = m(Y)\eta(X) - m(X)\eta(Y),$$

which together with (2.12) gives

$$l(\phi X) = m(X) + k\eta(X), \tag{2.13}$$

which tells us, using (2.11), that

$$m(\phi X) = -l(X), \tag{2.14}$$

where we have used (2.9) and (2.11).

Taking the inner product with LY to (2.6) and using (2.9), we obtain

$$g(KLX, Y) + g(LKX, Y) = -\{l(X)m(Y) + l(Y)m(X)\}. \tag{2.15}$$

We put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ be because of (2.5).

We put

$$A\xi = \alpha\xi + \mu W, \tag{2.16}$$

where W is a unit vector orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.17}$$

by virtue of (2.5). Thus, W is also orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.18}$$

From (2.16) and (2.17) we obtain

$$\phi U = -A\xi + \alpha\xi. \tag{2.19}$$

If we take account of (2.5), (2.10) and (2.19), then we find

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{2.20}$$

Since W is orthogonal to ξ, we can, using (2.5) and (2.17), see that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{2.21}$$

Differentiating (2.19) covariantly along M and using (2.4) and (2.5), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX. \tag{2.22}$$

In the rest of this paper we shall suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ and that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X,Y) = g(\phi X, Y) \tag{2.23}$$

for any vector fields X and Y and a certain scalar θ, where d denotes by the exterior differential operator. Then we can verify that (see, [18])

$$l = 0 \tag{2.24}$$

provided that $\theta - 2c \neq 0$ and hence

$$m(X) = -k\eta(X) \tag{2.25}$$

because of (2.13). Using these facts (2.8) and (2.9) turn out respectively to

$$K\xi = k\xi, \quad L\xi = 0. \tag{2.26}$$

Because of (2.24) and (2.25), we can also write respectively (2.6) and (2.7) as

$$KX = \phi LX + k\eta(X), \tag{2.27}$$

$$L = -\phi K. \tag{2.28}$$

Since $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$, we have

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X \\ &- g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY \\ &+ g(KY,Z)KX - g(KX,Z)KY + g(LY,Z)LX - g(LX,Z)LY. \end{aligned} \tag{2.29}$$

If we take account of (2.24) and (2.25), then equations of the Codazzi are given respectively by

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = k\{&\eta(Y)LX - \eta(X)LY\} \\ &+ c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \end{aligned} \tag{2.30}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = t(X)LY - t(Y)LX, \tag{2.31}$$

$$(\nabla_X L)Y - (\nabla_Y L)X = k\{\eta(X)AY - \eta(Y)AX\} - t(X)KY + t(Y)KX, \tag{2.32}$$

$$KAX - AKX = k\{\eta(X)t - t(X)\xi\}, \tag{2.33}$$

$$\begin{aligned} LAX - ALX &= (Xk)\xi - \eta(X)\nabla k + k(\phi AX + A\phi X), \\ g((LK - KL)X,Y) &= -2(\theta - c)g(\phi X,Y), \end{aligned} \tag{2.34}$$

which together with (2.15) and (2.24) yields

$$g(LKX,Y) = -(\theta - c)g(\phi X,Y). \tag{2.35}$$

From (2.28) and this, we obtain

$$L^2X = (\theta - c)(X - \eta(X)\xi). \tag{2.36}$$

By properties of the almost contact metric structure we have from (2.35)

$$T_r({}^tKK) - \|K\xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c),$$

where we have used (2.6), (2.9) and (2.10), and $\|F\|^2 = g(F, F)$ for any tensor field F on M. which connected to (2.8) gives

$$\|K - m \otimes \xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c). \tag{2.37}$$

In the same way, using (2.7), (2.11), (2.14), (2.35) we see that

$$\|K - k\xi\|^2 - \|L\xi\|^2 - Tr({}^tLL) = 2(n-1)(\theta - c). \tag{2.38}$$

Differentiating (2.23) covariantly along M and making use of (2.4) and the first Bianchi identity, we find

$$(X\theta)\omega(Y, Z) + (Y\theta)\omega(Z, X) + (Z\theta)\omega(X, Y) = 0,$$

which implies $(n-2)X\theta = 0$. Therefore, θ is a constant if $n > 2$.

For the case where $\theta = c$ in (2.23) we have $dt = 2c\omega$. In this case, the normal connection of M is said to be $L - flat$([25]).

Using (2.37) and (2.38) we can verify that the following lemma (see [17],[18]) :

Lemma 2.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c)$, $c \neq 0$. If $A\xi = \alpha\xi$, then we have $\nabla^{\perp}C = 0$ and $K = L = 0$ on M.*

Putting $X = \xi$ in (2.33) and using (2.26), we find

$$KA\xi = kA\xi + k\{t' - t(\xi)\xi\}, \tag{2.39}$$

where $g(t', X) = t(X)$. From now on we will use the same letter t instead of t'.

If we apply this by ϕ and use (2.19), (2.26) and (2.28), then we get

$$g(KU, X) = k\{t(\phi X) - u(X)\}, \tag{2.40}$$

where $u(X) = g(U, X)$ for any vector field X.

Replacing X by ξ in (2.34) and using (2.5), (2.26) and (2.28), we have

$$KU = (\xi k)\xi - \nabla k + kU. \tag{2.41}$$

which together with (2.40) gives

$$Xk = (\xi k)\eta(X) + k\{2u(X) - t(\phi X)\}. \tag{2.42}$$

If we apply (2.34) by ϕ and take account of (2.27) and the last equation, then we find

$$\begin{aligned}\phi ALX - KAX = -k\{&(t - t(\xi)\xi)\eta(X) + 2\eta(X)(A\xi - \alpha\xi) \\ &+ 2g(A\xi, X)\xi - AX + \phi A\phi X\},\end{aligned}$$

or, using (2.33) we have $\phi AL + LA\phi = 0$.

Since θ is constant if $n > 2$, differentiating (2.36) covariantly, we get

$$2L\nabla_X L = (c - \theta)\{\eta(X)\phi A + g(\phi A, X)\xi\},$$

or, using (2.32), (2.35) and (2.42), it is verified that (see, [17])

$$\begin{aligned}(\theta - c)(A\phi - \phi A)X + (k^2 + \theta - c)(u(X)\xi + \eta(X)U)\\ + k\{(AL + LA)X + k(-t(\phi X)\xi + \eta(X)\phi \circ t)\} = 0.\end{aligned} \tag{2.43}$$

In the previous paper [12], [18] the following lemma was proved.

Lemma 2.2 *If M satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ and $\mu = 0$ in $M_{n+1}(c)$, $c \neq 0$, then we have $k = 0$ on M.*

We set $\Omega = \{p \in M : k(p) \neq 0\}$, and suppose that Ω is not empty. In the rest of this paper, we discuss our arguments on the open subset Ω of M. So, by Lemma 2.2, we see that $\mu \neq 0$ on Ω.

3. Jacobi operators of semi-invariant submanifolds

We introduce the structure Jacobi operator R_ξ with respect to the structure vector field ξ which is defined by $R_\xi X = R(X, \xi)\xi$ for any vector field X. Then we have from (2.29)

$$\begin{aligned}R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + \eta(K\xi)KX - \eta(KX)K\xi\\ + \eta(L\xi)LX - \eta(LX)L\xi.\end{aligned}$$

Since l and m are dual 1-forms of $L\xi$ and $K\xi$ respectively because of (2.8) and (2.9), the last relationship is reformed as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX + m(X)K\xi - l(X)L\xi.$$

where we have used (2.8)$\sim$(2.12).

We will continue now, our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as in section 2. Then, by virtue of (2.25) and (2.26) we can write the last equation as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX - k^2\eta(X)\xi. \tag{3.1}$$

which implies

$$R_\xi KX = c(KX - k\eta(X)\xi) + \alpha AKX - \eta(AKX)A\xi + kK^2X + k^3\eta(X)\xi,$$

where we have used the first equation of (3.26), which together with (2.16), (2.23) and (2.39) gives

$$(R_\xi K - KR_\xi)X = k\mu\{t(X)W - w(X)t - t(\xi)(\eta(X)W - w(X)\xi)\}, \tag{3.2}$$

where $g(W, X) = w(X)$ for any vector field X.

According to (3.2) and Lemma 2,2, we then have

Lemma 3.1. *$R_\xi K = KR_\xi$ holds on Ω if and only if $t \in f(\xi, W)$, where $f(\xi, W)$ is denoted by a linear subspace spanned by ξ and W.*

Further suppose, throughout this paper, that $R_\xi K = KR_\xi$ and at the same time $\nabla_\xi R_\xi = 0$ hold on M. Then, from Lemma 3.1, we have

$$t(X) = t(\xi)\eta(X) + t(W)w(X) \tag{3.3}$$

for any vector field X.

From (2.17) and (3.3) we obtain $t(\phi X) = -\frac{1}{\mu}t(W)u(X)$, which together with (2.40) yields

$$KU = \tau U, \tag{3.4}$$

where τ is defined by $\mu\tau = -k(\mu + t(W))$, or using (2.27),

$$LU = \mu\tau W. \tag{3.5}$$

By virtue of (2.35) and the last two relationships, it follows that

$$\tau^2 = \theta - c. \tag{3.6}$$

τ is a nonnegative constant on Ω if $n > 2$.

In a direct consequence of (2.28) and (3.4), we verify that

$$\mu LW = \tau U. \tag{3.7}$$

Using (2.16) and (2.26), we can write (2.39) as

$$\mu KW = k\mu W + k(t - t(\xi)\xi),$$

which together with (3.2) and (3.3) gives

$$KW = -\tau W \tag{3.8}$$

because of Lemma 2.2.

Now, by using (2.41) and (3.4) it is verified that

$$t(\phi X) = (1 + \frac{\tau}{k})u(X) \tag{3.9}$$

on Ω, or using the property of the almost contact metric structure,

$$t(X) = t(\xi)\eta(X) - \mu(1 + \frac{\tau}{k})w(X) \tag{3.10}$$

for any vector field X.

If we take account of (3.4), then (2.41) can be written as

$$Xk = (\xi k)\eta(X) + (k - \tau)u(X) \tag{3.11}$$

for any vector field X.

On the other hand, if we use (2.19) and (2.30), then (2.22) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha + 2\eta(L\xi) - 2\eta(K\xi)L\xi,$$

which together with (2.26) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{3.12}$$

Putting $X = \xi$ in (2.22) and making use of (2.16) and (2.18), we get

$$\phi(\nabla_\xi A)\xi = \nabla_\xi U + \beta\xi - \alpha A\xi + \phi AU,$$

which together with (3.12) yields

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{3.13}$$

In the following, we see, using (2.16) and (2.19), that $\phi U = -\mu W$. Differentiating this covariantly and using (2.4), we find

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W. \tag{3.14}$$

Putting $X = \xi$ in this and using (3.13), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{3.15}$$

which tells us that

$$W\alpha = \xi\mu. \tag{3.16}$$

In the next place, differentiating the first equation of (2.26) covariantly and using (2.5), we find

$$(\nabla_X K)\xi + K\phi AX = (Xk)\xi + k\phi AX,$$

which together with (2.26) and (2.31) yields

$$(\nabla_\xi K)X = -K\phi AX + (Xk)\xi + k\phi AX + t(\xi)LX. \tag{3.17}$$

If we put $X = \xi$ in this and make use of (2.26) and (3.4), then we obtain

$$(\nabla_\xi K)\xi = (\xi k)\xi + (k - \tau)U. \tag{3.18}$$

Now, differentiating (3.1) covariantly along M and using (2.5), we find

$$\begin{aligned} &g((\nabla_X R_\xi)Y, Z) \\ &\quad = -(k^2 + c)(\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)) + (X\alpha)g(AY, Z) \\ &\quad + \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)(g((\nabla_X A)\xi, Y) - g(A\phi AY, X)) \\ &\quad - g(A\xi, Y)(g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)) + (Xk)g(KY, Z) \\ &\quad + kg((\nabla_X K)Y, Z) - 2k(Xk)\eta(Y)\eta(Z). \end{aligned}$$

Replacing X by ξ in this and using (2.5) and (3.12), we find

$$\begin{aligned} &g((\nabla_X R_\xi)Y, Z) \\ &\ = \ -(k^2 + c)(u(Y)\eta(Z) + u(Z)\eta(Y)) + (\xi\alpha)g(AY, Z) + \alpha g((\nabla_\xi A)Y, Z) \\ &\ - g(A\xi, Z)(3g(AU, Y) + Y\alpha) - g(A\xi, Y)(3g(AU, Z) + Z\alpha) \\ &\ + (\xi k)g(KY, Z) + kg((\nabla_\xi K)Y, Z) - 2k(\xi k)\eta(Y)\eta(Z), \end{aligned}$$

which shows

$$\begin{aligned} (\nabla_\xi R_\xi)X = &- (k^2 + c)(u(X)\xi + \eta(X)U) + (\xi\alpha)AX + \alpha(\nabla_\xi A)X \\ &- (3AU + \nabla\alpha)g(A\xi, X) - (3g(AU, X) + X\alpha)A\xi + (\xi k)KX \\ &+ k(\nabla_\xi K)X - 2k(\xi k)\eta(X)\xi. \end{aligned}$$

Thus, the second assumption $\nabla_\xi R_\xi = 0$ gives

$$\begin{aligned}\alpha(\nabla_\xi A)X &+ k(\nabla_\xi K)X + (\xi\alpha)AX + (\xi k)KX \\ &= (k^2+c)(u(X)\xi + \eta(X)U) + (3AU + \nabla\alpha)g(A\xi, X) \\ &+ (3g(AU,X) + X\alpha)A\xi + 2k(\xi k)\eta(X)\xi.\end{aligned} \tag{3.19}$$

Replacing X by ξ in this and using (2.26), we find

$$\alpha(\nabla_\xi A)\xi + k(\nabla_\xi K)\xi = (k^2+c)U + \alpha(3AU + \nabla\alpha) + k(\xi k)\xi, \tag{3.20}$$

which together with (3.18) gives

$$\alpha AU + (k\tau + c)U = 0, \tag{3.21}$$

where we have used (2.26) and (3.12).

Replacing X by U in (3.17) and using (2.19), (3.4) and (3.5), we find

$$(\nabla_\xi K)U = -K\phi AU + k\phi AU + \mu\tau t(\xi)W + (k-\tau)\mu^2\xi.$$

If we put $X = U$ in (3.19) and make use of the last equation, then we obtain

$$\begin{aligned}\alpha(\nabla_\xi A)U &+ k\{k\phi AU - K\phi AU + \mu\tau t(\xi)W\} + (\xi\alpha)AU + \tau(\xi k)U \\ &= \mu^2(k\tau + c)\xi + \{3g(AU,U) + U\alpha\}A\xi.\end{aligned} \tag{3.22}$$

Now, if we take account of (3.6) and (3.9), then (2.43) turns out to be

$$\tau^2(A\phi - \phi A)X + \tau(\tau - k)(u(X)\xi + \eta(X)U) + k(AL + LA)X = 0.$$

Putting $X = \mu W$ in this and using (3.7), we find

$$\tau(k+c)AU = \mu(\tau^2\phi AW - kLAW).$$

By the way, if we replace X by μW in the first equation of (2.34) and use (2.19) and (3.7), then we obtain

$$(k+\tau)AU = \mu(LAW - k\phi AW).$$

Combining this to the last equation, we have

$$(k+\tau)AU = \mu(\tau - k)\phi AW \tag{3.23}$$

because $k + \tau$ does not vanish with the aid of (3.11).

Lemma 3,2. *If $k - \tau = 0$, then we have on Ω the following :*

$$\begin{aligned}(t(\xi) + 2\alpha)\{u(X)\eta(Y) - u(Y)\eta(X) + g(\phi AX, Y) - g(\phi AY, X)\} \\ + 4\{cg(\phi X, Y) - g(A\phi AX, Y)\} = 0.\end{aligned} \tag{3.24}$$

Proof. Since we have $k - \tau = 0$, we see from (3.23) that $AU = 0$ because of (3.11). Thus, it follows that $k\tau + c = 0$ by virtue of (3.21), which tells us that $\tau^2 + c = 0$. So we have $\theta = 0$ because of (3.6). Since $k - \tau = 0$ was assumed, (3.10) reformed as $t(Y) = t(\xi)\eta(Y) + 2g(\phi U, Y)$ for any vector field Y. Differentiating this covariantly and using (2.4), (2.6) and the fact that $AU = 0$, we find

$$(\nabla_X t)(Y) = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) - 2g(\phi\nabla_X U, Y),$$

from which taking the skew-symmetric part and using (2.23) with $\theta = 0$

$$X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + t(\xi)(g(\phi AX, Y) - g(\phi AY, X)) + 2(g(\phi\nabla_Y U, X) - g(\phi\nabla_X U, Y)) = 0.$$

By the way, we see from (2.22) that

$$g(\phi\nabla_X U, Y) - g(\phi\nabla_Y U, X) + (Y\alpha)\eta(X) - (X\alpha)\eta(Y) = 2cg(\phi X, Y) - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X)),$$

where we have used (2.30). If we substitute this into the last relationship, then we obtain

$$4cg(\phi X, Y) + t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\} = Y(t(\xi))\eta(X) - X(t(\xi))\eta(Y) + 2\{2g(A\phi AX, Y) - \alpha(g(\phi AX, Y) - g(\phi AY, X)) - (X\alpha)\eta(Y) + (Y\alpha)\eta(X)\}.$$

Putting $Y = \xi$ in this and using the fact that $AU = 0$ and $\tau^2 + c = 0$, we get

$$X(t(\xi)) + 2(X\alpha) = (\xi(t(\xi)) + 2(\xi\alpha))\eta(X) + (t(\xi) + 2\alpha)u(X).$$

Substituting this into the last equation, we have (3.24). This completes the proof of Lemma 3.2. ∎

Remark 3.1. *$\alpha \neq 0$ on Ω.*

In fact, if not, then we have $\alpha = 0$ on this open subset of Ω. So we have $k\tau + c = 0$ because of (3.21) on the set. We discuss our arguments on such a place. From this and (3.11) we see that $k - \tau = 0$ and hence $\tau^2 + c = 0$. We also have $AU = 0$ because of (3.23). If we put $X = U$ in (3.24) and take account of these facts, then we have $t(\xi) = 0$. Therefore (3.24) will produce a contradiction by using $\alpha = 0$ and $AU = 0$. Accordingly $\alpha = 0$ is not impossible on Ω.

Remark 3.2. *$\tau \neq 0$ on Ω.*

In fact, if not, then we have $\tau = 0$. Thus, we see, using (2.27) and (2.36), that $KX = k\eta(X)\xi$ and $L = 0$. Consequently (2.32) is reduced to

$$k\{\eta(X)AY - \eta(Y)AX\} + \eta(X)t(Y) - \eta(Y)t(X) = 0,$$

which implies that $AX = \eta(X)A\xi + g(A\xi, X)\xi - \alpha\eta(X)\xi$. Accordingly we have $AU = 0$, which connected to (3.21) gives a contradiction. Hence $\tau \neq 0$ on Ω is proved.

Lemma 3.3. *$k - \tau \neq 0$ on Ω.*

Proof. Let Ω' be a set of points such that $k(p) - \tau \neq 0$ on Ω and suppose that Ω' be nonvoid. We discuss our arguments on such a place. Then we have (3.24). Furthermore, it is clear that $AU = 0$ and $\tau^2 + c = 0$ on Ω'. Putting $X = U$ in (3.24) and using these facts, we find

$$(t(\xi) + 2\alpha)(\mu^2\xi - \mu AW) = -4c\mu W,$$

which implies

$$AW = \mu\xi + g(AW, W)W$$

on Ω', where we have put $g(AW, W) = -4c/(t(\xi) + 2\alpha)$, which together with (2.16) implies that $A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi$, where we have put $\rho = \alpha + g(AW, W)$. Thus, it follows that

$$AW = \mu\xi + (\rho - \alpha)W \tag{3.25}$$

on Ω'.
Differentiating this covariantly along Ω', we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \tag{3.26}$$

If we take the inner product with ξ to this, and use (2.21) and (2.30), then we find

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu, \tag{3.27}$$

which implies that

$$W\mu = \xi\rho - \xi\alpha. \tag{3.28}$$

In the next step, differentiating (3.8) covariantly, we find

$$g((\nabla_X K)W, Y) + g(K\nabla_X W, Y) + \tau g(\nabla_X W, Y) = 0,$$

from which, taking the skew-symmetric part and using (2.31) and (3.7),

$$\begin{aligned}\frac{\tau}{\mu}(t(X)u(Y) - t(Y)u(X)) + g(K\nabla_X W, Y) - g(K\nabla_Y W, X)\\ = \tau((\nabla_Y W)X - (\nabla_X W)Y).\end{aligned} \tag{3.29}$$

Putting $X = \xi$ in this and taking account of (2.21) and the fact that $AU = 0$, we find $\tau t(\xi)U - \mu K\nabla_\xi W = \mu\tau\nabla_\xi W$, which together with (3.15) and $AU = 0$ yields

$$K\nabla\alpha + \tau\nabla\alpha = 2\tau(\xi\alpha)\xi + \tau(2\alpha + t(\xi))U,$$

where have used (2.26) and (3.8), which connected to (3.4) yields $2U\alpha = (t(\xi) + 2\alpha)\mu^2$. From this and $(t(\xi) + 2\alpha)g(AW, W) + 4c = 0$ it follows that

$$(\rho - \alpha)U\alpha = -2c\mu^2. \tag{3.30}$$

On the other hand, if we put $X = \mu W$ in (3.26) and make use of (3.25) and (3.27), then we obtain

$$\mu^2\nabla_W W - \mu\nabla\mu = (\alpha^2 - \alpha\rho - 2c)U - \mu(W\alpha)\xi - \mu(W\mu)W.$$

Since we have $t(W) = -2\mu$ because of (3.10), replacing X by W in (3.29) and making use of the last relationship, we obtain

$$\mu(K\nabla\mu + \tau\nabla\mu) = 2\tau(\mu^2 - \alpha^2 + \rho\alpha + 2c)U + 2\mu\tau(W\alpha)\xi,$$

which together with (3.4) gives

$$U\mu = (\mu^2 - \alpha^2 + \rho\alpha + 2c)\mu. \tag{3.31}$$

In the meanwhile, differentiating $AU = 0$ covariantly, we find $(\nabla_X A)U + A\nabla_X U = 0$, which shows $g(\nabla_X AU, U) = 0$ and hence $(\nabla_U A)U = 0$ because of

(2.30) and (3.5). Thus, it follows that $A\nabla_U U = 0$. We also have above equation that $(\nabla_\xi A)U + A\nabla_\xi U = 0$, which together with (3.13) yields

$$(\nabla_\xi A)U + \alpha A^2\xi - \beta A\xi + A\phi\nabla\alpha = 0,$$

or, using the fact that $A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi$ gives

$$\phi(\nabla_\xi A)U + (\rho\alpha - \beta)U + \phi A\phi\nabla\alpha = 0. \tag{3.32}$$

Putting $X = U$ and $Y = \xi$ in (2.30), we obtain

$$(\nabla_U A)\xi = (\nabla_\xi A)U \tag{3.33}$$

by virtue of (2.19), (2.26), (3.5) and the fact that $AU = 0$.

On the other hand, applying (2.22) by ϕ and using (2.20), we get

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW, X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi,$$

which together with the fact that $AU = 0$ gives $\nabla_U U = \phi(\nabla_U A)\xi$. Hence (3.33) becomes $\nabla_U U = \phi(\nabla_\xi A)U$. Using this, we can write (3.32) as

$$\nabla_U U = (\beta - \rho\alpha)U - \phi A\phi\nabla\alpha.$$

If we take the inner product with U to this and use $AU = 0$, then we find

$$\mu(U\mu) = (\beta - \rho\alpha)\mu^2 - (\rho - \alpha)U\alpha,$$

which together with (3.30) and (3.31) yields $\alpha(\rho - \alpha) = 0$. It is, using (3.30) and Remark 3.1, a contradictory. Hence, we have $\Omega' = \varnothing$, that is, $k - \tau \neq 0$ is proved on Ω. ■

Because of (3.21) and Remark 3.1, we can write (3.23) as

$$\mu\phi AW = \frac{k\tau + c}{\alpha(k - \tau)}U,$$

which together with (2.19) implies that

$$AW = \mu\xi + g(AW, W)W$$

on Ω, where we have define $g(AW, W)$ by

$$\alpha(k - \tau)g(AW, W) + k\tau + c = 0.$$

If we put $g(AW, W) = \rho - \alpha$, then we have on Ω the following :

$$AW = \mu\xi + (\rho - \alpha)W. \tag{3.34}$$

From (3.21) we have

$$AU = \lambda U, \qquad \alpha\lambda + k\tau + c = 0. \tag{3.35}$$

From this and (3.34) it is verified that (see, [17])

$$\xi\lambda = 0, \qquad W\lambda = 0. \tag{3.36}$$

Because of (3.34) and (3.35), the equation (3.23) implies that

$$\lambda(k + \tau) + (k - \tau)(\rho - \alpha) = 0. \tag{3.37}$$

Lemma 3.4. $\xi k = 0$ *on* Ω.

Proof. Differentiating (3.11) covariantly along Ω, we find

$$Y(Xk) = Y(\xi k)\eta(X) + (\xi k)g(\phi AY, X) + (Yk)u(X) + (k-\tau)\{g(\nabla_Y U, X) + u(\nabla_Y X)\},$$

from which taking the skew-symmetric part with respect to X and Y and using (3.11),

$$\begin{aligned} &Y(\xi k)\eta(X) - X(\xi k)\eta(Y) + (\xi k)\{g(\phi AY, X) - g(\phi AX, Y)\} \\ &\quad + (\xi k)(\eta(Y)u(X) - \eta(X)u(Y)) + (k-\tau)(g(\nabla_Y U, X) - g(\nabla_X U, Y)) = 0. \end{aligned} \tag{3.38}$$

On the other hand, differentiating (3.4) covariantly, we find

$$g((\nabla_Y K)U, X) + g(K\nabla_Y U, X) = \tau g(\nabla_Y U, X). \tag{3.39}$$

Putting $X = U$ in this, we find $g((\nabla_Y K)U, U) = 0$, which together with (2.33), (3.5) and (3.10) yields $(\nabla_U K)U = 0$. If we put $Y = U$ in (3.39) and use this fact, then we get $K\nabla_U U = \tau\nabla_U U$, which implies that $g(\nabla_U U, W) = 0$. If we put $Y = U$ and $X = W$ in (3.38) and use the last fact, then we obtain $(\xi k)\{g(\phi AU, W) - g(\phi AW, U)\} = 0$, which together with (3.34) and (3.35) gives $\xi k(\lambda + \rho - \alpha) = 0$. From this and (3.37) it follows that $\tau\lambda(\xi k) = 0$, which connected to (3.35) and Remark 3.2 implies that $(k\tau + c)(\xi k) = 0$. Accordingly $\xi k = 0$ is proved. ■

Owing to Lemma 3.3 and Lemma 3.4, we can write (3.38) as

$$g(\nabla_Y U, X) = g(\nabla_X U, Y). \tag{3.40}$$

Putting $Y = \xi$ in this, we find $g(\nabla_\xi U, X) + g(U, \nabla_X \xi) = 0$, which together with (2.20) and (3.13) gives

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0.$$

By virtue of (2.16), (2.19), (3.34) and (3.35), this is reformed as

$$\phi\nabla\alpha + (\rho - 3\lambda)\mu W = 0,$$

which tells us that

$$\nabla\alpha = (\xi\alpha)\xi + (\rho - 3\lambda)U.$$

If we differentiate the second equation of (3.35), and take account of (3.36), Lemma 3.3 and Lemma 3.4, then we find $\lambda\xi\alpha = 0$. But, λ does not vanish because of (3.11), (3.35) and Lemma 3.3. Thus, we have $\xi\alpha = 0$ on Ω. Accordingly we have

$$\nabla\alpha = (\rho - 3\lambda)U. \tag{3.41}$$

4. Theorems

We will continue our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, and at the same time $R_\xi K = KR_\xi$ and $\nabla_\xi R_\xi = 0$ as in section 3.

If we take the skew-symmetric part of (3.39) with respect to X and Y, and use (3.40), then we find

$$g(K\nabla_X U, Y) - g(K\nabla_Y U, X) + \mu\tau\{t(X)w(Y) - t(Y)w(X)\} = 0.$$

Putting $X = \xi$ in this and using (2.20), (3.13) and (3.35), we find

$$K(3\lambda\phi U + \alpha A\xi - \beta\xi + \phi\nabla\alpha) + k\mu AW + \mu\tau(\xi)W = 0,$$

which connected to (2.16), (2.19), (3.8), (3.34) and (3.41) gives

$$\tau t(\xi) + (\rho - \alpha)(k + \tau) = 0, \tag{4.1}$$

or using (3.37)

$$\tau(k - \tau)t(\xi) = \lambda(k + \tau)^2. \tag{4.2}$$

On the other side, differentiating (3.10) covariantly along Ω and taking account of (2.4), (2.5), (3.11), (3.14) ,(3.35) and Lemma 3.4, we find

$$\begin{aligned} X(t(Y)) = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) + \frac{\tau}{k^2}(k - \tau)\mu u(X)w(Y) \\ - (1 + \frac{\tau}{k})(\lambda u(X)\eta(Y) - g(\phi\nabla_X U, Y) + t(\nabla_X Y)), \end{aligned}$$

from which, taking the skew-symmetric part with respect to X and Y and using (2.22) and (2.23),

$$\begin{aligned} &2\theta g(\phi X, Y) + \frac{\tau}{k^2}(k - \tau)\mu(u(Y)w(X) - u(X)w(Y)) \\ &\quad + Y(t(\xi))\eta(X) - X(t(\xi))\eta(Y) + t(\xi)(g(\phi AY, X) - g(\phi AX, Y)) \\ &= (1 + \frac{\tau}{k})\{2cg(\phi X, Y) + (\rho - 3\lambda)(u(X)\eta(Y) - u(Y)\eta(X)) \\ &\quad - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X))\}, \end{aligned} \tag{4.3}$$

where we have used (226), (2.30), (3.35) and (3.41).
From this we can verify that (see, (6.8) of [17])

$$(k + \tau)\nabla\lambda = 6\tau\lambda U. \tag{4.4}$$

If we put $X = U$ and $Y = W$ in (4.3) and take account of (3.34), (3.35) and (3.37), then we find

$$2\theta k(k - \tau) + \frac{\tau}{k}(k - \tau)^2\mu^2 + 2\tau\lambda k t(\xi) = 2c(k^2 - \tau^2) + 2\lambda^2(k + \tau)^2 - 2\tau(k + \tau)\alpha\lambda. \tag{4.5}$$

On the other hand, because of (3.35) and (3.41) we can write (3.13) as

$$\nabla_\xi U = -\mu^2\xi + \mu(\alpha - \rho)W.$$

If we differentiate the first equation of (3.35) covariantly with respect to ξ and remember (3.36), then we find $(\nabla_\xi A)U + A\nabla_\xi U = \lambda\nabla_\xi U$, which together with (2.16), (3.34) and the last relationship yields

$$(\nabla_\xi A)U = \mu^2(\rho-\lambda)\xi + \mu\{\mu^2 + (\rho-\alpha)^2 + \lambda(\alpha-\rho)\}W.$$

Thus, it follows that

$$g((\nabla_\xi A)U, W) = \mu\{\mu^2 + (\rho-\alpha)^2 + \lambda(\alpha-\rho)\}. \tag{4.6}$$

In the meanwhile, using (3.35), (3.41) and Lemma 3.3, we can also write (3.22) as

$$g((\nabla_\xi A)U + k\mu\{\tau t(\xi) - \lambda(k+\tau)\}W = \mu^2(k\tau + c)\xi + \rho\mu^2 A\xi,$$

which together with (3.37) and (4.1) gives

$$\alpha(\nabla_\xi A)U = \mu^2(k^2 + c)\xi + \rho\mu^2 A\xi,$$

which implies that $\alpha g((\nabla_\xi A)U, W) = \rho\mu^3$.
Combining this to (4.6), we obtain

$$(\rho-\alpha)\{\mu^2 - \alpha(\rho-\alpha) + \alpha\lambda\} = 0. \tag{4.7}$$

However, $\rho - \alpha \neq 0$ on Ω. Indeed, if not, then we have $\rho = \alpha$ on this subset. Using (4.1) and (4.2), we obtain $(k+\tau)\lambda = 0$ and hence $\lambda = 0$ on the set by virtue of (3.11) with $\xi k = 0$. Hence, the second equation of (3.35) becomes $k\tau + c = 0$ and consequently $k - \tau = 0$ on the set, a contradiction because of Lemma 3.3. Thus, $\rho - \alpha = 0$ on Ω is impossible. Therefore (4.7) implies that $\mu^2 = \alpha(\rho-\alpha) - \alpha\lambda$, which connected to (3.37) gives $(k-\tau)\mu^2 = -2k\lambda\alpha$. Substituting this and (4.1) into (4.5), we obtain

$$\theta k(k-\tau) + 2\tau^2\alpha\lambda + \lambda k(\alpha-\rho)(k+\tau) = c(k^2-\tau^2) + \lambda^2(k+\tau)^2,$$

which together with (3.6) and (3.37)

$$(c+\tau^2)k(k-\tau)^2 + 2\tau^2\alpha\lambda(k-\tau) + k\lambda^2(k+\tau)^2 = c(k-\tau)^2(k+\tau) + \lambda^2(k+\tau)^2(k-\tau).$$

Because of (3.35), it follows that

$$\lambda^2(k+\tau)^2 - 2\tau(k-\tau)(c+k\tau) + (k\tau - c)(k-\tau)^2 = 0.$$

Differentiating this and using (3.11) with $\xi k = 0$ and (4.4), we find

$$\lambda^2(k+\tau)(k+9\tau) + (k-\tau)^2(2\tau k - 3\tau^2 - c) = 0.$$

If we eliminate λ to above two equations, then we can verify that k is a root of an algebraic equation with constant coefficients. So k is a constant, which together with (3.11) implies that $k - \tau = 0$, a contradiction because of Lemma 3.3. Therefore, we conclude that $\Omega = \varnothing$, that is $k = 0$ on M. Hence $m = 0$ because of (2.25). We also have $K\xi = 0$ be virtue of (2.26).

Combining (2.27) with $k = 0$ to (2.35), we get

$$K^2X = (\theta - c)(X - \eta(X)\xi). \tag{4.8}$$

By (2.41) we have $KU = 0$, we see from (4.8) that $(\theta - c)U = 0$. Thus, (2.43) turns out to be $(\theta - c)(A\phi - \phi A) = 0$.

In the following we assume that $\theta - c \neq 0$ on M. Then we have

$$A\phi - \phi A = 0. \tag{4.9}$$

From this we have $A\xi = \alpha\xi$, that is $U = 0$. If we take account of (2.30) with $k = 0$ and (4.9), then we can verify that (cf. [11])

$$A^2X = \alpha AX + c(X - \eta(X)\xi),$$

which implies that

$$h_{(2)} = \alpha h + 2(n-1)c. \tag{4.10}$$

On the other hand, differentiating (4.8) covariantly and using previously obtained formulas, we find (see, (4.13) of [18])

$$(\nabla_X K)Y = t(X)LY - \eta(X)ALY - \eta(Y)ALX - g(AX, LY)\xi,$$

from which, differentiating covariantly and using the Ricci identity for K, we obtain (for detail, see (4.20) and (4.22) of [18]):

$$(h + 3\alpha)(h - \alpha) = \delta, \tag{4.11}$$

where we have put $\delta = 4(n-1)\{(n+1)\theta - 2(n+2)c\}$ and

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)(h - \alpha) = 2(n-1)\{4c\alpha - (\theta - 2c)(h - \alpha)\}. \tag{4.12}$$

Combining (4.12) to (4.10), we obtain

$$(\theta - 3c)(h - \alpha) = 2(n-1)(\theta - 2c)\alpha. \tag{4.13}$$

Now, from (2.29) the Ricci tensor of type (1,1) of M is given by

$$SX = \{c(2n+1) - 2(\theta - c)\}X + \{2(\theta - c) - 3c\}\eta(X)\xi + hAX - A^2X, \tag{4.14}$$

where we have used (2.36) and (4.8), which implies that the scalar curvature $\bar{r}$ of M is given by

$$\bar{r} = 4c(n^2 - 1) - 4(n-1)(\theta - c) + h^2 - h_{(2)}.$$

Using (4.10), it follows that

$$\bar{r} = 2(n-1)(2n+1)c - 4(n-1)(\theta - c) + h(h - \alpha). \tag{4.15}$$

By the way, if we use (4.11) and (4.13), then we have (see, [17])

$$h(h - \alpha) = 2(n-1)(2n-1)(\theta - c) - 4n(n-1)c.$$

Thus, (4.15) becomes

$$\bar{r} - 2(n-1)c = 2(n-1)(2n-3)(\theta - c).$$

Therefore we have $\theta - c = 0$ if $\bar{r} - 2(n-1)c \leq 0$ and hence $K = L = 0$ because of (2.36) and (4.8). Thus, it follows that $\nabla^{\perp}_X C = 0$ for any vector field X on M.

Let $N_0(p) = \{\upsilon \in T_p^{\perp}(M) : A_\nu = 0\}$ and $H_o(p)$ be the maximal J-invariant subspace of $N_0(p)$. Since $K = L = 0$, the orthogonal complement of $N_0(p)$ is invariant under the parallel translation with respect to the normal connection because of $\nabla^{\perp} C = 0$. Thus, by the reduction theorem for $P_{n+1}\mathbb{C}$([26]) and $H_{n+1}\mathbb{C}$ ([10]), there exists a totally geodesic complex space form including M in $M_{n+1}(c)$([8]). Accordingly we conclude that

Theorem 4.1. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, where $\omega(X, Y) = g(\phi X, Y)$ for any vector fields X and Y on M. If M satisfies $R_\xi K = K R_\xi$ and at the same time $\nabla_\xi R_\xi = 0$, then M is a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$.*

Since $k = 0$ on M, (3.19) can be written as

$$\begin{aligned} \alpha(\nabla_\xi A)X + (\xi\alpha)AX = c(u(X)\xi + \eta(X)U) + \eta(AX)(3AU + \nabla\alpha) \\ + \{3g(AU, X) + X\alpha\}A\xi. \end{aligned} \tag{4.16}$$

Here, the distinguished normal C can be regard a unit normal vector field N on M in $M_n(c)$. Thus, the second fundamental form A with respect to C can also be regarded as that of N.

Since $k = 0$ was proved as above, we can write (2.30) as

$$(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi). \tag{4.17}$$

In the following, we will discuss our arguments on M of $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$. Further, we assume that $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. Then we have $\mu\nabla_W R_\xi = 0$. From now on we set $\Omega_0 = \{p \in M : \mu(p) \neq 0\}$ and suppose that $\Omega_0 \neq \varnothing$, that is, ξ is not principal curvature vector on M. We discuss our arguments on Ω_0. Then we have $\nabla_W R_\xi = 0$. So, if we put $X = W$ and $Y = \xi$ in the following equation of (3.18) in section 3, then we have

$$\begin{aligned} g((\nabla_W R_\xi)\xi, Z) = -cg(\phi AW, Z) - c(W\alpha)g(A\xi, Z) - \eta(AZ)g((\nabla_W A)\xi, \xi) \\ - \alpha g(A\phi AW, Z). \end{aligned}$$

Since $\nabla_W R_\xi = 0$ on Ω_0, it follows from the last equation that

$$\alpha A\phi AW + c\phi AW = 0 \tag{4.18}$$

by virtue of (3.12) and (4.17).

In the meanwhile, $\nabla_\xi R_\xi = 0$ was assumed, we see from (3.21) that

$$\alpha AU + cU = 0. \tag{4.19}$$

If we differentiate (4.19) covariantly along Ω_0 and use itself again, then we find

$$-c(X\alpha)U + \alpha^2(\nabla_X A)U + \alpha^2 A\nabla_X U + c\alpha\nabla_X U = 0, \tag{4.20}$$

which connected to (2.19) and (4.17) gives

$$\begin{aligned} c\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + c\alpha^2\mu(\{\eta(X)w(Y) - \eta(Y)w(X)\} \\ + \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + c\alpha du(X, Y) = 0, \end{aligned} \tag{4.21}$$

where d is the exterior differential operator.
Putting $X = U$ in (4.21), we obtain

$$c\{\mu^2\nabla\alpha - (U\alpha)U\} + \alpha^2 A\nabla_U U + c\alpha\nabla_U U = 0 \tag{4.22}$$

because U and W are mutually orthogonal.

Combining (2.22) to (4.16) and using (4.17), we have

$$\begin{aligned}\alpha^2\phi\nabla_X U =& \alpha^2(X\alpha)\xi - c\alpha u(X)\xi + \alpha(\xi\alpha)AX + c\alpha^2\phi X\\ &- \eta(AX)(\alpha\nabla\alpha - 3cU) - \{\alpha(X\alpha) - 3cu(X)\}A\xi\\ &- c\alpha\{u(X)\xi + \eta(X)U\} - \alpha^2 A\phi AX + \alpha^3\phi AX.\end{aligned}$$

Applying this by ϕ and using (2.20), we have

$$\begin{aligned}&\alpha^2\nabla_X U + \alpha^2\mu g(AW, X)\xi - \alpha\eta(AX)\phi\nabla\alpha\\ &= -\alpha(\xi\alpha)\phi AX + c\alpha^2\{X - \eta(X)\xi\} + 3c\mu\eta(AX)W + \alpha(X\alpha)U\\ &- 3cu(X)U + \alpha^3 AX - c\alpha\mu\eta(X)W - \alpha^3\eta(AX)\xi + \alpha^2\phi A\phi AX.\end{aligned} \tag{4.23}$$

Putting $X = U$ in (4.23) and using (2.16), (2.19) and (4.19), we get

$$\alpha^2\nabla_U U = -c\mu(\xi\alpha)W + \{\alpha(U\alpha) - 3c\mu^2\}U + c\mu\alpha\phi AW, \tag{4.24}$$

which shows that

$$\alpha^2 A\nabla_U U = -c\mu(\xi\alpha)AW + \{\alpha(U\alpha) - 3c\mu^2\}AU + c\mu\alpha\phi A\phi AW. \tag{4.25}$$

On the other hand, putting $X = \alpha U$ in (4.16) and taking account of (4.19), we find

$$\alpha^2(\nabla_\xi A)U - c(\xi\alpha)U = c\alpha\mu^2\xi + \{\alpha(U\alpha) - 3c\mu^2\}A\xi.$$

If we put $X = \alpha\xi$ in (4.21) and make use of this, then we have

$$c\alpha\mu^2\xi + \{\alpha(U\alpha) - 3c\mu^2\}A\xi + \alpha^2 A(\nabla_\xi U) + c\alpha\nabla_\xi U = 0.$$

This, together with (3.13) and (4.19), implies that

$$\begin{aligned}&\alpha\{\alpha A\phi\nabla\alpha + c\phi\nabla\alpha + (U\alpha)A\xi\}\\ &\qquad + \mu(\alpha^2 + 3c)\{\alpha(AW - \mu\xi) - (\mu^2 - c)W\} = 0.\end{aligned} \tag{4.26}$$

If we combine (4.22), (4.24), and (4.25) to (4.18), then we get

$$\alpha\mu^2\nabla\alpha = \alpha(U\alpha)U + \mu(\xi\alpha)(\alpha AW + cW), \tag{4.27}$$

which enables us to obtain

$$\mu(W\alpha) = (w(AW) + \frac{c}{\alpha})\xi\alpha. \tag{4.28}$$

Substituting (4.27) into (4.26) and making use of (4.18), we find

$$\{\alpha(U\alpha) - \mu^2(\alpha^2 + 3c)\}\{\alpha(AW - \mu\xi) - (\mu^2 - c)W\} = 0,$$

which implies that $\alpha(U\alpha) = \mu^2(\alpha^2 + 3c)$.

In fact, if not, then we have $AW = \mu\xi + (\rho - \alpha)W$, $\alpha(\rho - \alpha) = \mu^2 - c$, where we put $\rho - \alpha = w(AW)$. From these facts and (2.16), it follows that $A^2\xi = \rho A\xi + c\xi$, which connected to (3.1) with $K = 0$ yields $R_\xi A = AR_\xi$. Since

$\nabla_\xi R_\xi = 0$ was assumed, we verify that $\Omega_0 = \varnothing$ (see, [7]). Thus, (4.27) turns out to be

$$\alpha\nabla\alpha = (\alpha^2 + 3c)U + \mu(\xi\alpha)(\alpha AW + cW),$$

which shows that $\alpha\nabla\alpha = (\alpha^2 + 3c)U$ (see, (5.36) of [13]). From this fact we see that α is a constant and hence $\Omega_0 = \varnothing$ (cf.Theorem 3 of [13]). Therefore we verify that $A\xi = \alpha\xi$ and α is a constant (see, [20]). Accordingly (4.16) is reduced to $\alpha(\nabla_\xi A)X = 0$ for any vector field X on M. From this and (4.17) we see that $\alpha(A\phi - \phi A) = 0$ and hence $A\xi = 0$ or $A\phi = \phi A$. Since M is a Hopf hypersurface, $A\xi = 0$ means $\alpha = 0$. Here we note that $\alpha = 0$ corresponds to a tube of radius $\pi/4$ in $P_n\mathbb{C}$([5]). But, α never vanishes for Hopf hypersurface in $H_n\mathbb{C}$(cf. [23]) Thus, owing to Theorem 4.1 and Theorem O-MR, we have

Theorem 4.2. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ such that the third fundamental from t satisfies $dt(X,Y) = 2\theta(\phi X, Y)$ for a scalar $\theta(\neq 2c)$, and any vector fields X and Y, and the scalar curvature $\bar{r}$ of M satisfies $\bar{r} - 2(n-1)c \leq 0$. Suppose that the second fundamental form K satisfies $R_\xi K = KR_\xi$ and at the same time $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. Then $\nabla_\xi R_\xi = 0$ holds on M if and only if M is locally congruent to one of the following hypersurfaces :*

- (I) *in case that $M_n(c) = P_n\mathbb{C}$,*
  - (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
  - (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
  - (T) *a tube of radius $\pi/4$ over a certain complex submanifold in $P_n\mathbb{C}$,*
- (II) *in case that $M_n(c) = H_n\mathbb{C}$,*
  - (A_0) *a horosphere,*
  - (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*
  - (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$.*

References

[1] A. Bejancu, CR-submanifolds of a *Kähler* manifold I, *Proc. Amer. Math. Soc.* 69(1978), 135-142.

[2] J. Berndt, Real hypersurfaces with constant principal curvatures in a complex hyperbolic space, *J. Reine Angew. Math.* 395(1989), 132-141.

[3] J. Berndt and H. Tamaru, Cohomogeneity one actions on non compact symmetric space of rank one, *Trans. Amer. Math. Soc.* 359(2007), 3425-3438.

[4] D. E. Blair, G. D. Ludden and K. Yano, Semi-invariant immersion, *Kodai Math. Sem. Rep.* 27(1976), 313-319.

[5] T. E. Cecil and P. J. Ryan, Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* 269(1982), 481-499.

[6] T. E. Cecil and P. J. Ryan, Geometry of Hypersurfaces, Springer (2015).

[7] J. T. Cho and U-H. Ki, Real hypersurfaces in complex space forms with Reeb-flows symmetric Jacobi operator, *Canadian Math. Bull.* 51(2008), 359-371.

[8] J. Erbacher, Reduction of the codimension of an isometric immersion, *J. Diff. Geom.* 3(1971), 333-340
[9] J. I. Her, U-H. Ki and S.-B. Lee, Semi-invariant submanifolds of codimension 3 of a complex projective space in terms of the Jacobi operator, *Bull. Korean Math. Soc.* 42(2005), 93-119.
[10] S. Kawamoto, Codimension reduction for real submanifolds of a complex hyperbolic space,*Rev. Mat. Univ. Compul. Madrid* 7(1994), 119–128.
[11] U-H. Ki, Cyclic-parallel real hypersurfaces of a complex space form, *Tsukuba J. Math.* 12(1988), 259-268.
[12] U-H. Ki, Semi-invariant submanifold of codimension 3 satisfying $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ in a complex space form, *East Asian Math. J.* 37(2021), 47-78.
[13] U-H. Ki and H. Kurihara, Real hypersurfaces and ξ-parallel structure Jacobi operators in complex space forms, *J. Nat. Acad. Sci. ROK. Sci. Ser.* 48-1(2009), 53-78.
[14] U-H. Ki, H. Kurihara, S. Nagai and R. Takagi, Characterizations of real hypersurfaces of type A in a complex space form in terms of the structure Jacobi operator, Toyama Math. J. 32(2009), 5-23.
[15] U-H. Ki and S. J. Kim, Structure Jacobi operators of semi-invariant submanifolds in a complex space form, *East Asian Math. J.* 36(2020), 389-415.
[16] U-H. Ki and S. J. Kim, Semi-invariant submanifolds of codimension 3 in a complex space form concerned with Jacobi operators with respect to the structure vector, *J. Nat. Acad. ROK. Nat. Sci. Ser.* 60(2021), 79-107.
[17] U-H. Ki and H. Song, Submanifolds of codimension 3 in a complex space form with commuting structure Jacobi operator, *Kyungpook. Math. J.* 62(2022), 133-166.
[18] U-H. Ki, H. Song and R. Takagi, Submanifolds of codimension 3 admitting almost contact metric structure in a complex projective space, *Nihonkai Math J.* 11(2000), 57-86.
[19] U-H. Ki and H. Liu, Some characterizations of real hypersurfaces of type (A) in a nonflat complex space form, *Bull. Korean Math. Soc.* 44(2007), 152-172.
[20] U-H. Ki and Y. J. Suh, On real hypersurfaces of a complex space form, *Math. J. Okayama Univ.* 32(1990), 207-221.
[21] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space, *Trans. Amer. Math. Soc.* 296(1986), 137-149.
[22] S. Montiel and A.Romero, On some real hypersurfaces of a complex hyperbolic space, *Geom. Dedicata* 20(1986), 245-261.
[23] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space form, in Tight and Taut submanifolds, *Cambridge University Press* : (1998(T. E. Cecil and S.-S. Chern eds.)), 233-305.
[24] M. Okumura, On some real hypersurfaces of a complex projective space, *Trans. Amer. Math. Soc.* 212(1973), 355-364.
[25] M. Okumura, Normal curvature and real submanifold of the complex projective space. *Geom. Dedicata* 7(1978), 509-517.
[26] M. Okumura, Codimension reduction problem for real submanifolds of a complex projective space. *Colloq. Math János Bolyai.* 56(1989), 574-585.
[27] M. Ortega, J. D. Pérez and F. G. Santos, Non-existence of real hypersurface with parallel structure Jacobi operator in nonflat complex space forms, *Rocky Mountain J. Math.* 36(2006), 1603-1613.
[28] H. Song, Some differential-geometric properties of R-spaces, *Tsukuba J. Math.* 25(2001), 279-298.
[29] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* 19(1973), 495-506.
[30] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures I,II, *J. Math. Soc. Japan* 27(1975), 43-53, 507-516.

[31] Y. Tashiro, Relations between the theory of almost complex spaces and that of almost contact spaces (in Japanese), *Sũgaku* 16(1964), 34-61.
[32] K. Yano, and U-H. Ki, On $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$, *Kodai Math. Sem. Rep.* 29(1978), 285-307.
[33] K. Yano and M. Kon, CR submanifolds of Kaehlerian and Sasakian manifolds, *Birkhäuser* (1983).

U - Hang KI
The National Academy of Sciences,
Seoul 06579, Korea
Email address: uhangki2005@naver.com

Hyunjung SONG
Department of Mathematics
Hankuk University of Foreign Studies,
Seoul 02450, Korea
Email address: hsong@hufs.ac.kr