POSITIVE SOLUTIONS FOR THE SECOND ORDER DIFFERENTIAL SYSTEM WITH STRONGLY COUPLED INTEGRAL BOUNDARY CONDITION

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ABSTRACT. We establish the existence, multiplicity and uniqueness of positive solutions to nonlocal boundary value systems with strongly coupled integral boundary condition by using the global continuation theorem and Banach's contraction principle.

1. Introduction

In this paper, we study the existence of the following differential system;

$$
\begin{cases}
u''(t) + \lambda a_1(t) f_1(u(t), v(t)) = 0, \ t \in (0,1), \\
v''(t) + \lambda a_2(t) f_2(u(t), v(t)) = 0, \ t \in (0,1), \\
u(0) = 0 = v(0), \\
u(1) = \int_0^1 (g_1(s)u(s) + g_2(s)v(s))\, ds, \\
v(1) = \int_0^1 (g_3(s)u(s) + g_4(s)v(s))\, ds,
\end{cases}
\tag{P_λ}
$$

where $f_i \in C([0,\infty) \times [0,\infty)), (0,\infty))$ and $a_i,\ g_i \in L^1((0,1),[0,\infty))$, for $i \in \{1,2,3,4\}$. We further assume that there exists an interval $I \subset (0,1)$ with positive measure such that $a_i(t) > 0$ for all $t \in I$ and we notice that $f_i(0,0) > 0$ for $i = 1, 2$.

Such differential equations with an integral boundary condition arise in various areas of applied mathematics and physics like heat conduction, chemical engineering, underground water flow, thermo-elasticity, hydro dynamic problems and plasma phenomena. One may refer to [1], [2], [6], [7] and [10] for integral boundary value problems and the references therein. Recently, many works have been done for second order ordinary differential systems with integral boundary

conditions ([3], [4], [5], [8], [9], [12], [13]), but most of papers considered the differntial systems with uncoupled or weekly coupled boundary conditions.

In this paper, the problem (P_λ) has more general strongly coupled integral boundary conditions, which makes the operator $\mathbf{S}_\lambda$ (see Section 2 for definition) complicated and induces substantial difficulties in proving our results. According to the growth rates of f_i $(i = 1, 2)$, we proved the existence, nonexistence, multiplicity and uniqueness of the positive solutions of (P_λ). Throughout this paper, we assume the following hypotheses;

$(H0)$ $0 < \int_0^1 sg_i(s)ds < 1$ for $i = 1, 4$ and

$$(1 - \int_0^1 sg_1(s)ds)(1 - \int_0^1 sg_4(s)ds) - (\int_0^1 sg_2(s)ds)(\int_0^1 sg_3(s)ds) > 0.$$

$(H1')$ $f_{i,\infty} := \lim\limits_{|u|+|v|\to\infty} \dfrac{f_i(u,v)}{u+v} = 0$, for $i = 1, 2$.

$(H1'')$ $0 < f_{i,\infty} < \infty$, for $i = 1, 2$.

$(H1''')$ $f_{i,\infty} = \infty$, for $i = 1, 2$.

This paper is organized as follows. In Section 2, we present the solution operator to problem (P_λ) and introduce the well-known theorems such as global continuation theorem and eigenvalue theorem which will be used to prove our main result. In Section 3, the existence and multiplicity results are proven by using the solution continuum. In Section 4, the uniqueness results are proven. In Section 5, as an applications, the range of parameter for uniqueness are given by using matlab.

2. Preliminary

In this section, we set up the operator equation for the problem (P_λ). From $(H0)$, we know that $\det A \neq 0$ when

$$A := \begin{pmatrix} 1 - \int_0^1 sg_1(s)ds & - \int_0^1 sg_2(s)ds \\ - \int_0^1 sg_3(s)ds & 1 - \int_0^1 sg_4(s)ds. \end{pmatrix}.$$

Let

$$A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Here, we note that from $(H0)$, $a_{ij} > 0$ for all $i, j \in \{1, 2\}$. Let us denote $E := C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ where E is the usual Banach space with the norm $\|(u,v)\| = \|u\|_\infty + \|v\|_\infty$, when $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$. Now, we define $S_{1,\lambda}$ and $S_{2,\lambda}$ from E to $C([0,1], \mathbb{R})$ by

$$S_{1,\lambda}(u,v)(t) := \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds,$$

and

$$S_{2,\lambda}(u,v)(t) := \lambda \int_0^1 H_2(t,s)a_2(s)f_2(u(s),v(s)) + tK_2(s)a_1(s)f_1(u(s),v(s))ds,$$

where

$$H_1(t,s) \quad = G(t,s) + t\int_0^1 G(\tau,s)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau,$$
$$H_2(t,s) \quad = G(t,s) + t\int_0^1 G(\tau,s)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau,$$

$$K_1(s) = \int_0^1 G(\tau,s)(a_{11}g_2(\tau) + a_{12}g_4(\tau))d\tau,$$

$$K_2(s) = \int_0^1 G(\tau,s)(a_{21}g_1(\tau) + a_{22}g_3(\tau))d\tau,$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Now we define

$$\mathbf{S}_\lambda(u,v)(t) := (S_{1,\lambda}(u,v)(t), S_{2,\lambda}(u,v)(t)).$$

Then $\mathbf{S}_\lambda : E \to E$ is well defined and we notice that the problem (P_λ) is equivalent to the following operator equation;

$$(u,v) = \mathbf{S}_\lambda(u,v) \text{ on } E. \tag{1}$$

Let $\mathcal{P} = \{(u,v) \in E : u(t) \ge 0, v(t) \ge 0 \text{ for all } t \in [0,1]\}$. We recall $I \subset (0,1)$ is a nondegenerate interval such that $a_i(t) > 0$ for all $t \in I$ and $i = 1, 2$. Let $\gamma = \min\{j_*, 1 - j^*\}$ where $j_* = \inf I$ and $j^* = \sup I$. Here we define $\mathcal{K}$ by

$$\mathcal{K} = \{(w_1, w_2) \in \mathcal{P} : \min_I w_i(t) \ge \gamma\|w_i\|_\infty, \text{ for } i = 1, 2\}.$$

Then $\mathcal{P}$ and $\mathcal{K}$ are cones in E. It is clear that $\mathbf{S}_\lambda(\mathcal{P}) \subset \mathcal{K}$ and $\mathbf{S}_\lambda$ is completely continuous on E, by standard argument.

Following remark will be used in the proof for our results.

Remark 1. It is easy to check that

$$t(1-t)s(1-s) \le G(t,s) \le s(1-s), \ t,s \in (0,1). \tag{2}$$

By using (2), we have

$$\nu s(1-s)t \le H_i(t,s) \le \rho t, \ i = 1,2 \tag{3}$$

and

$$\nu s(1-s) \le K_i(s) \le \rho, \ i = 1,2, \tag{4}$$

where

$$\rho = max\{1 + C_1, 1 + C_2, C_3, C_4\},$$

and

$$\nu = min\{C_1, C_2, C_3, C_4\}$$

with $C_1 = \int_0^1 \tau(1-\tau)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau$, $C_2 = \int_0^1 \tau(1-\tau)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau$, $C_3 = \int_0^1 \tau(1-\tau)(a_{11}g_2(\tau) + a_{12}g_4(\tau))d\tau$ and $C_4 = \int_0^1 \tau(1-\tau)(a_{21}g_1(\tau) + a_{22}g_3(\tau))d\tau$.

To prove our main results, we use the following well known theorems for the existence of a global continuum of solutions and the existence of eigenvalues of operator .

Theorem 2.1. *([14], Corollary 14.12) Let E be a Banach space with $E \neq \{0\}$ and let $\mathcal{P}$ be an order cone in E. Consider*

$$x = H(\lambda, x), \tag{5}$$

where $\lambda \in \mathbb{R}_+$ and $x \in \mathcal{P}$. If $H : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ is completely continuous and $H(0, x) = 0$ for all $x \in \mathcal{P}$, then $\mathcal{C}_+(\mathcal{P})$, the component of solution set of (5) containing $(0, 0)$, is unbounded.

Definition 1. *([11]) Let E be a Banach space and $\mathcal{P} \subset E$ be a cone in E. Let $e \in \mathcal{P} \setminus \{0\}$. A mapping $T : \mathcal{P} \to \mathcal{P}$ is called e-positive if for every nonzero $x \in \mathcal{P}$ a natural number $n = n(x)$ and two positive number c_x, d_x can be found such that*

$$c_x e \leq T^n x \leq d_x e.$$

Recall that a real number λ is an eigenvalue of the operator T if there exists a non-zero element $x \in E$ such that $Tx = \lambda x$.

Theorem 2.2. *([11]) Suppose that $T : E \to E$ is a e-positive, completely continuous linear operator. If there exist $\psi \in E \setminus (-\mathcal{P})$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$, and $r(T)$ is the unique positive eigenvalue with its eigenfunction in $\mathcal{P}$.*

3. Existence and Multiplicity

In this section, we establish the existence and multiplicity results for positive solutions of (P_λ).

Lemma 3.1. *Assume (H1′). For any closed bounded interval $J = [\alpha, \beta] \subset [0, \infty)$, there exists $M_J > 0$ such that for all $\lambda \in J$, all possible solution (u, v) of (P_λ) satisfy $\|(u, v)\| < M_J$.*

Proof. Suppose on the contrary that there exist a sequence $(\lambda_n) \subset J$ such that (P_{λ_n}) has positive solutions (u_n, v_n) with $\|(u_n, v_n)\| \to \infty$. Let $\mu \in (0, \frac{1}{\beta(Q_1 + Q_2)})$, with

$$Q_1 = \int_0^1 h_1(s)a_1(s) + K_1(s)a_2(s)ds \text{ and}$$

$$Q_2 = \int_0^1 h_2(s)a_2(s) + K_2(s)a_1(s)ds,$$

where

$$h_1(s) \quad = G(s,s) + \int_0^1 G(\tau,s)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau \text{ and}$$

$$h_2(s) \quad = G(s,s) + \int_0^1 G(\tau,s)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau.$$

From $(H1')$, there exists a constant $l_\mu > 0$ such that $f_i(u,v) < \mu(u+v)$ for all $u + v > l_\mu$. Let

$$m_\mu := \max\{ \max_{(u,v)\in[0,l_\mu]\times[0,l_\mu]} f_1(u,v), \max_{(u,v)\in[0,l_\mu]\times[0,l_\mu]} f_2(u,v)\},$$

$$A_n \quad := \{t \in [0,1] \mid u_n(t) + v_n(t) \le l_\mu\} \text{ and}$$

$$B_n \quad := \{t \in [0,1] \mid u_n(t) + v_n(t) > l_\mu\}.$$

Then we obtain

$$u_n(t) \le \lambda_n \int_0^1 h_1(s)a_1(s)f_1(u_n(s), v_n(s)) + K_1(s)a_2(s)f_2(u_n(s), v_n(s))ds$$

$$\le \lambda_n \int_{A_n} m_\mu \left(h_1(s)a_1(s) + K_1(s)a_2(s)\right) ds$$

$$+ \lambda_n \int_{B_n} \mu \left(h_1(s)a_1(s) + K_1(s)a_2(s)\right)(u_n(s) + v_n(s))ds$$

$$\le \lambda_n m_\mu Q_1 + \lambda_n \mu Q_1(\|u_n\|_\infty + \|v_n\|_\infty)$$

and thus $\|u_n\|_\infty \le \lambda_n m_\mu Q_1 + \lambda_n \mu Q_1(\|u_n\|_\infty + \|v_n\|_\infty)$. By similar computation for v_n, we have $\|v_n\|_\infty \le \lambda_n m_\mu Q_2 + \lambda_n \mu Q_2(\|u_n\|_\infty + \|v_n\|_\infty)$. By adding two inequalities, we have $\|u_n\|_\infty + \|v_n\|_\infty \le \lambda_n m_\mu(Q_1 + Q_2) + \lambda_n \mu(Q_1 + Q_2)(\|u_n\|_\infty + \|v_n\|_\infty)$ and

$$\frac{1}{\beta} \le \frac{1}{\lambda_n} \le \frac{m_\mu(Q_1 + Q_2)}{\|u_n\|_\infty + \|v_n\|_\infty} + \mu(Q_1 + Q_2).$$

By taking limit $n \to \infty$, from the choice of μ, we have following contradiction,

$$\frac{1}{\beta} \le \mu(Q_1 + Q_2) < \frac{1}{\beta}.$$

$\square$

With this Lemma, we have the following first existence result.

Theorem 3.2. *Suppose that $(H0)$ and $(H1')$ hold. Then (P_λ) has a positive solution for all $\lambda > 0$.*

Proof. Define $H(\lambda, (u,v)) = \mathbf{S}_\lambda(u,v)$, then $H : \mathbb{R}_+ \times \mathcal{P} \to \mathcal{P}$ is completely continuous and $H(0, (u,v)) = 0$ for all $(u,v) \in \mathcal{P}$. By Theorem 2.1, there exists an unbounded continuum $\mathcal{C}_+(\mathcal{P})$, the component of the solution set of $(u,v) = H(\lambda, (u,v))$ containing $(0,(0,0))$. Since $f_i(0,0) > 0$, if $(\lambda, (u,v))$ in $\mathcal{C}_+(\mathcal{P})$ and $\lambda > 0$, then (u,v) is positive solution to problem (P_λ). By Lemma 3.1, (P_λ) has a positive solution for all $\lambda > 0$ (see Figure 1). $\square$
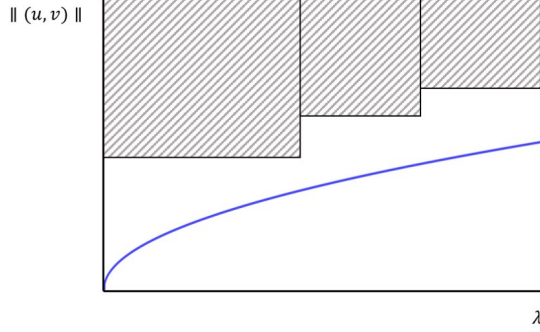
FIGURE 1. Solution continuum : $f_{i,\infty} = 0$

Lemma 3.3. *Suppose that either $(H1'')$ or $(H1''')$ hold. Then there exists a positive constant $\bar{\lambda} > 0$ such that (P_λ) has no positive solution for all $\lambda > \bar{\lambda}$.*

Proof. Suppose on the contrary that there exist $\lambda_n > 0$ such that $\lambda_n \to \infty$ and (P_{λ_n}) has a positive solution (u_n, v_n). From $f_i(0,0) > 0$ and $(H1'')$ (or $(H1''')$), we can choose a constant $\sigma > 0$ such that $f_i(u,v) > \sigma(u+v)$ for all $(u,v) \in [0,\infty) \times [0,\infty)$ and $i = 1,2$. By using (3), (4) and the fact $(u_n, v_n) = \mathbf{S}_{\lambda_n}(u_n, v_n) \in \mathcal{K}$, we obtain

$$\|u_n\|_\infty + \|v_n\|_\infty \geq \|u_n\|_\infty \geq |u_n(j_*)|$$

$$= \lambda_n \int_0^1 H_1(j_*, s)a_1(s)f_1(u_n(s), v_n(s)) + j_* K_1(s)a_2(s)f_2(u_n(s), v_n(s))ds$$

$$\geq \lambda_n \int_0^1 \nu j_* s(1-s)a_1(s)f_1(u_n(s), v_n(s)) + \nu j_* s(1-s)a_2(s)f_2(u_n(s), v_n(s))ds$$

$$\geq \lambda_n \nu j_* \int_{j_*}^{j^*} s(1-s)(a_1(s) + a_2(s))\sigma(u_n(s) + v_n(s))ds$$

$$\geq \lambda_n \gamma \nu j_* \sigma \left(\int_{j_*}^{j^*} s(1-s)(a_1(s) + a_2(s))ds \right) (\|u_n\|_\infty + \|v_n\|_\infty).$$

Thus, we have $\lambda_n \leq \left(\gamma \nu j_* \sigma(\int_{j_*}^{j^*} s(1-s)(a_1(s) + a_2(s))ds) \right)^{-1}$, which contradicts to $\lambda_n \to \infty$. $\square$

Lemma 3.4. *Assume $(H1''')$. For any closed bounded interval $J = [\alpha, \beta] \subset (0,\infty)$, there exists $B_J > 0$ such that for all $\lambda \in J$, all possible solutions (u,v) satisfy $\|(u,v)\| < B_J$.*

Proof. Suppose on the contrary that there exists a sequence $(\lambda_n) \subset J$ such that (P_{λ_n}) has a positive solution (u_n, v_n) with $\|u_n\|_\infty + \|v_n\|_\infty \to \infty$. Choose

constant $\eta > 0$ such that

$$\alpha\eta\nu j_* \gamma \left(\int_{j_*}^{j^*} s(1-s)(a_1(s) + a_2(s))ds \right) > 1.$$

From $(H1''')$, there exists $R_1 > 0$ such that $f_i(u, v) \geq \eta(u+v)$ for $u+v \geq R_1$ and $i = 1, 2$. From $\|u_n\|_\infty + \|v_n\|_\infty \to \infty$, $u_n(t) + v_n(t) \geq \gamma(\|u_n\|_\infty + \|v_n\|_\infty) \geq R_1$ for sufficiently large n and $t \in [j_*, j^*]$. For such a large n, from the choice of $\eta > 0$ and by the almost same calculation as in the proof of Lemma 3.3, we get the following contradiction,

$$\|u_n\|_\infty + \|v_n\|_\infty \geq \|u_n\|_\infty \geq |u_n(j_*)|$$

$$= \lambda_n \int_0^1 H_1(j_*, s)a_1(s)f_1(u_n(s), v_n(s)) + j_* K_1(s)a_2(s)f_2(u_n(s), v_n(s))ds$$

$$\geq \lambda_n \nu j_* \int_{j_*}^{j^*} s(1-s)a_1(s)f_1(u_n(s), v_n(s)) + s(1-s)a_2(s)f_2(u_n(s), v_n(s))ds$$

$$\geq \lambda_n \nu j_* \int_{j_*}^{j^*} (s(1-s)a_1(s) + s(1-s)a_2(s))\eta(u_n(s) + v_n(s))ds$$

$$\geq \alpha\eta\nu j_* \gamma \int_{j_*}^{j^*} s(1-s)(a_1(s) + a_2(s))ds(\|u_n\|_\infty + \|v_n\|_\infty)$$

$$> \|u_n\|_\infty + \|v_n\|_\infty.$$

$\square$

With those Lemmas, we have the following the multiplicity result.

Theorem 3.5. *Suppose that $(H0)$ and $(H1''')$ hold. Then there exist $\lambda^* \leq \bar{\lambda} < \infty$ such that (P_λ) has two positive solutions $(\bar{u}_\lambda, \bar{v}_\lambda)$ and $(\underline{u}_\lambda, \underline{v}_\lambda)$ for $0 < \lambda < \lambda^*$ and no positive solution for $\lambda > \bar{\lambda}$. Moreover, $\|(\underline{u}_\lambda, \underline{v}_\lambda)\| \to 0$ and $\|(\bar{u}_\lambda, \bar{v}_\lambda)\| \to \infty$ as $\lambda \to 0^+$.*

Proof. From $(H0)$ and $f_i(0, 0) > 0$, as in the proof of Theorem 3.2, we know that there exists an unbounded continuum $\mathcal{C}_+(\mathcal{P})$, the component of the positive solution set of $(u, v) = H(\lambda, (u, v)) = \mathbf{S}_\lambda(u, v)$ containing $(0, (0, 0))$. Then by Lemma 3.3 and Lemma 3.4, (P_λ) has two positive solutions $(\bar{u}_\lambda, \bar{v}_\lambda)$ and $(\underline{u}_\lambda, \underline{v}_\lambda)$ for $0 < \lambda < \lambda^*$ and no positive solution for $\lambda > \bar{\lambda}$. Moreover, $\|(\underline{u}_\lambda, \underline{v}_\lambda)\| \to 0$ and $\|(\bar{u}_\lambda, \bar{v}_\lambda)\| \to \infty$ as $\lambda \to 0^+$ (see Figure 2). $\square$

4. Uniqueness

In this section, we give a hypothesis which will be used for our uniqueness result.
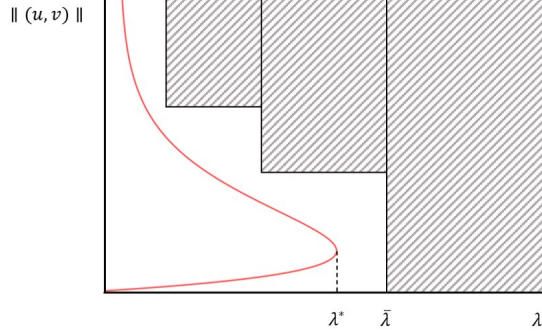
FIGURE 2. Solution continuum : $f_{i,\infty} = \infty$

(H2) There exists $\mathbf{a} = (a, b, c, d) \in [0, \infty)^4$ with $a^2 + b^2 + c^2 + d^2 \neq 0$ such that

$$|f_1(u_1, v_1) - f_1(u_2, v_2)| \leq a|u_1 - u_2| + b|v_1 - v_2| \text{ and}$$
$$|f_2(u_1, v_1) - f_2(u_2, v_2)| \leq c|u_1 - u_2| + d|v_1 - v_2|.$$

For $\mathbf{a} = (a, b, c, d) \in [0, \infty)^4$ with $a^2 + b^2 + c^2 + d^2 \neq 0$, define an operator $\mathbf{T_a} : E \to E$ by

$$\mathbf{T_a}(u, v) := (T_{\mathbf{a},1}(u, v), T_{\mathbf{a},2}(u, v)), \tag{6}$$

where operators $T_{\mathbf{a},1}$, and $T_{\mathbf{a},2} : E \to C[0,1]$ are defined by

$$T_{\mathbf{a},1}(u, v)(t) = \int_0^1 H_1(t, s)a_1(s)(au(s) + bv(s))ds + t\int_0^1 K_1(s)a_2(s)(cu(s) + dv(s))ds,$$

$$T_{\mathbf{a},2}(u, v)(t) = \int_0^1 H_2(t, s)a_2(s)(cu(s) + dv(s))ds + t\int_0^1 K_2(s)a_1(s)(au(s) + bv(s))ds.$$

Then $\mathbf{T_a} : E \to E$ is completely continuous and linear operator.

Lemma 4.1. *Suppose that (H2) holds. Then for the operator $\mathbf{T_a}$ defined by (6), there is a unique positive eigenvalue $r(\mathbf{T_a})$ with its eigenfunction in $\mathcal{P}$.*

Proof. For a nonzero $(u, v) \in \mathcal{P}$, by (3) and (4),

$$T_{\mathbf{a},1}(u, v)(t) = \int_0^1 H_1(t, s)a_1(s)(au(s) + bv(s))ds + t\int_0^1 K_1(s)a_2(s)(au(s) + bv(s))ds$$

$$\leq \rho t \int_0^1 a_1(s)(au(s) + bv(s))ds + \rho t \int_0^1 a_2(s)(cu(s) + dv(s))ds$$

$$= \rho t \int_0^1 (aa_1(s) + ca_2(s))u(s)ds + \rho t \int_0^1 (ba_1(s) + da_2(s))v(s)ds$$

$$\tag{7} = d_{(u,v)}t,$$

where $d_{(u,v)} = \rho \left(\int_0^1 (aa_1(s) + ca_2(s))u(s)ds + \int_0^1 (ba_1(s) + da_2(s))v(s)ds \right)$ and

$T_{\mathbf{a},1}(u,v)(t)$

$$\geq \nu t \int_0^1 s(1-s)a_1(s)(au(s) + bv(s))ds + \nu t \int_0^1 s(1-s)a_2(s)(cu(s) + dv(s))ds$$

$$= \nu t \int_0^1 s(1-s)(aa_1(s) + ca_2(s))u(s)ds + \nu t \int_0^1 s(1-s)(ba_1(s) + da_2(s))v(s)ds$$

$$(8) \quad = c_{(u,v)}t,$$

where $c_{(u,v)} = \nu \left(\int_0^1 s(1-s)(aa_1(s) + ca_2(s))u(s) + (ba_1(s) + da_2(s))v(s)ds \right).$
From (7) and (8), we have

$$c_{(u,v)}t \leq T_{\mathbf{a},1}(u,v)(t) \leq d_{(u,v)}t.$$

By the same way, we have

$$c_{(u,v)}t \leq T_{\mathbf{a},2}(u,v)(t) \leq d_{(u,v)}t.$$

Then

$$c_{(u,v)} \cdot \mathbf{e} \leq \mathbf{T_a}(u,v) \leq d_{(u,v)} \cdot \mathbf{e}, \tag{9}$$

when $\mathbf{e}(t) = (t,t)$. By Theorem 2.2, this completes the proof. $\qquad \square$

Remark 2. Let (φ, ψ) be a positive eigenfunction of $\mathbf{T_a}$ corresponding to $r(\mathbf{T_a})$. From $\mathbf{T_a}(\varphi, \psi) = r(\mathbf{T_a})(\varphi, \psi))$ and (9), we notice that there exists $c_{(\varphi,\psi)} > 0$ such that

$$t \leq \frac{r(\mathbf{T_a})}{c_{(\varphi,\psi)}}\varphi(t) \text{ and } t \leq \frac{r(\mathbf{T_a})}{c_{(\varphi,\psi)}}\psi(t). \tag{10}$$

Let $E_1 := \{(u,v) \in E : \frac{|u(t)|}{\varphi(t)} \text{ and } \frac{|v(t)|}{\psi(t)} \text{ are bounded for } t \in [0,1]\}$. Then $E_1 \subset E$ and E_1 is a Banach space with the norm

$$\| (u,v) \|_1 = \max \left\{ \sup_{t \in [0,1]} \frac{|u(t)|}{\varphi(t)}, \sup_{t \in [0,1]} \frac{|v(t)|}{\psi(t)} \right\}$$

and thus for $(u,v) \in E_1$,

$$|u(t)| \leq \| (u,v) \|_1 \varphi(t) \qquad \text{and} \qquad |v(t)| \leq \| (u,v) \|_1 \psi(t). \tag{11}$$

Lemma 4.2. *Suppose (H0) and (H2). Then* $\mathbf{S}_\lambda(E) \subset E_1$.

Proof. For $(u, v) \in E$, by using (3), (4) and (10), we have

$$|S_{1,\lambda}(u,v)(t)| = \lambda \left| \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \right|$$

$$\leq \lambda \left| \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s))ds - \int_0^1 H_1(t,s)a_1(s)f_1(0,0)ds \right|$$

$$+ \lambda \left| \int_0^1 H_1(t,s)a_1(s)f_1(0,0)ds \right|$$

$$+ \lambda t \left| \int_0^1 K_1(s)a_2(s)f_2(u(s),v(s))ds - \int_0^1 K_1(s)a_2(s)f_2(0,0)ds \right|$$

$$+ \lambda t \left| \int_0^1 K_1(s)a_2(s)f_2(0,0)ds \right|$$

$$\leq \lambda \rho t \left| \int_0^1 a_1(s)(f_1(u(s),v(s)) - f_1(0,0))ds \right| + \lambda \rho t \left| \int_0^1 a_1(s)f_1(0,0)ds \right|$$

$$+ \lambda \rho t \left| \int_0^1 a_2(s)(f_2(u(s),v(s)) - f_2(0,0))ds \right| + \lambda \rho t \left| \int_0^1 a_2(s)f_2(0,0)ds \right|$$

$$\leq \lambda \rho t \left(\int_0^1 a_1(s)(a|u(s)| + b|v(s)|)ds + \int_0^1 a_1(s)f_1(0,0)ds \right.$$

$$+ \left. \int_0^1 a_2(s)(c|u(s)| + d|v(s)|)ds + \int_0^1 a_2(s)f_2(0,0)ds \right)$$

$$\leq \frac{r(\mathbf{T_a})}{c_{(\varphi,\psi)}} \lambda \rho \left(\int_0^1 a_1(s)(a|u(s)| + b|v(s)| + f_1(0,0)) \right.$$

$$+ a_2(s)(c|u(s)| + d|v(s)| + f_2(0,0))ds \Big) \varphi(t).$$

Similarly,

$$|S_{2,\lambda}(u,v)(t)| \leq \frac{r(\mathbf{T_a})}{c_{(\varphi,\psi)}} \lambda \rho \left(\int_0^1 a_1(s)(a|u(s)| + b|v(s)| + f_1(0,0)) \right.$$

$$+ a_2(s)(c|u(s)| + d|v(s)| + f_2(0,0))ds \Big) \psi(t).$$

Thus $\mathbf{S}_\lambda$ maps all of E into E_1. $\square$

Now, we give the following uniqueness result.

Theorem 4.3. *Suppose* (H0) *and* (H2). *If* $\lambda < \frac{1}{r(\mathbf{T_a})}$, *then differential system* (P_λ) *has a unique solution in* E.

Proof. From Lemma 4.2, it suffices to show the uniqueness of a fixed point of $\mathbf{S}_\lambda$ in E_1. Note that $\mathbf{T_a}(\varphi,\psi) = r(\mathbf{T_a})(\varphi,\psi)$ means

$$r(\mathbf{T_a})\varphi(t) = \int_0^1 H_1(t,s)a_1(s)(a\varphi(s) + b\psi(s))ds + t \int_0^1 K_1(s)a_2(s)(c\varphi(s) + d\psi(s))ds$$

$$(12)$$

and

$$r(\mathbf{T_a})\psi(t) = \int_0^1 H_2(t,s)a_2(s)(c\varphi(s) + d\psi(s))ds + t\int_0^1 K_2(s)a_1(s)(a\varphi(s) + b\psi(s))ds.$$

For (u_1, v_1), $(u_2, v_2) \in E_1$, by using (11) and (12), we have

$$|S_{1,\lambda}(u_1, v_1)(t) - S_{1,\lambda}(u_2, v_2)(t)|$$

$$\leq \lambda \left| \int_0^1 H_1(t,s)a_1(s)f_1(u_1(s), v_1(s))ds - \int_0^1 H_1(t,s)a_1(s)f_1(u_2(s), v_2(s))ds \right|$$

$$+ \lambda t \left| \int_0^1 K_1(s)a_2(s)f_2(u_1(s), v_1(s))ds - \int_0^1 K_1(s)a_2(s)f_2(u_2(s), v_2(s))ds \right|$$

$$\leq \lambda a \int_0^1 H_1(t,s)a_1(s)|u_1(s) - u_2(s)|ds + \lambda b \int_0^1 H_1(t,s)a_1(s)|v_1(s) - v_2(s)|ds$$

$$+ \lambda tc \int_0^1 K_1(s)a_2(s)|u_1(s) - u_2(s)|ds + \lambda td \int_0^1 K_1(s)a_2(s)|v_1(s) - v_2(s)|ds$$

$$\leq \lambda a \int_0^1 H_1(t,s)a_1(s) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1 \varphi(s)ds$$

$$+ \lambda b \int_0^1 H_1(t,s)a_1(s) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1 \psi(s)ds$$

$$+ \lambda tc \int_0^1 K_1(s)a_2(s) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1 \varphi(s)ds$$

$$+ \lambda td \int_0^1 K_1(s)a_2(s) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1 \psi(s)ds$$

$$\leq \lambda \left(\int_0^1 H_1(t,s)a_1(s)(a\varphi(s) + b\psi(s))ds \right.$$

$$\left. + t\int_0^1 K_1(s)a_2(s)(c\varphi(s) + d\psi(s))ds \right) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1$$

$$= \lambda r(\mathbf{T_a}) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1 \cdot \varphi(t).$$

Similarly,

$$|S_{2,\lambda}(u_1, v_1)(t) - S_{2,\lambda}(u_2, v_2)(t)| \leq \lambda r(\mathbf{T_a}) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1 \cdot \psi(t).$$

For all (u_1, v_1), $(u_2, v_2) \in E_1$,

$$\parallel \mathbf{S}_\lambda(u_1, v_1) - \mathbf{S}_\lambda(u_2, v_2) \parallel_1 \leq \lambda r(\mathbf{T_a}) \parallel (u_1, v_1) - (u_2, v_2) \parallel_1.$$

Since $\lambda r(\mathbf{T_a}) < 1$, by Banach's Contraction Principle, $\mathbf{S}_\lambda$ has a unique fixed point in E_1 and the proof is done. □

We remark that the hypothesis $(H2)$ can be assumed with $(H1')$ or $(H1'')$ but it cannot be assumed with $(H1''')$ together. Thus we have following two corollary results containing uniqueness.

Corollary 4.4. *Suppose that* $(H0)$, $(H1')$ *and* $(H2)$ *hold. Then* (P_λ) *has a unique positive solution for* $\lambda \in (0, \frac{1}{r(T_a)})$ *and at least one positive solution for* $\lambda \in [\frac{1}{r(T_a)}, \infty)$.

Proof. From Theorem 3.2 and Theorem 4.3, the result can be obtained (See Figure 3). ∎

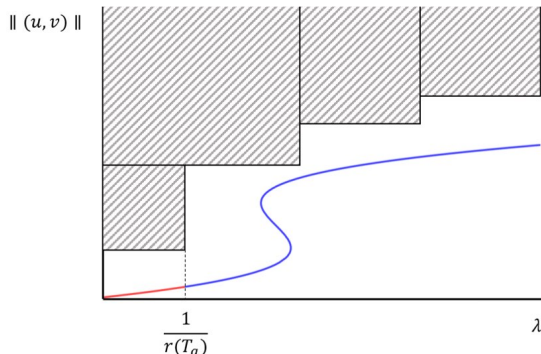FIGURE 3. Solution continuum with uniqueness : $f_{i,\infty} = 0$

Corollary 4.5. *Suppose that* $(H0)$, $(H1'')$ *and* $(H2)$ *hold. Then there exist* $\frac{1}{r(T_a)} \leq \tilde{\lambda} \leq \bar{\lambda}$ *such that* (P_λ) *has at least one positive solution for* $0 < \lambda < \tilde{\lambda}$, *a unique positive solution for* $0 < \lambda < \frac{1}{r(T_a)}$ *and no positive solution for* $\lambda > \bar{\lambda}$.

Proof. From $(H0)$ and $f_i(0,0) > 0$, as in the proof of Theorem 3.2 and Theorem 3.5, we know that there exists an unbounded continuum $\mathcal{C}_+(\mathcal{P})$, the component of the positive solution set of $(u,v) = H(\lambda, (u,v)) = \mathbf{S}_\lambda(u,v)$ containing $(0, (0,0))$. By Lemma 3.3 and Theorem 4.3, the result can be obtained (See Figure 4). ∎

5. Example

We give an example in which we can get the eigenvalue $r(\mathbf{T_a})$ and the range of the parameter for guaranting unique poisitve solution of the certain integral boundary valued system.
Consider the system

$$
\begin{cases}
u''(t) + \lambda(\cos u(t) + \ln(1 + v^2(t)) + 2) = 0, & t \in (0,1), \\
v''(t) + \lambda(\ln(1 + u^2(t)) + \arctan v(t) + 1) = 0, & t \in (0,1), \\
u(0) = 0 = v(0), \\
u(1) = \int_0^1 su(s) + v(s)ds, \\
v(1) = \int_0^1 u(s) + sv(s)ds.
\end{cases} \qquad (E_\lambda)
$$
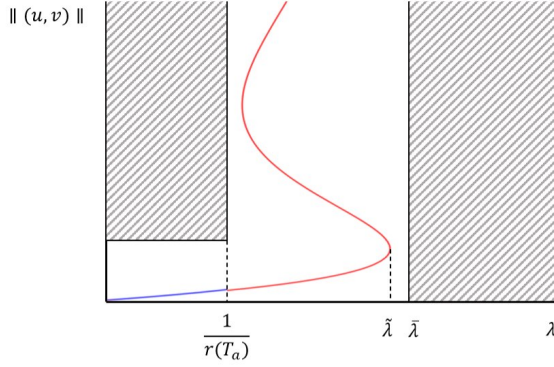
FIGURE 4. Solution continuum with uniqueness : $0 < f_{i,\infty} < 0$

Let $f_1(u,v) = \cos u + \ln(1 + v^2) + 2$, $f_2(u,v) = \ln(1 + u^2) + \arctan v + 1$, $g_1(t) = g_4(t) = t$, $g_2(t) = g_3(t) = 1$ and $a_1(t) = a_2(t) = 1$. Then $f_i(0,0) > 0$, $(H0)$, $(H1')$ and $(H2)$ hold with $\mathbf{a} = (1, 1, 1, 1)$. Let (φ, ψ) be an eigenfunction of $\mathbf{T_a}$. i.e., $\mathbf{T_a}(\varphi, \psi) = r(\mathbf{T_a})(\varphi, \psi)$. For $\mu = \frac{1}{r(T_\mathbf{a})}$, (φ, ψ) satisfies

$$\begin{cases} -\varphi''(t) = \mu\varphi(t) + \mu\psi(t), & t \in (0,1), \\ -\psi''(t) = \mu\varphi(t) + \mu\psi(t), & t \in (0,1), \\ \varphi(0) = 0 = \psi(0), \\ \varphi(1) = \int_0^1 s\varphi(s) + \psi(s)ds, \\ \psi(1) = \int_0^1 \varphi(s) + s\psi(s)ds. \end{cases}$$

By ordinary method, we have

$$\varphi(t) = \frac{c_1}{2}\sin\sqrt{2\mu}t + \frac{c_2}{2}t \text{ and}$$

$$\psi(t) = \frac{c_1}{2}\sin\sqrt{2\mu}t - \frac{c_2}{2}t.$$

From the boundary conditions,

$$\varphi(1) = \int_0^1 t(\frac{c_1}{2}\sin\sqrt{2\mu}t + \frac{c_2}{2}t) + (\frac{c_1}{2}\sin\sqrt{2\mu}t - \frac{c_2}{2}t)dt \text{ and}$$

$$\psi(1) = \int_0^1 (\frac{c_1}{2}\sin\sqrt{2\mu}t + \frac{c_2}{2}t) + t(\frac{c_1}{2}\sin\sqrt{2\mu}t - \frac{c_2}{2}t)dt.$$

From the value of $\varphi(1) + \psi(1)$, we have the equation

$$\sin\sqrt{2\mu} = -\frac{2}{\sqrt{2\mu}}\cos\sqrt{2\mu} + \frac{1}{2\mu}\sin\sqrt{2\mu} + \frac{1}{\sqrt{2\mu}}$$

and by using Matlab, $\mu \sim 1.0369$. By Corollary 4.4, (E_λ) has a positive solution for all $\lambda > 0$ and (E_λ) has a unique solution for $\lambda < \frac{1}{r(T_\mathbf{a})} = \mu \sim 1.0369$.

References

[1] J.R. Cannon, *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math. 21 (1963), no. 2, 155-160.

[2] R. Yu. Chegis, *Numerical solution of a heat conduction problem with an integral boundary condition*, Litovsk. Mat. Sb. 24 (1984), 209-215.

[3] X. Cheng and C. Zhong, *Existence of positive solutions for a second-order ordinary differential system*, J. Math. Anal. Appl. 312 (2005), no. 1, 14-23.

[4] Y. Cui and J. Sun, *On existence of positive solutions of coupled integral boundary vlaue problems for a nonlinear singular superlinear differential system*, Electron. J. Qual. Theory Differ. Equ. (2012), no. 41, 1-13.

[5] Y. Cui, W. Ma, X. Wang and X. Su, *Uniqueness theorem of differential system with coupled integral boundary conditions*, Electron. J. Qual. Theory Differ. Equ. (2018), no. 9, 1-10.

[6] M. Feng, D. Ji and W. Ge, *Positive solutions for a class of boundary-value problem with integral boundary conditions in Banach spaces*, J. Comput. Appl. Math. 222 (2008), no. 2, 351-363.

[7] J.M. Gallardo, *Second-order differential operators with integral boundary conditions and generation of analytic semigroups*, Rocky Mountain J. Math. 30 (2000) no. 4, 1265-1291.

[8] J. Henderson and R. Luca, *Positive solutions for systems of second-order integral boundary value problems*, Electron. J. Qual. Theory Differ. Equ. (2013), no. 70, 1-21.

[9] G. Infante and P. Pietramala, *Eigenvalues and nonnegative solutions of a system with nonlocal BCs*, Nonlinear Stud. 16 (2009), no.2, 187-196.

[10] N.I. Ionkin, *Solution of a boundary value problem in heat conduction theory with nonlocal boundary conditions*, Diff. Equ. 13 (1977), 294-304.

[11] M. A. Krasnosel'skii, *Positive solutions of operator equations*, P. Noordhoff, Groningen, The Netherlands, 1964.

[12] W.J. Song and W.J. Gao, *Positive solutions for a second-order system with integral boundary conditions*, Electron. J. Differential Equations, 2011 (2011), no.13, 1-9.

[13] Z.L. Yang, *Positive solutions to a system of second-order nonlocal boundary value problems*, Nonlinear Anal. 62 (2005), no. 7, 1251-1265.

[14] E. Zeidler, *Nonlinear Functional Analysis and its Applications I*, Springer -Verlag, New York, 1985.

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