East Asian Math. J.
Vol. 40 (2024), No. 1, pp. 025–036
http://dx.doi.org/10.7858/eamj.2024.002

UNIVARIATE TRUNCATED MOMENT PROBLEMS VIA WEAKLY ORTHOGONAL POLYNOMIAL SEQUENCES

Seonguk Yoo

Abstract. Full univariate moment problems have been studied using continued fractions, orthogonal polynomials, spectral measures, and so on. On the other hand, the truncated moment problem has been mainly studied through confirming the existence of the extension of the moment matrix. A few articles on the multivariate moment problem implicitly presented about some results of this note, but we would like to rearrange the important results for the existence of a representing measure of a moment sequence. In addition, new techniques with orthogonal polynomials will be introduced to expand the means of studying truncated moment problems.

1. Univariate Moment Problems

For an infinite real sequence $\beta = \{\beta_n\}_{n\geq 0}$, the *full moment problem* entails finding a representing measure μ such that $\beta_n = \int x^n \, d\mu$, $n \geq 0$. According to the location of the support of the measure, the problem is classified as:

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger moment problem)} \\ \text{supp } \mu \subseteq [a,b] & \text{(Hausdorff moment problem)} \\ \text{supp } \mu \subseteq [0,\infty) & \text{(Stieltjes moment problem)} \end{array}$$

Given a finite real sequence $\{\beta_n\}_{n=0}^{m}$, the *truncated moment problem* (in short, TMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ satisfying $\beta_n = \int x^n \, d\mu$ $(0 \leq n \leq m)$.

Solutions to full and truncated moment promes are described based on the Hankel matrices consisting of their moments:

Received September 6, 2023; Revised December 6, 2023; Accepted December 13, 2023.
2020 *Mathematics Subject Classification.* Primary 47A57, 44A60; Secondary 15-04, 47A20, 32A60.
Key words and phrases. moment problem, recursively generated, consistency, weakly orthogonal polynomial sequence.
The author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (NRF-2020R1F1A1A01070552).

(pISSN 1226-6973, eISSN 2287-2833)

$$A(k)=\begin{pmatrix}\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_k\\ \beta_1 & \beta_2 & \beta_3 & \iddots & \beta_{k+1}\\ \beta_2 & \beta_3 & \iddots & \iddots & \beta_{k+2}\\ \vdots & \iddots & \iddots & \iddots & \vdots\\ \beta_k & \beta_{k+1} & \beta_{k+2} & \cdots & \beta_{2k}\end{pmatrix},\quad B(k)=\begin{pmatrix}\beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{k+1}\\ \beta_2 & \beta_3 & \beta_4 & \iddots & \beta_{k+2}\\ \beta_3 & \beta_4 & \iddots & \iddots & \beta_{k+3}\\ \vdots & \iddots & \iddots & \iddots & \vdots\\ \beta_{k+1} & \beta_{k+2} & \beta_{k+3} & \cdots & \beta_{2k+1}\end{pmatrix}.$$

For the case of full moment problems, Hamburger proved that there is a representing measure μ for $\{\beta_n\}_{n\geq 0}$ such that $\beta_n=\int x^n\,d\mu$ and supp $\mu\in\mathbb{R}$ if and only if $A(k)\geq 0$ for all $k\geq 0$. Moreover, Stieltjes verified that the existence of a representing measure supported on $[0,\infty]$ is equivalent to the fact that $A(k)\geq 0$ and $B(k)\geq 0$ for all $k\geq 0$.

The j-th column of $A(k)$ will be denoted by $\mathbf{v}_j=(\beta_{j+\ell})_{\ell=0}^{k}$, $0\leq j\leq k$, so that we may write $A(k)=\begin{pmatrix}\mathbf{v}_0 & \cdots & \mathbf{v}_k\end{pmatrix}$.

The *(Hankel) rank* of β, denoted rank β, is now defined as follows: If $A(k)$ is nonsingular, rank $\beta=k+1$; if $A(k)$ is singular, rank β is the smallest integer i, $1\leq i\leq k$, such that $\mathbf{v}_i\in\operatorname{span}(\mathbf{v}_0\ \cdots\ \mathbf{v}_{i-1})$. Thus, if $A(k)$ is singular, there exists a unique $(\phi_0,\dots,\phi_{i-1})$ such that $\mathbf{v}_i=\phi_{i-1}\mathbf{v}_0+\phi_{i-2}\mathbf{v}_1+\cdots+\phi_0\mathbf{v}_{i-1}$. The polynomial

$$g_\beta(t)=t^i-\phi_0t^{i-1}-\cdots-\phi_{i-1}t-\phi_{i-1}\tag{1.1}$$

is called the *generating function* of β. This polynomial has a very important role, because the zeros of this polynomial become the atoms of a representing measure for a truncated moment sequence.

A key to prove the coming well-known solutions to TMP is:

Proposition 1.1. [4] *Let $\beta=(\beta_0,\dots,\beta_{2k+1})$, $\beta_0>0$. Assume $A(k)$ is positive definite. Then the generating function g_β has $k+1$ distinct real roots, $x_0,\dots,x_k$. Thus the Vandermonde matrix W of the points $x_0,\dots,x_k$ is invertible, and if $\rho=\begin{pmatrix}\rho_0 & \cdots & \rho_k\end{pmatrix}=W^{-1}\mathbf{v}_0$, then $\rho_j>0$ for $0\leq j\leq k$. Moreover, if $\mu=\sum_{i=0}^k\rho_i\delta_{x_i}$, where δ_{x_i} is the point mass at x_i, then $\beta_j=\int x^j\,d\mu$, $0\leq j\leq 2k+1$.*

Solutions to the truncated moment problem are introduced in detail by classifying several cases as follows:

Theorem 1.2. [4, Hamburger TMP, Existence of Odd Cases] *Let $\beta=(\beta_0,\dots,\beta_{2k+1})$, $\beta_0>0$, and let $r=$ rank β. The following are equivalent:*

(i) *There exists a (r-atomic) positive Borel measure μ satisfying $\beta_j=\int x^j\,d\mu$ ($j=0,\dots,2k+1$), and* supp $\mu\subseteq\mathbb{R}$;
(ii) $A(k)\geq 0$, $\mathbf{v}_{k+1}\in\operatorname{Ran}\ A(k)$;
(iii) $A(k+1)\geq 0$ *for some choice of $\beta_{2k+2}\in\mathbb{R}$, that is, $A(k)$ has a positive Hankel extension.*

Theorem 1.3. [4, Hamburger TMP, Existence of Even Cases] *Let $\beta=(\beta_0,\dots,\beta_{2k})$, $\beta_0>0$, and let $r=$ rank β. The following are equivalent:*

(i) *There exists a (r-atomic) positive Borel measure μ satisfying $\beta_j=\int x^j\,d\mu$, ($j=0,\dots,2k$), and* supp $\mu\subseteq\mathbb{R}$;

(ii) $A(k) \geq 0$, rank $A(k) =$ rank β;
(iii) $A(k)$ *has a positive Hankel extension.*

Theorem 1.4. [4, Stieltjes TMP, Existence of Odd Cases] *Let* $\beta = (\beta_0, \ldots, \beta_{2k+1})$, $\beta_0 > 0$, *and let* $r =$ rank β. *The following are equivalent:*

(i) *There exists a positive Borel measure* μ *satisfying* $\beta_j = \int x^j d\mu$ $(j = 0, \ldots, 2k+1)$, *and* supp $\mu \subseteq [0, \infty)$;
(ii) *There exists a* r*-atomic representing measure* μ *for* β *satisfying* supp $\mu \subseteq [0, \infty)$;
(iii) $A(k) \geq 0$, $B(k) \geq 0$, *and* $\mathbf{v}(k+1, k) = (\beta_{k+1} \cdots \beta_{2k+1})^T \in$ Ran $A(k)$.

Theorem 1.5. [4, Stieltjes TMP, Existence of Even Cases] *Let* $\beta = (\beta_0, \ldots, \beta_{2k})$, $\beta_0 > 0$, *and let* $r =$ rank β. *The following are equivalent:*

(i) *There exists a positive Borel measure* μ *satisfying* $\beta_j = \int x^j d\mu$ $(j = 0, \ldots, 2k)$, *and* supp $\mu \subseteq [0, \infty)$;
(ii) *There exists a* r*-atomic representing measure* μ *for* β *satisfying* supp $\mu \subseteq [0, \infty)$;
(iii) $A(k) \geq 0$, $B(k-1) \geq 0$, *and* $\mathbf{v}(k+1, k-1) = (\beta_{k+1} \cdots \beta_{2k})^T \in$ Ran $B(k-1)$.

Theorem 1.6. [4, Hausdorff TMP, Existence of Odd Cases]
Let $\beta = (\beta_0, \ldots, \beta_{2k+1})$, $\beta_0 > 0$, *and let* $r =$ rank β, *and let* g_β *as in (1.1). There exists a positive Borel measure* μ *satisfying* $\beta_j = \int x^j d\mu$, $(j = 0, \ldots, 2k+1)$ *and* supp $\mu \subseteq [a, b]$ *if and only if* $A(k) \geq 0$, $bA(k) \geq B(k) \geq aA(k)$, *and* $\mathbf{v}(k+1, k) = (\beta_{k+1} \cdots \beta_{2k+1})^T \in$ Ran $A(k)$.

Theorem 1.7. [4, Hausdorff TMP, Existence of Even Cases]
Let $\beta = (\beta_0, \ldots, \beta_{2k})$, $\beta_0 > 0$, *and let* $r =$ rank β. *There exists a positive Borel measure* μ *satisfying* $\beta_j = \int x^j d\mu$ *if and only if* $A(k) \geq 0$, $bA(k) \geq B(k) \geq aA(k)$, *and there exists* β_{2k+1} *such that* $\mathbf{v}(k+1, k) = (\beta_{k+1} \cdots \beta_{2k+1})^T \in$ Ran $A(k)$.

Let us illustrate how the above results are applied in the following example:

Example 1.8. *Consider an example:* $\beta \equiv \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\} = \{1, 1, 2, 3, 5, 8\}$, *which is the beginning part of the Fibonacci sequence. This is an odd case with* $k = 2$; *thus,*

$$A(2) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix} \quad \text{and} \quad B(2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 8 \end{pmatrix}$$

Note that $A(2) \geq 0$ *but* $B(2) \not\geq 0$; *thus, in the view of the Stieltjes moment problem, this sequence has no solution. However, in the view of the Hausdorff moment problem, it may have a representing measure on some* $[a, b]$, *where* $bA(2) \geq B(2) \geq aA(2)$. *There are infinitely many desired* a *and* b. *Indeed,* $A(2)$ *has a unique column relation* $\mathbf{v}_2 = \mathbf{v}_0 + \mathbf{v}_1$. *Thus, the generating function*

$g_\beta(t) = t^2 - 1 - t$ has the two roots $(1 \pm \sqrt{5})/2$ that are the atoms of the unique representing measure μ. Solving the equation of the Vandermonde system

$$\begin{pmatrix} 1 & 1 \\ (1-\sqrt{5})/2 & (1+\sqrt{5})/2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we find the densities $\rho_1 = (5-\sqrt{5})/10$ and $\rho_2 = (5+\sqrt{5})/10$. We finally can write the representing measure as $\rho_1 \delta_{(1-\sqrt{5})/2} + \rho_2 \delta_{(1+\sqrt{5})/2}$ for β.

2. Orthogonal Polynomial Sequences

In this section we introduce powerful tools for the study of univariate moment problems: the Riesz functional and orthogonal polynomials. Readers are referred to the references [3, 13] for a deeper treatment of the contents.

For a sequence, $\beta = (\beta_n)_{n\geq 0}$, define a *Riesz functional* $\mathcal{L}_\beta$ acting on $\mathbb{R}[x]$ as

$$\mathcal{L}_\beta\Big[\sum c_n x^n\Big] = \sum c_n \beta_n.$$

We say that $\mathcal{L}_\beta$ is *K-positive* if

$$\mathcal{L}_\beta(p) \geq 0 \quad \text{for all } p \in \mathbb{R}[x] : p|_K \geq 0. \tag{2.1}$$

If the conditions, $p|_K \geq 0$ and $p|_K \not\equiv 0$, imply $\mathcal{L}_\beta(p) > 0$, then $\mathcal{L}_\beta$ is said to be *strictly K-positive*. When $K = \mathbb{R}$, we use the term *positive* instead of K-positive. The K-positivity of $\mathcal{L}_\beta$ is a necessary condition for β to admit a K-representing measure. Conversely, the classical theorem of M. Riesz says the K-positivity is also sufficient for the existence of K-measures and Haviland generalized the result in $\mathbb{R}^n$.

Theorem 2.1 (Riesz-Haviland's Theorem). *A sequence $\beta = (\beta_n)_{n\geq 0}$ admits a representing measure supported in the closed set $K \subset \mathbb{R}$ if and only if $\mathcal{L}_\beta$ is K-positive.*

From now on, we will collect well-known results about the Riesz functional and orthogonal polynomials, and will see the role of positivity of a sequence. A sequence $\{p_n(x)\}_{n\geq 0}$ is called an *orthogonal polynomial sequence*(in short, OPS) with respect to a linear functional L if it satisfies that

$$\deg(p_n) = n \quad \text{and} \quad L[p_m p_n] = K_n \delta_{mn} \ (K_n \neq 0) \quad \text{for all } m, n \in \mathbb{N}.$$

When $K_n = 1$ for all $n \in \mathbb{N}$, such an OPS is called an *orthonormal polynomial sequence*. There exists an explicit formula for the orthogonal polynomial sequences (see [13, Proposition 5.3]).

Theorem 2.2. *For a sequence $\beta = (\beta_n)_{n\geq 0}$, let $\mathcal{L}_\beta$ be the Riesz functional of β. Then the monic OPS for $\mathcal{L}_\beta$ is expressed as*

$$p_n(x) = \frac{1}{\Delta_{n-1}(\beta)} \det \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_n \\ \beta_1 & \beta_2 & \cdots & \beta_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \beta_n & \cdots & \beta_{2n-1} \\ 1 & x & \cdots & x^n \end{bmatrix}, \tag{2.2}$$

provided that $\Delta_n(\beta) \neq 0$ for all $n \in \mathbb{N}$, where $\Delta_n(\beta) = \det A(n)$.

The condition, $\Delta_n(\beta) \neq 0$ for all $n \in \mathbb{N}$, is a necessary and sufficient condition for the existence of an OPS for $\mathcal{L}_\beta$; such an $\mathcal{L}_\beta$ is called *quasi-definite* and its OPS has a 3-term recurrence relation as follows:

Theorem 2.3 ([3], Theorem 4.1, Chapter 1)**.** *Let $\mathcal{L}_\beta$ be a quasi-definite Riesz functional of a sequence $\beta = \{\beta_n\}_{n\geq 0}$ and let $\{p_n(x)\}_{n\geq 0}$ be the corresponding monic OPS with respect to $\mathcal{L}_\beta$. Then there exist $\sigma = \{s_n\}_{n\geq 0}$ and $\tau = \{t_n\}_{n\geq 1}$ with $t_n \neq 0$ for all $n \geq 0$ such that*

$$p_n(x) = (x - s_n)p_{n-1}(x) - t_n p_{n-2}(x), \quad n \geq 1, \tag{2.3}$$

where we assume $p_{-1}(x) = 0$ and t_1 is arbitrary. Furthermore, for each $n \in \mathbb{N}$

$$s_n = \frac{\mathcal{L}_\beta[xp_{n-1}^2(x)]}{\mathcal{L}_\beta[p_{n-1}^2(x)]} \quad \text{and} \quad t_{n+1} = \frac{\mathcal{L}_\beta[p_n^2(x)]}{\mathcal{L}_\beta[p_{n-1}^2(x)]} = \frac{\Delta_{n-2}(\beta)\Delta_n(\beta)}{(\Delta_{n-1}(\beta))^2}, \tag{2.4}$$

where $\Delta_{-1}(\beta) = 1$. Moreover, if $\mathcal{L}_\beta$ is strictly positive, then $s_n \in \mathbb{R}$ and $t_{n+1} > 0$ for all $n \in \mathbb{N}$.

The converse of the proceeding result is referred to as the Favard's Theorem:

Theorem 2.4 ([3], Theorem 4.4, Chapter 1)**.** *[Favard's Theorem] Let $\sigma = \{s_n\}_{n\geq 1}$ and $\tau = \{t_n\}_{n\geq 1}$ be arbitrary sequences of complex numbers and let $\{p_n(x)\}_{n\geq 0}$ be defined by the recurrence formula*

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad p_n(x) = (x - s_n)p_{n-1}(x) - t_n\, p_{n-2}(x), \; n \geq 1. \tag{2.5}$$

Then, there exists a unique moment functional L such that

$$L[1] = t_1, \quad L[p_m(x)p_n(x)] = 0 \quad \text{for all } m, n \geq 0 \text{ with } m \neq n. \tag{2.6}$$

Moreover, L is quasi-definite and $\{p_n(x)\}_{n\geq 0}$ is the corresponding monic OPS if and only if $t_n \neq 0$ for all $n \in \mathbb{N}$. In addition, L is strictly positive if and only if $s_n \in \mathbb{R}$ and $t_n > 0$ for all $n \in \mathbb{N}$.

What the last sentence of the above theorem means is that if the linear functional is set to be the Riesz functional, the existence of such an OPS generated by the conditions, $s_n \in \mathbb{R}$ and $t_n > 0$, $n \in \mathbb{N}$, can be a solution to the full moment problem.

In the proof of this theorem, we may observe that

$$L[x^k p_n(x)] = 0, \qquad k = 0, 1, \ldots, n-1, \tag{2.7}$$

$$L[x^n p_n(x)] = t_{n+1} L[x^{n-1} p_{n-1}(x)], \qquad n \geq 1, \tag{2.8}$$

$$L[p_n^2(x)] = L[x^n p_n(x)] = t_1 t_2 \cdots t_{n+1}, \qquad n \geq 0. \tag{2.9}$$

If $t_{N+1} = 0$ for some (minimal) N, then $L[p_n^2(x)] = 0$ for $n \geq N$, and hence $\{p_n(x)\}_{n\geq 0}$ cannot be an OPS; but it still satisfies the primary orthogonality condition, $L[p_m(x)p_n(x)] = 0$ for $m \neq n$. When this situation arises, $\{p_n(x)\}_{n\geq 0}$ is said to be *weakly orthogonal polynomial sequence* (in short, WOPS) *of order* N, with respect to a linear functional L. This notion has received some study as appeared in [7] and [12]; the following recurrence relations are known as a Favard theorem for a WOPS.

Theorem 2.5. [12] *A monic polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is a WOPS of finite order m if and only if $\{p_n(x)\}_{n\geq 0}$ satisfies recurrence relations:*

$$p_{n+1}(x) = (x - b_n)p_n(x) - c_n p_{n-1}(x), \;\; 0 \leq n \leq m \;\; (p_0 = 1, \;\; p_{-1} = 0);$$

$$p_{m+2}(x) = (x - b_{m+1})p_{m+1}(x);$$

$$p_{n+1}(x) = (x - b_n)p_n(x) - c_n p_{n-1}(x) + \sum_{k=m+1}^{n-2} d_n^k \, p_k(x), \;\; n \geq m+1,$$

where b_n, c_n, and d_n^k are some constants with $c_n \neq 0$, $1 \leq n \leq m$.

Example 2.6. *Consider a sequence $\beta = \left(\frac{2^n+3^n}{2}\right)_{n\geq 0}$ with its Riesz functional and we can see that*

$$p_0(x) = 1, \quad p_1(x) = x - \frac{5}{2}, \quad p_2(x) = x^2 - 5x + 6,$$

$$p_n(x) = x^{n-2}(x^2 - 5x + 6), \quad n \geq 3;$$

$$t_1 = 1, \;\; t_2 = \frac{1}{4}, \;\; t_3 = t_4 = \cdots = 0.$$

Thus, $\{p_n(x)\}_{n\geq 0}$ is a WOPS of order 2.

From now on, we will focus on truncated moment sequences, and it can be seen that these definitions of OPS and WOPS apply equally well up to certain degrees; we may have an OPS or WOPS up to degree k for both cases of $\beta = (\beta_0, \ldots, \beta_{2k})$ or $\beta = (\beta_0, \ldots, \beta_{2k+1})$. Here is a formal definition:

Definition 1. For finite sequences $\beta = (\beta_0, \ldots, \beta_{2k})$ or $\beta = (\beta_0, \ldots, \beta_{2k+1})$, let $\mathcal{L}_\beta$ be the Riesz functional of β.

(i) When $\Delta_n(\beta) \neq 0$ for $n = 0, 1, \ldots, k$, we call $\{p_n(x)\}_{n=0}^k$ given by (2.2) an *orthogonal polynomial sequence* (in short OPS) for β.

(ii) If $\Delta_n(\beta) \neq 0$ for $n = 0, 1, \ldots, r-1 (\leq k-1)$ and $\Delta_n(\beta) = 0$ for $n = r, r+1, \ldots, k$, and if $\mathcal{L}_\beta \left[x^i p_r(x)\right] = 0$ for all $i = 0, \ldots, 2k-r$, where $\{p_n(x)\}_{n=0}^r$ given by (2.2), then we call $\left\{p_0(x), \ldots, p_r(x), xp_r(x), x^2 p_r(x), \ldots, x^{k-r} p_r(x)\right\}$

a *weak orthogonal polynomial sequence* (in short WOPS) *of order* r for β.

We may observe that if $\beta = (\beta_0, \ldots, \beta_{2k})$ or $\beta = (\beta_0, \ldots, \beta_{2k+1})$ admits an OPS, then the Riesz functional $\mathcal{L}_\beta$ of β satisfies

$$\begin{aligned} \mathcal{L}_\beta[p_m(x)p_n(x)] &= 0, \qquad m \neq n,\ m, n \leq k, \\ \mathcal{L}_\beta[p_n^2(x)] &\neq 0, \qquad n = 0, 1, \ldots, k. \end{aligned}$$

On the other hand, if $\beta = (\beta_0, \ldots, \beta_{2k})$ or $\beta = (\beta_0, \ldots, \beta_{2k+1})$ admits an WOPS of order r, then

$$\begin{aligned} \mathcal{L}_\beta[p_m(x)p_n(x)] &= 0, \qquad m \neq n;\ m,\ n \leq r \leq k-1, \\ \mathcal{L}_\beta[p_n^2(x)] &\neq 0, \qquad n = 0, 1, \ldots, r-1, \\ \mathcal{L}_\beta[p_n^2(x)] &= 0, \qquad n = r, r+1, \ldots, k. \end{aligned}$$

It can be seen from the discussion so far that the definition of an OPS or WOPS for a finite sequence is compatible with the case of an infinite sequence.

Example 2.7. *The corresponding Hankel matrix of a turncated moment sequence $\beta = (1, 1, 2, 3, t)$ is*

$$A(2) = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & t \end{pmatrix}.$$

If $t \neq 5$, then $P = \left\{1, -x+1, x^2 - x - 1\right\}$ is an OPS for β; on the other hand, if $t = 5$, then P bocomes a WOPS for β.

Example 2.8. *For a turncated moment sequence $\beta = (1, 1, 1, s, t)$, the corresponding Hankel matrix is*

$$A(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & s \\ 1 & s & t \end{pmatrix}.$$

In this case, $\Delta_1(\beta) = 0$ and $\Delta_2(\beta) = -(-1+s)^2$; we may take $p_0(x) = 1$ and $p_1(x) = -x + 1$. Then $\mathcal{L}_\beta\left[x^2 p_1(x)\right] = s - 1$ and $\mathcal{L}_\beta\left[x^3 p_1(x)\right] = t - s$. If $s \neq 1$ or $s \neq t$, then β cannot have a WOPS.

For $\beta = (\beta_0, \ldots, \beta_{2k})$ or $\beta = (\beta_0, \ldots, \beta_{2k+1})$, let $r = \text{rank } \beta$. When the columns of $A(k)$ are labeled as $\mathit{1}, X, X^2, \ldots, X^k$ and $A(k)$ is singular, let $r = \min\left\{i : X^i \in \langle 1, X, X^2, \ldots, X^{i-1}\rangle\right\}$ $(1 \leq r \leq k)$; that is, there exist real numbers $a_0, a_1, \ldots, a_{r-1}$ such that $X^r = a_0\mathit{1} + a_1 X + \cdots + a_{r-1}X^{r-1}$. In the case where β satisfies

$$X^{r+s} = a_0 X^s + a_1 X^{s+1} + \cdots + a_{r-1}X^{r+s-1}\ (0 \leq s \leq k-r),$$

we say that β is *recursively generated.*

If β is recursively generated, and if the first column relation in $A(k)$ is written as $p(X) = \mathbf{0}$, then we set $V(\beta) \equiv \mathcal{Z}(p) = \{x \in \mathbb{R} : p(x) = 0\}$, which is called the *algebraic variety* of β.

In the sequel, $\hat{p}$ denotes the cofficient vector of the polynomial p and let $\mathcal{P}_n = \{p \in \mathbb{R}[x] : \deg p \leq n\}$. Since $A(k)$ is a real symmetric matrix, we can define the sesquilinear form as follows:

$$\langle A(k)\hat{p}, \hat{q}\rangle = \mathcal{L}_\beta\left[pq\right], \qquad p, q \in \mathcal{P}_k.$$

Recall that in the presence of a (positive) representing measure μ for a positive β, Proposition 3.1 in [5] states that

$$\hat{p} \in \ker A(k) \iff p(X) = \mathbf{0} \iff \text{supp } \mu \subseteq \mathcal{Z}(p).$$

This result provides an evidence that where the atoms of μ lie for a singular $A(k)$; that is, the algebraic variety of $A(k)$ must contain the support of a representing measure.

We now recall well-known important necessary conditions for the existence of a representing measure for β [6]:

$$\begin{array}{rll} \text{(Weak Consistency)} & p \in \mathcal{P}_n,\, p|_{V(\beta)} \equiv 0 & \Longrightarrow \mathcal{L}_\beta(p) = 0. \\ \text{(Consistency)} & p \in \mathcal{P}_{2n},\, p|_{V(\beta)} \equiv 0 & \Longrightarrow \mathcal{L}_\beta(p) = 0. \end{array}$$

It is obvious that the following holds:

$$\beta \text{ consistent} \Longrightarrow \beta \text{ weakly consistent} \Longrightarrow \beta \text{ recursively generated}.$$

We will see later that there is not much difference between the above three concepts for univariate moment problems. However, as the examples in [6] show, this is not the case for multivariate moment problems.

As can be seen in the below result, consistency of a moment sequence guarantees the existence of an interpolating (or signed) measure.

Lemma 2.9. ([6, Lemma 2.3]) *Let $W \subseteq \mathbb{R}$. If $L : \mathcal{P}_{2n} \to \mathbb{R}$ is a linear functional, then the following statements are equivalent:*

(i) *There exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$ and there exist $\mathbf{w}_1, \ldots, \mathbf{w}_\ell \in W$ such that for all $p \in \mathcal{P}_{2n}$*

$$L(p) = \sum_{k=1}^{\ell} \alpha_k p(\mathbf{w}_k). \tag{2.10}$$

(ii) *If $p \in \mathcal{P}_{2n}$ and $p|_W \equiv 0$, then $L(p) = 0$.*

If L is the Riesz functional of the moment sequence β, then Lemma 2.9(ii) is just as consistency of β and $\sum_{k=1}^{\ell} \alpha_k \delta_{\mathbf{w}_k}$ is an interpolating measure for β.

Before we have our main result, note the following key tool; a column X^r in $A(k)$ is linearly dependent if and only if there are some real numbers $a_0, a_1, \ldots, a_{r-1}$ such that

$$\mathcal{L}_\beta\left[x^i(x^r - a_0 - a_1 x - \cdots - a_{r-1}x^{r-1})\right] = 0, \qquad i = 0, \ldots, k.$$

There is an interesting class of moment sequences called *extremal*, which are cases where rank $A(k) = \text{card } V(\beta)$. One relevant and important result is the following:

Theorem 2.10. [6, Theorem 1.3] *For an extremal $\beta = (\beta_0, \ldots, \beta_{2k})$, the following are equivalent:*

(i) *β has a representing measure;*
(ii) *β has a unique representing measure, which is* rank *β-atomic;*
(iii) *$A(k)$ is positive semidefinite and β is consistent.*

Now we are about to look at the main results.

Theorem 2.11. *$\beta = (\beta_0, \ldots, \beta_{2k})$ or $\beta = (\beta_0, \ldots, \beta_{2k+1})$ is recursively generated if and only if β admits a WOPS.*

Proof. ($\Longleftarrow$) When β admits a WOPS of order r, we may write the sequence as $\{p_0(x), \ldots, p_r(x), xp_r(x), x^2p_r(x), \ldots, x^{k-r}p_r(x)\}$.

We can actually find a column relation $p_r(X) = \mathbf{0}$ in $A(k)$. In details, since $\Delta_r(\beta) = 0$, $A(r)$ has a column relation, which can be written as

$$X_r^r = a_0 \mathit{1}_r + a_1 X_r + \cdots + a_{r-1} X_r^{r-1} \tag{2.11}$$

for some real numbers $a_0, a_1, \ldots, a_{r-1}$, where X_j^i denotes the $(i+1)$-th columns in $A(r)$. We can readily see that $p_r(x) = x^r - \left(a_0 + a_1 x + \cdots + a_{r-1}x^{r-1}\right)$. By the weak orthogonality, we also get that $\mathcal{L}_\beta\left[x^i p_r(x)\right] = 0$ for all $i = 0, \ldots, 2k-r$, which shows that $A(k)$ has the column relation $X^r = a_0 \mathit{1} + a_1 X + \cdots + a_{r-1}X^{r-1}$; equivalently, $p_r(X) = \mathbf{0}$. Furthermore, it is possible to proceed to the argument:

$$\begin{aligned} X^{r+s} = a_0 X^s + a_1 X_{r+1} + \cdots + a_{r-1} X_r^{r+s-1}, \qquad & s = 0, 1, \ldots, k-r \\ \Longleftrightarrow \mathcal{L}_\beta\left[x^i p_r(x)\right] = 0, \qquad & i = 0, \ldots, 2k-r. \end{aligned} \tag{2.12}$$

Thus, β is recursively generated.

($\Longrightarrow$) Suppose β is recursively generated and apply (2.12) again. Then one can easily complete the proof.

□

Combining all the results so far, the following main results are presented.

Theorem 2.12. *Let $\beta = (\beta_0, \ldots, \beta_{2k})$ with $\beta_0 > 0$, and let $r =$ rank $\beta \le k$; that is, $A(k)$ is singular. The following are equivalent:*

(i) *There exists a unique r-atomic positive Borel measure μ satisfying $\beta_j = \int x^j d\mu$ $(j = 0, \ldots, 2k)$, and* supp *$\mu \subseteq \mathbb{R}$;*
(ii) *$A(k) \ge 0$ and β admits a WOPS;*
(iii) *$A(k) \ge 0$ and β is recursively generated;*
(iv) *$A(k) \ge 0$ and β is weakly consistent;*
(v) *$A(k) \ge 0$ and β is consistent.*

Proof. The implications $(iii) \Longrightarrow (iv) \Longrightarrow (v)$ are immediate; $(ii) \iff (iii)$ is given by Theorem 2.11 and $(i) \Longrightarrow (iii)$ can be shown by Theorem 1.3.

It remains to prove that $(v) \Longrightarrow (i)$. If we assum that (v) is true, then β is extremal and so, by Theorem 2.10, it admits a unique r-atomic representing measure. This completes the proof. $\square$

Theorem 2.13. *Let $\beta = (\beta_0, \ldots, \beta_{2k})$ with $\beta_0 > 0$. The following are equivalent:*

(i) *There exists a $(k+1)$-atomic positive Borel measure μ satisfying $\beta_j = \int x^j d\mu$ $(j = 0, \ldots, 2k)$, and* $\text{supp } \mu \subseteq \mathbb{R}$*;*
(ii) *$A(k)$ is positive definite;*
(iii) *$A(k)$ is positive definite and β admits an OPS.*

Proof. The equivalence of (i) and (ii) is obvious, and the implication $(iii) \Longrightarrow (ii)$ is obtained by Theorem 1.3. It suffice to prove that $(i) \Longrightarrow (iii)$. If β admits a $(k+1)$-atomic positive Borel measure, then $A(k)$ is positive semidefinite. The nested determinant test for $A(k)$ guarantees that $\Delta_n(\beta) \neq 0$ for $n = 0, \ldots, k$. Thus, β admits an OPS. $\square$

Theorem 2.14. *Let $\beta = (\beta_0, \ldots, \beta_{2k}, \beta_{2k+1})$ with $\beta_0 > 0$. Suppose $A(k) > 0$ and assume β_{2k+2} is a real number to make $A(k+1)$ singular. The following are equivalent:*

(i) *There exists a unique k-atomic positive Borel measure μ satisfying $\beta_j = \int x^j d\mu$ $(j = 0, \ldots, 2k+2)$, and* $\text{supp } \mu \subseteq \mathbb{R}$*;*
(ii) *$\tilde{\beta} = (\beta_0, \ldots, \beta_{2k+1}, \beta_{2k+2})$ admits a WOPS;*
(iii) *$\tilde{\beta}$ is recursively generated.*

Proof. First of all, it is easy to know that the new moment β_{2k+2} in the hypothesis always exists due to the basic property of the determinant of matrices. Applying Theorem 1.2 and 2.11, we can complete the proof.

$\square$

Theorem 2.15. *Let $\beta = (\beta_0, \ldots, \beta_{2k}, \beta_{2k+1})$ with $\beta_0 > 0$. Suppose $A(k) > 0$ and assume β_{2k+2} is a real number to make* $\det A(k+1)$ *positive. The following are equivalent:*

(i) *There exists a $(k+2)$-atomic positive Borel measure μ satisfying $\beta_j = \int x^j d\mu$ $(j = 0, \ldots, 2k+2)$, and* $\text{supp } \mu \subseteq \mathbb{R}$*;*
(ii) *$\tilde{\beta} = (\beta_0, \ldots, \beta_{2k+1}, \beta_{2k+2})$ admits an OPS;*

Proof. The equivalence can be obtained from Theorem 2.13 and 2.11.

$\square$

Example 2.16. *For a turncated moment sequence $\beta = (1, 1, 2, 3, 6, 7)$, the corresponding Hankel matrix is*

$$A(3) = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 6 & 7 \\ 3 & 6 & 7 & \beta_6 \end{pmatrix}.$$

Note that $A(2)$ is positive definite. If $\beta_6 = 22$, then $A(3)$ is singular and it follows from Theorem 2.14 that $\tilde{\beta} = (1, 1, 2, 3, 6, 7, 22)$ has a unique 3-atomic representing measure. The set $P = \left\{1, x-1, x^2 - x - 1, x^3 + 2x^2 - 5x - 2\right\}$ is a WOPS for β.

On the other hand, if $\beta_6 > 22$, then $A(3)$ is positive definite and we conclude from Theorem 2.15 that $\tilde{\beta} = (1, 1, 2, 3, 6, 7, \beta_6)$ has a 4-atomic representing measure. It is different from the case above so that P is now an OPS for $\tilde{\beta}$. Lastly, we conclude that β admits infinitely many representing measures.

It is well-known that the most important solution to a multivariate moment problem is the existence of a flat tension (rank-preserving moment matrix extension) for a given moment sequence. Most information for solving truncated moment problems was obtained by observing the polynomials appearing from the column dependence relations in the moment matrix. The main result of this note can be interpreted as confirming the existence of a positive expansion of the moment matrix using weakly orthogonal polynomials based on moments. For lower degree moment problems, building positive moment matrix extensions is not difficult; but as the degree increases (in other words, when the given moment sequence becomes longer), it is highly nontrivial to determine whether an extension actually exists due to the increment of parameters. Since the method using weakly orthogonal polynomials is expected to be a new approach for multivariate moment problems, it should be a valuable topic for the future research.

Acknowledgment. The author is deeply grateful to the referees for many suggestions that led to significant improvements in the presentation.

References

[1] C. Bayer and J. Teichmann, *The proof of Tchakaloff's Theorem*, Proc. Amer. Math. Soc. 134 (2006), 3035–3040.

[2] R. P. Boas, Jr., *The Stieltjes moment problem for functions of bounded variation*, Bull. Amer. Math. Soc. 45 (1939), no. 6, 399–404.

[3] T. S. Chihara, An introduction to orthogonal polynomials, *Gordon and Breach Science Publishers*, New York-London-Paris, 1978.

[4] R. Curto and L. Fialkow, A. *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. 17 (1991), no. 4, 603–635.

[5] R. Curto and L. Fialkow, Solution of the truncated complex moment problem for flat data, *Mem. Amer. Math. Soc.* 119 (1996), no. 568, x+52 pp.

[6] R. Curto, L. Fialkow and H.M. Möller, *The extremal truncated moment problem*, Integral Equations Operator Theory 60 (2008), 177–200.

[7] L. Fernández, T. Pérez, and M. Piñar, *Weak classical orthogonal polynomials in two variables*, J. Comput. Appl. Math. 178 (2005), no.1-2, 191–203.

[8] L. Fialkow, *Truncated multivariable moment problems with finite variety*, J. Operator Theory 60 (2008), 343–377.

[9] L. Fialkow, *Solution of the truncated moment problem with variety $y = x^3$*, Trans. Amer. Math. Soc. 363 (2011), 3133–3165

[10] G. P. Flessas, W. K. Burton, and R. R. Whitehead, *On the moment problem for non-positive distributions*, J. Phys. A 15 (1982), no. 10, 3119–3130.

[11] R. Horn and C. Johnson, Matrix Analysis, *Cambridge University Press*, 2013.

[12] D. W. Lee, *Characterizations of distributional weights for weak orthogonal polynomials satisfying a second-order differential equation*, Ann. Mat. Pura Appl. (4) 194 (2015), no.5, 1319–1348.

[13] K. Schmüdgen, The Moment Problem, *Springer*, 2017. 225–234.

[14] S. Yoo, *Sextic moment problems with a reducible cubic column relation*, Integral Equations Operator Theory 88 (2017), no. 1, 45–63.

Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, 52828, Korea.

Email address: seyoo@gnu.ac.kr