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EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH A SINGULAR WEIGHT

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Abstract. In this work, we study the existence of a positive solution for nonlinear fractional differential equation with a singular weight. For the proof, we introduce newly defined solution operator and use well-known Krasnoselski's fixed point theorem. We also give an example with a singular weight which may not be integrable.

1. Introduction

In this paper, we study the existence and multiplicity of positive solutions to the following boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + h(t)f(u(t)) = 0, & t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{FDE}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative of order $\alpha \in (1,2]$, $f \in C([0,\infty),[0,\infty))$ and $h \in C([0,1),[0,\infty))$.

Throughout this paper, we assume the following hypotheses, unless otherwise stated.

(H_1) $h \in \mathcal{A}_\alpha$, where

$$\mathcal{A}_\alpha := \{k \in C([0,1),[0,\infty)) : \textstyle\int_0^1 (1-s)^{\alpha-1}k(s)ds < \infty\}.$$

(H_2) For some $n \in \mathbb{N}$, there exist $h_i \in \mathcal{B}_\alpha$ and $g_i \in C([0,\infty),[0,\infty))$ $(i = 1,2,\cdots n)$ such that

$$h(t)f(t^{\alpha-2}y) = \sum_{i=1}^{n} h_i(t)g_i(y) \text{ for } t \in (0,1) \text{ and } y \in [0,\infty),$$

where

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$$\mathcal{B}_\alpha := \Big\{ k \in C((0,1),[0,\infty)) : \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}k(s)ds < \infty \\ \text{and } k \not\equiv 0 \text{ on some compact subinterval of } \left[\tfrac{1}{4}, \tfrac{3}{4}\right] \Big\}.$$

Note that (H_2) implies the following assumption

$(H_2)'$ there exist $\underline{h}, \overline{h} \in \mathcal{B}_\alpha$ and $g \in C([0,\infty),[0,\infty))$ such that

$$\underline{h}(t)g(y) \le h(t)f(t^{\alpha-2}y) \le \overline{h}(t)g(y) \text{ for } t \in (0,1) \text{ and } y \in [0,\infty).$$

Indeed, if we assume that (H_2) is satisfied, then $(H_2)'$ is satisfied with

$$\underline{h} := \min_{1\le i\le n} h_i,\ \overline{h} := \max_{1\le i\le n} h_i \text{ and } g := \sum_{1\le i\le n} g_i.$$

Let

$$G(t,s) := \begin{cases} \dfrac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1 \end{cases} \tag{1}$$

be the Green's function for the boundary value problem $D^\alpha_{0+}u = 0$ and $u(0) = u(1) = 0$. Here Γ is the gamma function.

2. Preliminaries

In this section, we introduce some definitions of fractional calculus and some important lemmas, and a theorem that will be used later.

Definition 1. ([3]) For $\alpha > 0$, the integral

$$I^\alpha_{0+}v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\alpha}} d\tau,\ t > 0$$

is called the Riemann-Liouville fractional integral of order α.

Definition 2. ([3]) For $\alpha > 0$, the expression

$$D^\alpha_{0+}v(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{-\alpha+n-1} v(\tau)d\tau$$

is called the Riemann-Liouville fractional derivative of order α. Here $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of number of α.

By Lemma 2.2 in [3], we have the following lemma

Lemma 2.1. *Let $a > 0$ and $\alpha \in (1,2]$ be given. Assume that $v \in C(0,a) \cap L^1(0,a)$ and $D^\alpha_{0+}v \in C(0,a) \cap L^1(0,a)$. Then there exist $c_1, c_2 \in \mathbb{R}$ such that*

$$I^\alpha_{0+}D^\alpha_{0+}v(t) = v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \textit{ for } t \in (0,a).$$

Remark 1. It is well known that $G(t,s) \in C([0,1]\times[0,1])$ and

$$0 < G(t,s) \le \max_{0\le\tau\le1} G(\tau,s) = G(s,s) = \frac{1}{\Gamma(\alpha)} s^{\alpha-1}(1-s)^{\alpha-1} \text{ for } t,s \in (0,1).$$

For $k \in \mathcal{A}_\alpha$, consider the following equation

$$\begin{cases} D^\alpha_{0+}u(t) + k(t) = 0, \quad t \in (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{FDE_1}$$

Lemma 2.2. *Assume $k \in \mathcal{A}_\alpha$ with $\alpha \in (1,2]$. Then u is a solution to the problem (FDE_1) if and only if $u(t) = \int_0^1 G(t,s)k(s)ds$ for $t \in [0,1]$.*

Proof. Let $k \in \mathcal{A}_\alpha$ with $\alpha \in (1,2]$ be given. First we prove that the problem (FDE_1) has at most one solution. Assume that there exists u_1 and u_2 are solutions to the problem (FDE_1). Then $D^\alpha_{0+}(u_1(t) - u_2(t)) = 0$ for $t \in (0,1)$. By Lemma 2.1, there exist $c_1, c_2 \in \mathbb{R}$ such that $u_2(t) - u_1(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$ for $t \in (0,1)$. By the boundary conditions in (FDE_1), $c_1 = c_2 = 0$, and thus the problem (FDE_1) has at most one solution.

Now we prove that $u(t) = \int_0^1 G(t,s)k(s)ds$ is a solution of the problem (FDE_1). Let $u(t) = \int_0^1 G(t,s)k(s)ds$ for $t \in [0,1]$. Since $|G(t,s)k(s)| \le \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}k(s) \in L^1(0,1)$, by Lebesgue dominated convergence theorem, $u \in C[0,1]$ and $u(0) = u(1) = 0$. Next we show that u satisfies $D^\alpha_{0+}u(t) = -k(t)$ for $t \in (0,1)$. Note that $D^\alpha_{0+}u(t) = \frac{d^2}{dt^2} I^{2-\alpha}_{0+}u(t)$ for $t \in (0,1)$ and

$$I^{2-\alpha}_{0+}u(t) = \frac{1}{\Gamma(2-\alpha)} \left(\int_0^t (t-s)^{1-\alpha} u(s)ds \right) \quad \text{for } t \in (0,1).$$

Let $t \in (0,1)$ be given. Then

$$\begin{aligned}
&\int_0^t (t-s)^{1-\alpha} u(s)ds = \int_0^t (t-s)^{1-\alpha} \int_0^1 G(s,\tau)k(\tau)d\tau ds \\
= \; & \frac{1}{\Gamma(\alpha)} \Big(\int_0^t \int_0^s (t-s)^{1-\alpha} [(s(1-\tau))^{\alpha-1} - (s-\tau)^{\alpha-1}] k(\tau) d\tau ds \\
& + \int_0^t \int_s^1 (t-s)^{1-\alpha} (s(1-\tau))^{\alpha-1} k(\tau) d\tau ds \Big) \\
= \; & \frac{1}{\Gamma(\alpha)} \Big(\int_0^t \int_\tau^t (t-s)^{1-\alpha} [(s(1-\tau))^{\alpha-1} - (s-\tau)^{\alpha-1}] k(\tau) ds d\tau \\
& + \int_0^t \int_0^\tau (t-s)^{1-\alpha} (s(1-\tau))^{\alpha-1} k(\tau) ds d\tau \\
& + \int_t^1 \int_0^t (t-s)^{1-\alpha} (s(1-\tau))^{\alpha-1} k(\tau) ds d\tau \Big)
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)}\Big(\int_0^t\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}(1-\tau)^{\alpha-1}k(\tau)dsd\tau$$
$$-\int_0^t\int_\tau^t (t-s)^{1-\alpha}(s-\tau)^{\alpha-1}k(\tau)dsd\tau$$
$$+\int_t^1\int_0^t (t-s)^{1-\alpha}s^{\alpha-1}(1-\tau)^{\alpha-1}k(\tau)dsd\tau\Big)$$

$$= \frac{1}{\Gamma(\alpha)}\Big(\int_0^t\int_0^t (1-\frac{s}{t})^{1-\alpha}(\frac{s}{t})^{\alpha-1}(1-\tau)^{\alpha-1}k(\tau)dsd\tau$$
$$-\int_0^t\int_\tau^t (1-\frac{s-\tau}{t-\tau})^{1-\alpha}(\frac{s-\tau}{t-\tau})^{\alpha-1}k(\tau)dsd\tau$$
$$+\int_t^1\int_0^t (1-\frac{s}{t})^{1-\alpha}(\frac{s}{t})^{\alpha-1}(1-\tau)^{\alpha-1}k(\tau)dsd\tau\Big)$$
$$= \frac{1}{\Gamma(\alpha)}\Big(\int_0^t\int_0^1 (1-\theta)^{1-\alpha}\theta^{\alpha-1}(1-\tau)^{\alpha-1}k(\tau)td\theta d\tau$$
$$-\int_0^t\int_0^1 (1-\theta)^{1-\alpha}\theta^{\alpha-1}k(\tau)(t-\tau)d\theta d\tau$$
$$+\int_t^1\int_0^1 (1-\theta)^{1-\alpha}\theta^{\alpha-1}(1-\tau)^{\alpha-1}k(\tau)td\theta d\tau\Big).$$

Since $\int_0^1 \theta^{x_1-1}(1-\theta)^{x_2-1}d\theta = \frac{\Gamma(x_1)\Gamma(x_2)}{\Gamma(x_1+x_2)}$ for positive constants x_1, x_2,

$$\int_0^1 (1-\theta)^{1-\alpha}\theta^{\alpha-1}d\theta = \Gamma(2-\alpha)\Gamma(\alpha).$$

Thus we get

$$I_{0+}^{2-\alpha}u(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)ds$$
$$= \Big(t\int_0^t (1-\tau)^{\alpha-1}k(\tau)d\tau - \int_0^t (t-\tau)k(\tau)d\tau + t\int_t^1 (1-\tau)^{\alpha-1}k(\tau)d\tau\Big)$$
$$= \Big(t\int_0^1 (1-\tau)^{\alpha-1}k(\tau)d\tau - \int_0^t (t-\tau)k(\tau)d\tau\Big) \text{ for } t\in(0,1),$$

which implies

$$\begin{aligned} D_{0+}^{\alpha}u(t) &= \frac{d^2}{dt^2}I_{0+}^{2-\alpha}u(t) \\ &= \frac{d^2}{dt^2}\Big(t\int_0^1 (1-\tau)^{\alpha-1}k(\tau)d\tau - \int_0^t (t-\tau)k(\tau)d\tau\Big) \\ &= \frac{d}{dt}\left(\int_0^1 (1-\tau)^{\alpha-1}k(\tau)d\tau - \int_0^t k(\tau)d\tau\right) \\ &= -k(t) \text{ for } t\in(0,1), \end{aligned}$$

and this completes the proof. □

Lemma 2.3. ([1, Theorem 2]) *For $\alpha \in (1,2]$, the continuous function $G^*(t,s) := t^{2-\alpha}G(t,s)$ has the following properties:*

$$\frac{\alpha-1}{\Gamma(\alpha)}t(1-t)s(1-s)^{\alpha-1} \le G^*(t,s) \le \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1} \text{ for } t,s \in [0,1].$$

Let $\mathcal{K} := \{u \in C[0,1] : u(t) \ge (\alpha-1)t(1-t)\|u\|_\infty \text{ for } t \in [0,1]\}$. Here, $\|\cdot\|_\infty$ is the usual maximum norm in the Banach space $C[0,1]$, i.e., $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$ for $u \in C[0,1]$. Then $\mathcal{K}$ is a cone in $C[0,1]$.

Define $T : \mathcal{K} \to C[0,1]$ by, for $y \in \mathcal{K}$ and $t \in [0,1]$,

$$Ty(t) := \int_0^1 G^*(t,s)h(s)f(s^{\alpha-2}y(s))ds.$$

Then T is well defined. Indeed, for $y \in \mathcal{K}$ and $t \in [0,1]$, by Lemma 2.3 and $(H_2)'$, we get

$$0 \le G^*(t,s)h(s)f(s^{\alpha-2}y(s)) \le \frac{\|g(y)\|_\infty}{\Gamma(\alpha)}s^{\alpha-1}(1-s)^{\alpha-1}\overline{h}(s) \in L^1(0,1).$$

Thus, by Lebesgue Dominated Convergence Theorem, $Ty \in C[0,1]$ for all $y \in \mathcal{K}$ and T is well defined.

Lemma 2.4. *Assume that (H_2) is satisfied. Then $T : \mathcal{K} \to \mathcal{K}$ is completely continuous.*

Proof. First, we show that $T : \mathcal{K} \to \mathcal{K}$. Let $y \in \mathcal{K}$. By Lemma 2.3,

$$Ty(t) \ge \frac{\alpha-1}{\Gamma(\alpha)}t(1-t)\int_0^1 s(1-s)^{\alpha-1}h(s)f(s^{\alpha-2}y(s))ds \text{ for } t \in [0,1]$$

and

$$\|Ty\|_\infty \le \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}h(s)f(s^{\alpha-2}y(s))ds.$$

Consequently, $Ty(t) \ge (\alpha-1)t(1-t)\|Ty\|_\infty$ for $t \in [0,1]$ and $Ty \in \mathcal{K}$.

Next, we show that T is continuous. Let $y_n \to y$ in $\mathcal{K}$ as $n \to \infty$ and $\epsilon > 0$ be given. Then there exists $M > 0$ such that $\|y_n\|_\infty < M$ for all n and $\|y\|_\infty < M$. Since g_i is uniformly continuous on $[0,M]$ for $i = 1,2,\cdots,n$, there exists $\delta > 0$ such that if $z_1, z_2 \in [0,M]$ and $|z_1 - z_2| < \delta$, then $|g_i(z_1) - g_i(z_2)| < \frac{\Gamma(\alpha)\epsilon}{nC_i}$ for any $i = 1,2,\cdots,n$. Here, $C_i = \int_0^1 s(1-s)^{\alpha-1}h_i(s)ds > 0$ for $i = 1,2,\cdots,n$. Since $y_n \to y$ in $\mathcal{K}$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\|y_n - y\|_\infty < \delta$ for all $n \geq N$. By (H_2) and Lemma 2.3, for $n \geq N$ and $t \in [0,1]$,

$$\begin{aligned} |Ty_n(t) - Ty(t)| &\leq \int_0^1 G^*(t,s)h(s)|f(s^{\alpha-2}y_n(s)) - f(s^{\alpha-2}y(s))|ds \\ &= \int_0^1 G^*(t,s)\sum_{i=1}^n h_i(s)|g_i(y_n(s)) - g_i(y(s))|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^n \int_0^1 s(1-s)^{\alpha-1}h_i(s)|g_i(y_n(s)) - g_i(y(s))|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^n C_i \frac{\Gamma(\alpha)\epsilon}{nC_i} = \epsilon. \end{aligned}$$

Consequently, $Ty_n \to Ty$ in $\mathcal{K}$ as $n \to \infty$.

Finally, we show that T is compact. Let D be a bounded set in $\mathcal{K}$, i.e., there exists $L > 0$ such that $\|y\|_\infty \leq L$ for any $y \in D$. For $y \in D$ and $t \in [0,1]$, by (H_2) and Lemma 2.3,

$$\begin{aligned} |Ty(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}\sum_{i=1}^n h_i(s)g_i(y(s)) \\ &\leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^n M_i \int_0^1 s(1-s)^{\alpha-1}h_i(s)ds < \infty, \end{aligned}$$

where $M_i := \max_{0\leq z\leq L} |g_i(z)|$ for $i = 1, 2, \cdots, n$. Thus $T(D)$ is bounded.

Let $t_1, t_2 \in [0,1]$ be given. By Lemma 2.3, for $s \in [0,1]$,

$$\begin{aligned} G^*(t_2,s) - G^*(t_1,s) &\leq \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1} - \frac{\alpha-1}{\Gamma(\alpha)}t_1(1-t_1)s(1-s)^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-1}(1-(\alpha-1)t_1(1-t_1)) \\ &\leq \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leq \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

Similarly, it can be shown that $G^*(t_1,s) - G^*(t_2,s) \leq \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $s \in [0,1]$. Thus,

$$|G^*(t_1,s) - G^*(t_2,s)| \leq \frac{1}{\Gamma(\alpha)}s^{\alpha-1}(1-s)^{\alpha-1} \text{ for } s \in [0,1]. \tag{2}$$

By (H_2), there exists $\delta_1 > 0$ such that

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^n M_i \int_0^{\delta_1} s^{\alpha-1}(1-s)^{\alpha-1}h_i(s)ds < \frac{\epsilon}{3} \tag{3}$$

and

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^n M_i \int_{1-\delta_1}^1 s^{\alpha-1}(1-s)^{\alpha-1}h_i(s)ds < \frac{\epsilon}{3}. \tag{4}$$

Since $G^*(t,s)$ is uniformly continuous, there exists $\delta > 0$ such that

$$\text{if } |t_2 - t_1| < \delta, \text{ then } |G^*(t_1,s) - G^*(t_2,s)| < \frac{\epsilon}{3\sum_{i=1}^n M_i \hat{C}_i} \text{ for all } s \in [0,1]. \quad (5)$$

Here $\hat{C}_i = \max_{\delta_1 \le s \le 1-\delta_1} h_i(s)$ for $i = 1, 2, \cdots, n$. Let t_1 and t_2 be given with $|t_1 - t_2| < \delta$ and $y \in D$. Then, by (2), (3), (4) and (5),

$$\begin{aligned}
|Ty(t_1) - Ty(t_2)| &= \left| \int_0^1 (G^*(t_1,s) - G^*(t_2,s))h(s)f(s^{\alpha-2}y(s))ds \right| \\
&\le \int_0^1 |G^*(t_1,s) - G^*(t_2,s)| \sum_{i=1}^n h_i(s)g_i(y(s))ds \\
&\le \sum_{i=1}^n M_i \int_0^{\delta_1} |G^*(t_1,s) - G^*(t_2,s)|h_i(s)ds \\
&\quad + \sum_{i=1}^n M_i \int_{1-\delta_1}^1 |G^*(t_1,s) - G^*(t_2,s)|h_i(s)ds \\
&\quad + \sum_{i=1}^n M_i\hat{C}_i \int_{\delta_1}^{1-\delta_1} |G^*(t_1,s) - G^*(t_2,s)|ds \\
&\le \frac{1}{\Gamma(\alpha)} \sum_{i=1}^n M_i \int_0^{\delta_1} s^{\alpha-1}(1-s)^{\alpha-1}h_i(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^n M_i \int_{1-\delta_1}^1 s^{\alpha-1}(1-s)^{\alpha-1}h_i(s)ds \\
&\quad + \sum_{i=1}^n M_i\hat{C}_i \frac{\epsilon}{3\sum_{i=1}^n M_i C_i} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

which implies TD is equicontinuous. Thus, by Arzelà-Ascoli theorem, T is compact. □

Lemma 2.5. *Assume that (H_1) and (H_2) are satisfied. If y is a fixed point of T, then u is a solution to the problem (FDE). Here $u(t) := t^{\alpha-2}y(t)$ for $t \in (0,1]$ and $u(0) := 0$.*

Proof. Let y be a fixed point of T. Then

$$y(t) = Ty(t) = t^{2-\alpha} \int_0^1 G(t,s)h(s)f(s^{\alpha-2}y(s))ds \text{ for } t \in [0,1].$$

For $\alpha = 2$, by Lemma 2.2, the proof is clear. Let $\alpha \in (1,2)$ be given and let

$$u(t) := \begin{cases} t^{\alpha-2}y(t), & \text{for } t \in (0,1], \\ 0, & \text{for } t = 0. \end{cases}$$

Then $u(t) = t^{\alpha-2}y(t) = \int_0^1 G(t,s)h(s)f(u(s))ds$ for $t \in (0,1]$. From the facts that $u(0) = 0$ and $G(0,s) = 0$ for all $s \in [0,1]$, it follows that

$$u(t) = \int_0^1 G(t,s)h(s)f(u(s))ds \text{ for } t \in [0,1].$$

By $(H_2)'$ and Lebesgue's dominated convergence theorem,

$$\begin{aligned} 0 \le \lim_{t\to 0^+} u(t) &= \lim_{t\to 0^+} \int_0^1 G(t,s)h(s)f(s^{\alpha-2}y(s))ds \\ &\le \|g(y)\|_\infty \lim_{t\to 0^+} \int_0^1 G(t,s)\overline{h}(s)ds = 0, \end{aligned}$$

which implies that $u \in C[0,1]$. Since $h(\cdot)f(u(\cdot)) \in \mathcal{A}_\alpha$, by Lemma 2.2, we can conclude that u is a solution to the problem (FDE). $\square$

Theorem 2.6. ([2]) *Let E be a Banach space and let K be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Assume that $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ is completely continuous such that either*

(1) $\|Tu\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*
(2) $\|Tu\| \ge \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main results

For convenience, we introduce the following notations

$$g_0 = \lim_{s\to 0} \frac{g(s)}{s} \text{ and } g_\infty = \lim_{s\to\infty} \frac{g(s)}{s},$$

where g is the function in the assumption $(H_2)'$.

Theorem 3.1. *Assume that (H_1) and (H_2) are satisfied.*

(1) *If $g_0 = 0$ and $g_\infty = \infty$, then the problem (FDE) has a positive solution.*
(2) *If $g_0 = \infty$ and $g_\infty = 0$, then the problem (FDE) has a positive solution.*

Proof. (1) Let $\epsilon := \Gamma(\alpha)\left(\int_0^1 s(1-s)^{\alpha-1}\overline{h}(s)ds\right)^{-1} > 0$. By $g_0 = 0$, we may choose $r > 0$ satisfying

$$g(z) \le \varepsilon z \text{ for all } z \in [0,r]. \tag{6}$$

Let $B_r = \{y \in C[0,1] : \|y\|_\infty < r\}$. From Lemma 2.3 and (6), it follows that for $y \in \mathcal{K} \cap \partial B_r$ and $t \in [0,1]$,

$$\begin{aligned} Ty(t) &= \int_0^1 G^*(t,s)h(s)f(s^{\alpha-2}y(s))ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}\overline{h}(s)g(y(s))ds \\ &\le \frac{\epsilon}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}\overline{h}(s)ds\|y\|_\infty. \end{aligned}$$

Therefore, by the choice of ϵ, $\|Ty\|_\infty \le \|y\|_\infty$ for all $y \in \mathcal{K} \cap \partial B_r$.

Let

$$\rho := \frac{64\Gamma(\alpha)}{(\alpha-1)^2}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}\underline{h}(s)ds\right)^{-1}.$$

Since $g_\infty = \infty$, there exists $M > 0$ such that $g(v) \geq \rho v$ for $v \geq M$. Take $R > \max\{\frac{16}{\alpha-1}M, r\}$ and $B_R = \{y \in C[0,1] : \|y\|_\infty < R\}$. For $y \in \mathcal{K} \cap \partial B_R$,

$$y(t) \geq \frac{\alpha-1}{16}\|y\|_\infty > M \text{ for } t \in [\tfrac{1}{4}, \tfrac{3}{4}].$$

Consequently,

$$g(y(t)) \geq \rho y(t) \text{ for } y \in \mathcal{K} \cap \partial B_R \text{ and } t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{7}$$

By Lemma 2.3 and (7),

$$\begin{aligned} \|Ty\|_\infty \geq Ty\left(\frac{1}{2}\right) &= \int_0^1 G^*\left(\frac{1}{2}, s\right) h(s) f(s^{\alpha-2}y(s))ds \\ &\geq \frac{\alpha-1}{4\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}\underline{h}(s)g(y(s))ds \\ &\geq \frac{\rho(\alpha-1)}{4\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}\underline{h}(s)y(s)ds \\ &\geq \frac{\rho(\alpha-1)^2}{64\Gamma(\alpha)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\alpha-1}\underline{h}(s)ds\|y\|_\infty. \end{aligned}$$

Therefore, by the choice of ρ, $\|Ty\|_\infty \geq \|y\|_\infty$ for all $y \in \mathcal{K} \cap \partial B_R$. By Theorem 2.6, T has a fixed point y in $\mathcal{K} \cap (\overline{B}_R \setminus B_r)$. Consequently, the problem (FDE) has a positive solution u in view of Lemma 2.5.

(2) Let

$$L = \frac{16}{\alpha-1}\left(\int_{\frac{1}{4}}^{\frac{3}{4}} G^*\left(\frac{1}{2}, s\right)\underline{h}(s)ds\right)^{-1}.$$

By $g_0 = \infty$, we may choose r_1 so that $g(z) \geq Lz$ for $0 < z \leq r_1$. Let $B_{r_1} = \{y \in C[0,1] : ||y||_\infty < r_1\}$. For $y \in \mathcal{K} \cap \partial B_{r_1}$,

$$\begin{aligned} ||Ty||_\infty \geq Ty\left(\frac{1}{2}\right) &= \int_0^1 G^*\left(\frac{1}{2}, s\right) h(s) f(s^{\alpha-2} y(s)) ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G^*\left(\frac{1}{2}, s\right) \underline{h}(s) g(y(s)) ds \\ &\geq L \int_{\frac{1}{4}}^{\frac{3}{4}} G^*\left(\frac{1}{2}, s\right) \underline{h}(s) y(s) ds \\ &\geq \frac{L(\alpha-1)}{16} \int_{\frac{1}{4}}^{\frac{3}{4}} G^*\left(\frac{1}{2}, s\right) \underline{h}(s) ds ||y||_\infty. \end{aligned}$$

Therefore, by the choice of L, $\|Ty\|_\infty \geq \|y\|_\infty$ for all $y \in \mathcal{K} \cap \partial B_{r_1}$.

Let $\zeta > 0$ be a constant satisfying

$$\Gamma(\alpha) - \zeta \int_0^1 s(1-s)^{\alpha-1} \overline{h}(s) ds > 0.$$

Since $g_\infty = 0$, there exists $L_1 > 0$ such that $g(z) \leq \zeta z$ for $z > L_1$. Choose $R_1 (> r_1)$ satisfying

$$R_1 > \max\left\{ L_1, \frac{\max_{0 \leq z \leq L_1} |g(z)| \int_0^1 s(1-s)^{\alpha-1} \overline{h}(s) ds}{\Gamma(\alpha) - \zeta \int_0^1 s(1-s)^{\alpha-1} \overline{h}(s) ds} \right\}$$

and let $B_{R_1} = \{y \in C[0,1] : ||y||_\infty < R_1\}$. By Lemma 2.2 and $(H_2)'$, for $y \in \mathcal{K} \cap \partial B_{R_1}$ and $t \in [0,1]$,

$$\begin{aligned} Ty(t) &= \int_0^1 G^*(t,s) h(s) f(s^{\alpha-2} y(s)) ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} \overline{h}(s) g(y(s)) ds \\ &= \frac{1}{\Gamma(\alpha)} \Big[\int_{A_1} s(1-s)^{\alpha-1} \overline{h}(s) g(y(s)) ds + \int_{A_2} s(1-s)^{\alpha-1} \overline{h}(s) g(y(s)) ds \Big] \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\max_{0 \leq z \leq L_1} |g(z)| \int_{A_1} s(1-s)^{\alpha-1} \overline{h}(s) ds + \zeta \int_{A_2} s(1-s)^{\alpha-1} \overline{h}(s) y(s) ds \right] \\ &\leq \frac{1}{\Gamma(\alpha)} \Big(\max_{0 \leq z \leq L_1} |g(z)| + \zeta R_1 \Big) \int_0^1 s(1-s)^{\alpha-1} \overline{h}(s) ds \\ &\leq R_1 = ||y||_\infty. \end{aligned}$$

Here $A_1 = \{s : 0 \leq y(s) \leq L_1\}$ and $A_2 = \{s : L_1 \leq y(s) \leq R_1\}$. Thus $||Ty||_\infty \leq ||y||_\infty$ for $y \in \mathcal{K} \cap \partial B_{R_1}$. By Theorem 2.6, T has a fixed point y in

$\mathcal{K} \cap (\overline{B}_{R_1} \setminus B_{r_1})$. Consequently, the problem (FDE) has a positive solution u in view of Lemma 2.5.

□

Example 3.2. Let $\alpha \in (1, 2]$ be given and $h(t) = (1-t)^{-q}$ with $q < \alpha$. Then $h \in \mathcal{A}_\alpha$, i.e., (H_1) is satisfied. Let $f(u) = u^a$ with $a \in (0, \alpha^*)$. Here, $\alpha^* = \frac{\alpha}{2-\alpha}$ for $\alpha \in (1, 2)$ and $\alpha^* = \infty$ for $\alpha = 2$. Then $h(t)f(t^{\alpha-2}y) = t^{a(\alpha-2)}(1-t)^{-q}y^a$ and (H_2) is satisfied with $n = 1$, $h_1(t) = t^{a(\alpha-2)}(1-t)^{-q} \in \mathcal{B}_\alpha$ and $g_1(y) = y^a$.
(i) If $a \in (0, 1)$, then $(g_1)_0 = \infty$ and $(g_1)_\infty = 0$.
(ii) If $a \in (1, \alpha^*)$, then $(g_1)_0 = 0$ and $(g_1)_\infty = \infty$.

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