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ARTIN SYMBOLS OVER IMAGINARY QUADRATIC FIELDS

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Abstract. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ and N be a positive integer. By $K_{(N)}$ we mean the ray class field of K modulo $N\mathcal{O}_K$. In this paper, for each prime $\mathfrak{p}$ of K relatively prime to $N\mathcal{O}_K$ we explicitly describe the action of the Artin symbol $(\frac{K_{(N)}/K}{\mathfrak{p}})$ on special values of modular functions of level N. Furthermore, we extend the Kronecker congruence relation for the elliptic modular function j to some modular functions of higher level.

1. Introduction

Let L/K be a Galois extension of number fields. Let $\mathcal{O}_K$ be the ring of algebraic integers in K and $\mathfrak{p}$ be a prime of K (i.e., a nontrivial prime ideal of $\mathcal{O}_K$) which is unramified in L. For a prime $\mathfrak{P}$ of L lying above $\mathfrak{p}$, its decomposition group is defined by

$$D_{\mathfrak{P}}\,(=D_{\mathfrak{P}/K}) = \{\sigma \in \mathrm{Gal}(L/K) \mid \mathfrak{P}^{\sigma} = \mathfrak{P}\}.$$

Then, $D_{\mathfrak{P}}$ is isomorphic to the Galois group of residue fields, that is,

$$D_{\mathfrak{P}} \simeq \widetilde{G} = \mathrm{Gal}((\mathcal{O}_L/\mathfrak{P})/(\mathcal{O}_K/\mathfrak{p})).$$

Thus there is a unique element $\sigma \in D_{\mathfrak{P}}$ which maps to the Frobenius automorphism of $\widetilde{G}$, and so σ satisfies

$$\nu^{\sigma} \equiv \nu^{\mathrm{N}(\mathfrak{p})} \pmod{\mathfrak{P}} \quad \text{for all } \nu \in \mathcal{O}_L$$

where $\mathrm{N}(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ is the norm of $\mathfrak{p}$ ([3, Lemma 5.19]). This unique element σ is called the *Artin symbol* and is denoted by $\left(\frac{L/K}{\mathfrak{P}}\right)$. In particular, if L/K is an abelian extension, then the Artin symbol depends only on $\mathfrak{p}$ and hence it can be written as $\left(\frac{L/K}{\mathfrak{p}}\right)$. Some concrete examples of Artin symbols for $K = \mathbb{Q}$ can be found in [4, §9.1].

In what follows, we let K be an imaginary quadratic field of discriminant d_K and N be a positive integer. Let $\mathcal{C}(N\mathcal{O}_K)$ denote the ray class group of K

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modulo $N\mathcal{O}_K$, namely, $\mathcal{C}(N\mathcal{O}_K) = I(N\mathcal{O}_K)/P_1(N\mathcal{O}_K)$ where $I(N\mathcal{O}_K)$ is the group of fractional ideals of K relatively prime to $N\mathcal{O}_K$ and $P_1(N\mathcal{O}_K)$ is its subgroup defined by

$$P_1(N\mathcal{O}_K) = \langle \nu\mathcal{O}_K \mid \nu \in \mathcal{O}_K \smallsetminus \{0\} \text{ and } \nu \equiv 1 \pmod{N\mathcal{O}_K}\rangle.$$

Let $K_{(N)}$ be the ray class field of K modulo $N\mathcal{O}_K$ so that all primes of K ramified in $K_{(N)}$ divide $N\mathcal{O}_K$ and the Artin map

$$\left(\frac{K_{(N)}/K}{\cdot}\right) : I(N\mathcal{O}_K) \to \mathrm{Gal}(K_{(N)}/K)$$

induces an isomorphism $\mathcal{C}(N\mathcal{O}_K) \xrightarrow{\sim} \mathrm{Gal}(K_{(N)}/K)$. In particular, the Hilbert class field $H_K = K_{(1)}$ is the maximal unramified abelian extension of K. One may refer to [3, §8] or [8, Chapter V] for class field theory.

For a lattice Λ in $\mathbb{C}$, let $j(\Lambda)$ be the j-invariant of any elliptic curve over $\mathbb{C}$ isomorphic to $\mathbb{C}/\Lambda$. Let $\mathfrak{a}$ be a nontrivial ideal of $\mathcal{O}_K$. By the theory of complex multiplication, Hasse ([7]) proved that for all but a finite number of primes $\mathfrak{p}$ of K satisfying $\mathfrak{p} \neq \overline{\mathfrak{p}}$

$$j(\mathfrak{p}^{-1}\mathfrak{a}) \equiv j(\mathfrak{a})^p \pmod{\mathfrak{P}} \quad \text{with } p = \mathrm{N}(\mathfrak{p}) \tag{1}$$

for any prime $\mathfrak{P}$ of H_K lying above $\mathfrak{p}$. This congruence is called the *Kronecker congruence relation.* We also notice that there is an analog of (1) for the Weber function. For a positive integer m, let $\Phi_m(x, y) \in \mathbb{Z}[x, y]$ be the modular polynomial for which $\Phi_m(j(\tau), j(m\tau)) = 0$. Here, j stands for the elliptic modular function defined on the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Prior to the work of Hasse, Weber ([15]) had derived a weaker form of (1) in such a way that for each prime p

$$\Phi_p(x, y) \equiv (x^p - y)(x - y^p) \pmod{p\mathbb{Z}[x, y]}. \tag{2}$$

See also [2] for a generalization of (2) to certain Hauptmoduln including j.

In this paper, we shall deal with the following three topics related to Artin symbols.

(i) For a prime $\mathfrak{p}$ of K which is relatively prime to $N\mathcal{O}_K$, we shall explicitly describe the Artin symbol $\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)$ by utilizing the extended form class group of level N which was developed by Eum, Koo and Shin in [5] (Theorem 4.2).

(ii) From the description of $\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)$ we shall obtain a certain extension of the Kronecker congruence relation (1) when $\mathfrak{a} = \mathcal{O}_K$ to meromorphic modular functions of level N (Corollaries 5.1 and 5.3).

(iii) For a prime $\mathfrak{P}$ of $K_{(N)}$ such that $\mathfrak{P} \cap \mathbb{Q}$ is unramified in $K_{(N)}$, we shall investigate $\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)$ or $D_{\mathfrak{P}/\mathbb{Q}}$. (Theorem 6.2).

2. Theory of complex multiplication

In this section, we shall review some necessary facts of the theory of complex multiplication.

Proposition 2.1. *Let* $\mathfrak{a}$ *be a nontrivial ideal of* $\mathcal{O}_K$*. Then,* $j(\mathfrak{a})$ *is an algebraic integer which generates* H_K *over* K*.*

Proof. See [11, Theorem 4 in Chapter 5 and Theorem 1 in Chapter 10]. □

The idelic formalization of the theory of complex multiplication owing to Shimura and A. Robert yields the following result.

Proposition 2.2. *Let* $\mathfrak{a}$ *be a nontrivial ideal of* $\mathcal{O}_K$*. For any nontrivial ideal* $\mathfrak{b}$ *of* $\mathcal{O}_K$*, we have*

$$j(\mathfrak{a})^{\left(\frac{H_K/K}{\mathfrak{b}}\right)} = j(\mathfrak{b}^{-1}\mathfrak{a}).$$

Proof. See [11, Theorem 5 in Chapter 10] or [13, Theorem 5.7]. □

Remark 1. We observe by Proposition 2.2 that the Kronecker congruence relation (1) holds for every prime $\mathfrak{p}$ of K such that $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Furthermore, we may let $\mathfrak{a} = \mathcal{O}_K$ in (1) because the action of $\mathrm{Gal}(H_K/K)$ transitively permutes primes $\mathfrak{P}$ of H_K lying above $\mathfrak{p}$ ([8, Theorem 6.8 in Chapter I]).

For a prime p we mean by $\left(\frac{d_K}{p}\right)$ the Kronecker symbol. For $\nu_1, \nu_2 \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$, we shall denote by $[\nu_1, \nu_2]$ the lattice generated by ν_1 and ν_2, namely, $[\nu_1, \nu_2] = \mathbb{Z}\nu_1 + \mathbb{Z}\nu_2$.

Lemma 2.3. *Let* p *be a prime.*

(i) p *splits completely in* K *if and only if* $\left(\frac{d_K}{p}\right) = 1$*. In this case, there is an integer* u *such* $u^2 \equiv d_K \pmod{4p}$*. Furthermore,* $\mathfrak{p} = \left[\frac{-u+\sqrt{d_K}}{2}, p\right]$ *is a prime of* K *such that* $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$*.*

(ii) p *is inert in* K *if and only if* $\left(\frac{d_K}{p}\right) = -1$*.*

(iii) p *is ramified in* K *if and only if* $\left(\frac{d_K}{p}\right) = 0$ *(i.e.,* $p \mid d_K$*). In this case,*

$$\mathfrak{p} = \begin{cases} \left[\frac{2+\sqrt{d_K}}{2}, 2\right] & \text{if } p = 2 \text{ and } \frac{d_K}{4} \equiv 3 \pmod{4}, \\ \left[\frac{-d_K+\sqrt{d_K}}{2}, p\right] & \text{otherwise} \end{cases}$$

is a prime of K *satisfying* $p\mathcal{O}_K = \mathfrak{p}^2$*.*

Proof. See [1, Theorems 3 and 4 in §9.5]. □

Let τ_K be the element of $\mathbb{H}$ defined by

$$\tau_K = \begin{cases} \frac{-1+\sqrt{d_K}}{2} & \text{if } d_K \equiv 1 \pmod{4}, \\ \frac{\sqrt{d_K}}{2} & \text{if } d_K \equiv 0 \pmod{4}, \end{cases}$$

and so $\mathcal{O}_K = [\tau_K, 1]$ ([3, Lemma 7.2]). For a prime $\mathfrak{p}$ of K, the Artin symbol $\left(\frac{H_K/K}{\mathfrak{p}}\right)$ can be expressed in more detail as follows.

Proposition 2.4. *Let p be a prime and $\mathfrak{p}$ be a prime of K lying above p.*

(i) *If p splits completely in K and so $\mathfrak{p} = [\frac{-u+\sqrt{d_K}}{2}, p]$ for some integer u such that $u^2 \equiv d_K \pmod{4p}$ by Lemma 2.3 (i), then we have*

$$j(\tau_K)^{\left(\frac{H_K/K}{\mathfrak{p}}\right)} = j\left(\frac{u+\sqrt{d_K}}{2p}\right).$$

(ii) *If p is inert in K, then $\left(\frac{H_K/K}{\mathfrak{p}}\right)$ is the identity map on H_K.*

(iii) *If p is ramified in K, then we get that*

$$j(\tau_K)^{\left(\frac{H_K/K}{\mathfrak{p}}\right)} = \begin{cases} j\left(\frac{-2+\sqrt{d_K}}{4}\right) & \text{if } p = 2 \text{ and } \frac{d_K}{4} \equiv 3 \pmod 4, \\ j\left(\frac{d_K+\sqrt{d_K}}{2p}\right) & \text{otherwise.} \end{cases}$$

Proof. Note that if Λ and Λ' are homothetic lattices in $\mathbb{C}$, then $j(\Lambda) = j(\Lambda')$ ([3, Theorem 10.9]). By Proposition 2.1, $j(\mathcal{O}_K) = j(\tau_K)$ generates H_K over K.

(i) Since $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, we obtain by Proposition 2.2 that

$$j(\mathcal{O}_K)^{\left(\frac{H_K/K}{\mathfrak{p}}\right)} = j(\mathfrak{p}^{-1}) = j(p^{-1}\overline{\mathfrak{p}}) = j\left(\frac{u+\sqrt{d_K}}{2p}\right).$$

(ii) Observe by Proposition 2.2 and the fact $\mathfrak{p} = p\mathcal{O}_K$ that

$$j(\mathcal{O}_K)^{\left(\frac{H_K/K}{\mathfrak{p}}\right)} = j(\mathfrak{p}^{-1}) = j(p^{-1}\mathcal{O}_K) = j(\mathcal{O}_K).$$

This implies that $\left(\frac{H_K/K}{\mathfrak{p}}\right)$ is the identity map on H_K.

(iii) Since $p\mathcal{O}_K = \mathfrak{p}^2$, we derive by Proposition 2.2 that

$$j(\mathcal{O}_K)^{\left(\frac{H_K/K}{\mathfrak{p}}\right)} = j(\mathfrak{p}^{-1}) = j(p^{-1}\mathfrak{p}) = j(\mathfrak{p}) = j(\overline{\mathfrak{p}}).$$

Then the result follows from Lemma 2.3 (iii).

□

Let $\mathcal{F}_N$ be the field of meromorphic modular functions for the principal congruence subgroup

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I_2 \pmod{NM_2(\mathbb{Z})}\}$$

whose Fourier expansions with respect to $q_\tau^{\frac{1}{N}}$ $(q_\tau = e^{2\pi \mathrm{i}\tau})$ have coefficients in the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$ where $\zeta_N = e^{\frac{2\pi \mathrm{i}}{N}}$. As is well known, $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ whose Galois group is isomorphic to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I_2\rangle$.

Proposition 2.5. *Let $\alpha \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I_2\rangle$ and $f \in \mathcal{F}_N$ with Fourier expansion $f = \sum_{n\gg-\infty} c_n q_\tau^{\frac{n}{N}}$ $(c_n \in \mathbb{Q}(\zeta_N))$.*

(i) *If $\alpha \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I_2\rangle$, then $f^\alpha = f \circ \widetilde{\alpha}$ where $\widetilde{\alpha}$ is any preimage of α via the reduction $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I_2\rangle$.*

(ii) *If* $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ *for some integer* d *relatively prime to* N*, then* $f^\alpha = \sum_{n \gg -\infty} c_n^{\sigma_d} q_\tau^{\frac{n}{N}}$ *where* σ_d *is the automorphism of* $\mathbb{Q}(\zeta_N)$ *defined by* $\zeta_N \mapsto \zeta_N^d$.

Proof. See [11, Theorem 3 in Chapter 6] and [13, Proposition 6.9 (1)]. □

Essentially due to Hasse ([7]) we get the following proposition.

Proposition 2.6. *We have* $K_{(N)} = K\left(f(\tau_K) \mid f \in \mathcal{F}_N \text{ is finite at } \tau_K\right)$.

Proof. See [11, Corollary to Theorem 2 in Chapter 10]. □

3. The extended form class group of level N

We shall introduce the extended form class group of level N and its action on special values of modular functions of level N.

Let $\mathcal{Q}_N(d_K)$ be the set of binary quadratic forms given by

$$\mathcal{Q}_N(d_K) = \left\{ Q(x, y) = Q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y] \;\middle|\; \begin{array}{l} b^2 - 4ac = d_K,\ a > 0, \\ \gcd(a, b, c) = \gcd(a, N) = 1 \end{array} \right\}.$$

The congruence subgroup

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod{NM_2(\mathbb{Z})} \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$ acts on the set $\mathcal{Q}_N(d_K)$ from the right so as to have

$$Q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)^\gamma = Q\left(\gamma \begin{bmatrix} x \\ y \end{bmatrix}\right) \quad (Q \in \mathcal{Q}_N(d_K),\ \gamma \in \Gamma_1(N)).$$

This action of $\Gamma_1(N)$ naturally defines the equivalence relation $\sim_N$ on $\mathcal{Q}_N(d_K)$ as follows: for $Q, Q' \in \mathcal{Q}_N(d_K)$

$$Q \sim_N Q' \quad \Longleftrightarrow \quad Q' = Q^\gamma \text{ for some } \gamma \in \Gamma_1(N).$$

Denote by $\mathcal{C}_N(d_K)$ the set of equivalence classes, that is, $\mathcal{C}_N(d_K) = \mathcal{Q}_N(d_K)/\sim_N$. For each $Q = ax^2 + bxy + cy^2 \in \mathcal{Q}_N(d_K)$, let ω_Q be the zero of the quadratic polynomial $Q(x, 1)$ lying in $\mathbb{H}$ so that

$$\omega_Q = \frac{-b + \sqrt{d_K}}{2a}.$$

Proposition 3.1. *One can equip the set* $\mathcal{C}_N(d_K)$ *with the group structure in such a way that the mapping*

$$\begin{array}{ccc} \mathcal{C}_N(d_K) & \to & \mathcal{C}(N\mathcal{O}_K) \\ {[Q]} & \mapsto & [[\omega_Q, 1]] \end{array}$$

becomes a well-defined isomorphism.

Proof. See [5, Theorem 2.9 and Remark 2.10 (iv)]. □

Remark 2. (i) We shall call the group $\mathcal{C}_N(d_K)$ the *extended form class group* of discriminant d_K and level N.

(ii) For a negative integer D such that $D \equiv 0$ or $1 \pmod 4$, let $\mathcal{Q}(D)$ be the set of primitive binary quadratic forms over $\mathbb{Z}$ of discriminant D. Then, the modular group $\mathrm{SL}_2(\mathbb{Z})$ gives rise to the proper equivalence ~ on the set $\mathcal{Q}(D)$. Gauss' direct composition (or, the Dirichlet composition) makes the set of equivalence classes $\mathcal{C}(D) = \mathcal{Q}(D)/\sim$ an abelian group, which is now called the form class group ([6]). Moreover, if $\mathcal{O}$ is the order of discriminant D in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$, then $\mathcal{C}(D)$ is isomorphic to the $\mathcal{O}$-ideal class group ([3, Theorem 7.7]). Jung et al. quite recently generalized $\mathcal{C}(D)$ and constructed a form class group isomorphic to the ray class group of $\mathcal{O}$ modulo $N\mathcal{O}$ ([10, Definition 5.7 and Theorem 9.4]).

Let $\min(\tau_K, \mathbb{Q}) = x^2 + b_K x + c_K$ $(\in \mathbb{Z}[x])$. By virtue of Shimura's reciprocity law ([13, Theorem 6.31]), we get an extension of Proposition 2.2 over the Hilbert class field H_K to something over the ray class field $K_{(N)}$.

Proposition 3.2. *Let* $\mathfrak{a} \in I(N\mathcal{O}_K)$. *By* Proposition 3.1, *there exists a quadratic form* $Q = ax^2+bxy+cy^2 \in \mathcal{Q}_N(d_K)$ *so that* $[\mathfrak{a}] = [[\omega_Q, 1]]$ *in* $\mathcal{C}(N\mathcal{O}_K)$. *If* $f \in \mathcal{F}_N$ *is finite at* τ_K, *then we establish*

$$f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{a}}\right)} = f^{\begin{bmatrix} 1 & -a'\left(\frac{b+b_K}{2}\right) \\ 0 & a' \end{bmatrix}}(-\overline{\omega}_Q)$$

where a' *is an integer satisfying* $aa' \equiv 1 \pmod N$.

Proof. See Proposition 2.6 and [16, Theorem 3.5]. □

Remark 3. Note that b and b_K have the same parity for $Q = ax^2 + bxy + cy^2 \in \mathcal{Q}_N(d_K)$ because $b^2 - 4ac = d_K = b_K^2 - 4c_K$.

4. Description of Artin symbols

In this section, for a prime $\mathfrak{p}$ of K we shall describe the Arin symbol $\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)$ in a concrete way as some generalization of Proposition 2.4. Due to Stevenhagen we get the following explicit version of Shimura's reciprocity law ([13, Theorem 6.31]).

Lemma 4.1. *Let* s *and* t *be integers not both zero such that* $(s\tau_K + t)\mathcal{O}_K$ *is relatively prime to* $N\mathcal{O}_K$. *If* $f \in \mathcal{F}_N$ *is finite at* τ_K, *then we have*

$$f(\tau_K)^{\left(\frac{K_{(N)}/K}{(s\tau_K+t)\mathcal{O}_K}\right)} = f^{\begin{bmatrix} t-b_K s & -c_K s \\ s & t \end{bmatrix}}(\tau_K).$$

Proof. See [14, (3.4)]. □

Remark 4. (i) Let $\widehat{\mathbb{Z}} = \prod_{p:\,\mathrm{primes}} \mathbb{Z}_p$ and $\widehat{K} = K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Let $[\ \cdot, K] : \widehat{K}^* \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the Artin reciprocity map for (finite) ideles, where K^{ab} is the maximal abelian extension of K. Then the class field theory asserts that $[\,\cdot\,,K]$ is a surjection with kernel K^* ([3, §15.F] and [12, §IV.6]). Here we observe that

$$\left(\frac{K_{(N)}/K}{(s\tau_K+t)\mathcal{O}_K}\right)=[(x_p)_p,K]|_{K_{(N)}}\quad\text{where } x_p=\begin{cases}1 & \text{if } p\,|\,N,\\ s\tau_K+t & \text{if } p\nmid N.\end{cases}$$

(ii) More generally, if $\mathcal{O}$ is an order in K and $H_{\mathcal{O}}$ is the ring class field of $\mathcal{O}$, then Stevenhagen actually expressed $\mathrm{Gal}(K^{\mathrm{ab}}/H_{\mathcal{O}})$ as the image of a subset of $\widehat{K}^*$ via the reciprocity map $[\,\cdot\,,K]$ ([14, §3]).

Theorem 4.2. *Let p be a prime relatively prime to N and $\mathfrak{p}$ be a prime of K lying above p. Let $f\in\mathcal{F}_N$ be finite at τ_K.*

(i) *If p splits completely in K and so $\mathfrak{p}=[\frac{-u+\sqrt{d_K}}{2},p]$ for some integer u satisfying $u^2\equiv d_K \pmod{4p}$ by* Lemma 2.3 (i), *then*

$$f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)}=f^{\left[\begin{smallmatrix}p & -\frac{u+b_K}{2}\\ 0 & 1\end{smallmatrix}\right]}\left(\frac{u+\sqrt{d_K}}{2p}\right).$$

(ii) *If p is inert in K, then*

$$f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)}=f^{\left[\begin{smallmatrix}p & 0\\ 0 & p\end{smallmatrix}\right]}(\tau_K).$$

(iii) *If p is ramified in K, then*

$$f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)}=\begin{cases}f^{\left[\begin{smallmatrix}2 & -\frac{-2+b_K}{2}\\ 0 & 1\end{smallmatrix}\right]}\left(\frac{-2+\sqrt{d_K}}{4}\right) & \text{if } p=2 \text{ and } \frac{d_K}{4}\equiv 3 \pmod 4,\\ f^{\left[\begin{smallmatrix}p & -\frac{d_K+b_K}{2}\\ 0 & 1\end{smallmatrix}\right]}\left(\frac{d_K+\sqrt{d_K}}{2p}\right) & \text{otherwise.}\end{cases}$$

Proof. (i) We see that

$$\mathfrak{p}=p\mathcal{O}_K\left[\frac{-u+\sqrt{d_K}}{2p},1\right]=p\mathcal{O}_K[\omega_Q,1]\quad\text{with } Q=px^2+uxy+\frac{u^2-d_K}{4p}y^2.$$

Thus we achieve that

$$\begin{aligned}f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)} &= f(\tau_K)^{\left(\frac{K_{(N)}/K}{p\mathcal{O}_K}\right)\left(\frac{K_{(N)}/K}{[\omega_Q,1]}\right)}\\ &= f^{\left[\begin{smallmatrix}p & 0\\ 0 & p\end{smallmatrix}\right]}(\tau_K)^{\left(\frac{K_{(N)}/K}{[\omega_Q,1]}\right)}\quad\text{by Lemma 4.1}\\ &= f^{\left[\begin{smallmatrix}p & 0\\ 0 & p\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & -p'(\frac{u+b_K}{2})\\ 0 & p'\end{smallmatrix}\right]}\left(\frac{u+\sqrt{d_K}}{2p}\right)\\ &\qquad\text{where } p' \text{ is an integer such that } pp'\equiv 1 \pmod N \text{ by Proposition 3.2}\\ &= f^{\left[\begin{smallmatrix}p & -\frac{u+b_K}{2}\\ 0 & 1\end{smallmatrix}\right]}\left(\frac{u+\sqrt{d_K}}{2p}\right).\end{aligned}$$

(ii) Since $\mathfrak{p}=p\mathcal{O}_K$, the result directly follows from Lemma 4.1.

(iii) By Lemma 2.3 (iii), we have $p\mathcal{O}_K = \mathfrak{p}^2$ where

$$\mathfrak{p} = \begin{cases} 2\mathcal{O}_K[\frac{2+\sqrt{d_K}}{4}, 1] & \text{if } p=2 \text{ and } \frac{d_K}{4} \equiv 3 \pmod 4, \\ p\mathcal{O}_K[\frac{-d_K+\sqrt{d_K}}{2p}, 1] & \text{otherwise} \end{cases}$$

$$= p\mathcal{O}_K[\omega_Q, 1] \text{ with } Q = \begin{cases} 2x^2 - 2xy + \frac{4-d_K}{8}y^2 & \text{if } p=2 \text{ and } \frac{d_K}{4} \equiv 3 \pmod 4, \\ px^2 + d_K xy + \frac{d_K^2 - d_K}{4p}y^2 & \text{otherwise.} \end{cases}$$

In a similar way to (i), one can derive by using Lemma 4.1 and Proposition 3.2 that

$$f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)} = f(\tau_K)^{\left(\frac{K_{(N)}/K}{p\mathcal{O}_K}\right)\left(\frac{K_{(N)}/K}{[\omega_Q,1]}\right)}$$

$$= \begin{cases} f^{\left[\begin{smallmatrix} 2 & -\frac{-2+b_K}{2} \\ 0 & 1 \end{smallmatrix}\right]}\left(\frac{-2+\sqrt{d_K}}{4}\right) & \text{if } p=2 \text{ and } \frac{d_K}{4} \equiv 3 \pmod 4, \\ f^{\left[\begin{smallmatrix} p & -\frac{d_K+b_K}{2} \\ 0 & 1 \end{smallmatrix}\right]}\left(\frac{d_K+\sqrt{d_K}}{2p}\right) & \text{otherwise.} \end{cases}$$

□

5. The Kronecker congruence relations

As corollaries of Theorem 4.2, we shall exhibit the Kronecker congruence relations extending (1) to modular functions of higher level.

Corollary 5.1. *Let p be a prime relatively prime to N and $\mathfrak{P}$ be a prime of $K_{(N)}$ lying above p. Let $f \in \mathcal{F}_N$ be integral over $\mathbb{Z}[j]$.*

(i) *If p splits completely in K and so $\mathfrak{P} \cap K = \left[\frac{-u+\sqrt{d_K}}{2}, p\right]$ for some integer u such that $u^2 \equiv d_K \pmod{4p}$ by* Lemma 2.3*, then*

$$f^{\left[\begin{smallmatrix} p & -\frac{u+b_K}{2} \\ 0 & 1 \end{smallmatrix}\right]}\left(\frac{u+\sqrt{d_K}}{2p}\right) \equiv f(\tau_K)^p \pmod{\mathfrak{P}}.$$

(ii) *If p is inert in K, then*

$$f^{\left[\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right]}(\tau_K) \equiv f(\tau_K)^{p^2} \pmod{\mathfrak{P}}.$$

(iii) *If p is ramified in K, then*

$$\begin{cases} f^{\left[\begin{smallmatrix} 2 & -\frac{-2+b_K}{2} \\ 0 & 1 \end{smallmatrix}\right]}\left(\frac{-2+\sqrt{d_K}}{4}\right) \equiv f(\tau_K)^p \pmod{\mathfrak{P}} & \textit{if } p=2 \textit{ and } \frac{d_K}{4} \equiv 3 \pmod 4, \\ f^{\left[\begin{smallmatrix} p & -\frac{d_K+b_K}{2} \\ 0 & 1 \end{smallmatrix}\right]}\left(\frac{d_K+\sqrt{d_K}}{2p}\right) \equiv f(\tau_K)^p \pmod{\mathfrak{P}} & \textit{otherwise.} \end{cases}$$

Proof. Since f is integral over $\mathbb{Z}[j]$ and $j(\tau_K)$ is an algebraic integer by Proposition 2.1, $f(\tau_K)$ is also an algebraic integer. Furthermore, $f(\tau_K)$ belongs to $K_{(N)}$ by Proposition 2.6. Now, the corollary follows from Theorem 4.2. □

Lemma 5.2. *The field $K_{(N)}$ is Galois over $\mathbb{Q}$.*

Proof. See [9, Lemma 9.1]. □

Corollary 5.3. *Let p be a prime such that $p \equiv 1$ or $-1 \pmod N$. Let f be a meromorphic modular function for $\Gamma_1(N)$ with rational Fourier coefficients which is integral over $\mathbb{Z}[j]$. If p splits completely in K and so there is an integer u satisfying $u^2 \equiv d_K \pmod{4p}$ by* Lemma 2.3, *then we have*

$$\left(f(\omega)^p - f\left(\frac{\omega}{p}\right)\right)\left(f(\omega) - f\left(\frac{\omega}{p}\right)^p\right) \equiv 0 \pmod{p\mathcal{O}_{K_{(N)}}} \quad \textit{where } \omega = \frac{u+\sqrt{d_K}}{2}.$$

Proof. Since f is integral over $\mathbb{Z}[j]$ and belongs to $\mathcal{F}_N$, $f(\tau_K)$ is an algebraic integer in $K_{(N)}$ by Propositions 2.1 and 2.6. If we let $\mathfrak{p} = \left[\frac{-u+\sqrt{d_K}}{2}, p\right]$, then we get $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ by Lemma 2.3 (i) and derive that

$$\begin{aligned}
f(\tau_K)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)} &= f^{\left[\begin{smallmatrix} p & -\frac{u+b_K}{2} \\ 0 & 1\end{smallmatrix}\right]}\left(\frac{\omega}{p}\right) \quad \text{by Theorem 4.2 (i)}\\
&= f^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & p\end{smallmatrix}\right]\left[\begin{smallmatrix} p & -\frac{u+b_K}{2} \\ 0 & p'\end{smallmatrix}\right]}\left(\frac{\omega}{p}\right) \quad \text{where } p' \text{ is an integer such that } pp' \equiv 1 \pmod N\\
&= f^{\left[\begin{smallmatrix} p & -\frac{u+b_K}{2} \\ 0 & p'\end{smallmatrix}\right]}\left(\frac{\omega}{p}\right) \quad \text{by Proposition 2.5 (ii)}\\
&\qquad\qquad \text{because } f \text{ has rational Fourier coefficients}\\
&= f\left(\frac{\omega}{p}\right) \quad \text{by Proposition 2.5 (i)}\\
&\qquad\qquad \text{since } p \equiv \pm 1 \pmod N \text{ and } f \text{ is modular for } \Gamma_1(N).
\end{aligned}$$

This assertion also implies that $f(\frac{\omega}{p})$ is an algebraic integer in $K_{(N)}$. Moreover, we see that

$$f(\tau_K) = f\left(\omega - \frac{u+b_K}{2}\right) = f\left(\begin{bmatrix} 1 & -\frac{u+b_K}{2} \\ 0 & 1\end{bmatrix}(\omega)\right) = f(\omega)$$

because f is modular for $\Gamma_1(N)$. Hence we obtain that

$$f(\omega)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)} = f\left(\frac{\omega}{p}\right). \tag{3}$$

We further find by (3) and the fact $p \equiv \pm 1 \pmod N$ that

$$f\left(\frac{\omega}{p}\right)^{\left(\frac{K_{(N)}/K}{\overline{\mathfrak{p}}}\right)} = f(\omega)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)\left(\frac{K_{(N)}/K}{\overline{\mathfrak{p}}}\right)} = f(\omega)^{\left(\frac{K_{(N)}/K}{p\mathcal{O}_K}\right)} = f(\omega). \tag{4}$$

On the other hand, it follows from Lemma 5.2 that for any prime $\mathfrak{P}$ of $K_{(N)}$ lying above $\mathfrak{p}$, $\overline{\mathfrak{P}}$ is also a prime of $K_{(N)}$ lying above $\overline{\mathfrak{p}}$ which is different from $\mathfrak{P}$. Now that

$$f(\omega)^{\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)} \equiv f(\omega)^p \pmod{\mathfrak{P}} \quad \text{and} \quad f\left(\frac{\omega}{p}\right)^{\left(\frac{K_{(N)}/K}{\overline{\mathfrak{p}}}\right)} \equiv f\left(\frac{\omega}{p}\right)^p \pmod{\overline{\mathfrak{P}}},$$

we deduce by (3) and (4) that

$$\left(f(\omega)^p - f\left(\frac{\omega}{p}\right)\right)\left(f(\omega) - f\left(\frac{\omega}{p}\right)^p\right) \equiv 0 \pmod{\mathfrak{P}\overline{\mathfrak{P}}}.$$

Therefore we conclude by the fact

$$p\mathcal{O}_{K_{(N)}} = (\mathfrak{p}\mathcal{O}_{K_{(N)}})(\overline{\mathfrak{p}}\mathcal{O}_{K_{(N)}}) = (\mathfrak{P}_1\mathfrak{P}_2\cdots\mathfrak{P}_g)(\overline{\mathfrak{P}}_1\overline{\mathfrak{P}}_2\cdots\overline{\mathfrak{P}}_g)$$

for some distinct primes $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_g$ of $K_{(N)}$ and the Chinese remainder theorem that

$$\left(f(\omega)^p - f\left(\frac{\omega}{p}\right)\right)\left(f(\omega) - f\left(\frac{\omega}{p}\right)^p\right) \equiv 0 \pmod{p\mathcal{O}_{K_{(N)}}}.$$

□

6. Artin symbols over $\mathbb{Q}$

The field $K_{(N)}$ is a Galois extension of $\mathbb{Q}$, however, it is not necessarily abelian. In this last section, we shall consider the Artin symbol $\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)$ for a prime $\mathfrak{P}$ of $K_{(N)}$.

Lemma 6.1. *If t is a nonzero integer relatively prime to N, then the order of $\left(\frac{K_{(N)}/K}{t\mathcal{O}_K}\right)$ in $\mathrm{Gal}(K_{(N)}/K)$ is the smallest positive integer ℓ such that $t^\ell \equiv 1$ or $-1 \pmod{N}$.*

Proof. See Propositions 2.5, 2.6 and [14, (3.5)]. □

Let $\mathfrak{c}$ denote the complex conjugation on $K_{(N)}$.

Theorem 6.2. *Let p be a prime relatively prime to Nd_K, $\mathfrak{p}$ be a prime of K lying above p, and $\mathfrak{P}$ be a prime of $K_{(N)}$ lying above $\mathfrak{p}$. Let ℓ be the smallest positive integer such that $p^\ell \equiv 1$ or $-1 \pmod{N}$, and let σ be an element of $\mathrm{Gal}(K_{(N)}/K)$ which maps $\overline{\mathfrak{P}}$ to $\mathfrak{P}$.*

(i) *If p splits completely in K, then*

$$\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right) = \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right).$$

(ii) *If p is inert in K, then*

$$D_{\mathfrak{P}/\mathbb{Q}} = \left\langle (\mathfrak{c}\sigma)^{\frac{\ell}{2^e}} \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{2^e} \right\rangle$$

where $e \geq 0$ is the exponent of 2 in the prime factorization of ℓ.

(iii) *If p is inert in K and ℓ is odd, then*

$$\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right) = (\mathfrak{c}\sigma)^\ell \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{\frac{\ell+1}{2}}.$$

Proof. Note that $\mathrm{Gal}(K_{(N)}/K)$ is a subgroup of $\mathrm{Gal}(K_{(N)}/\mathbb{Q})$ and $\mathrm{Gal}((\mathcal{O}_{K_{(N)}}/\mathfrak{P})/(\mathcal{O}_K/\mathfrak{p}))$ is a subgroup of $\mathrm{Gal}((\mathcal{O}_{K_{(N)}}/\mathfrak{P})/(\mathbb{Z}/p\mathbb{Z}))$.

(i) Since p splits completely in K, $\mathcal{O}_K/\mathfrak{p}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Thus $\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)$ coincides with $\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)$.

(ii) Since p is inert in K, we get $\mathfrak{p} = p\mathcal{O}_K$. Then we achieve by Lemma 6.1 that

$$|D_{\mathfrak{P}/K}| = \ell \quad \text{and so} \quad |D_{\mathfrak{P}/\mathbb{Q}}| = 2\ell. \tag{5}$$

Moreover, since $\mathfrak{c}\sigma \in D_{\mathfrak{P}/\mathbb{Q}} \setminus D_{\mathfrak{P}/K}$, we obtain

$$D_{\mathfrak{P}/\mathbb{Q}} = D_{\mathfrak{P}/K} \cup (\mathfrak{c}\sigma) D_{\mathfrak{P}/K},$$

which yields that

$$\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right) = (\mathfrak{c}\sigma)\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^v \quad \text{for some integer } v.$$

Write $\ell = 2^e g$ for an odd positive integer g. If we let n be the order of $(\mathfrak{c}\sigma)^g$, then we see by (5) that n is a divisor of 2^{e+1}. Furthermore, we deduce again by (5) that

$$2 = \left|\left\langle \left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)^{\ell} \right\rangle\right| = \left|\left\langle (\mathfrak{c}\sigma)^{\ell} \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{v\ell} \right\rangle\right| = |\langle (\mathfrak{c}\sigma)^{\ell} \rangle| = |\langle \{(\mathfrak{c}\sigma)^g\}^{2^e} \rangle| = \frac{n}{\gcd(n,\, 2^e)},$$

from which it follows that $n = 2^{e+1}$. Since the order of $\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{2^e}$ is g which is relatively prime to 2^{e+1}, the order of $(\mathfrak{c}\sigma)^g\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{2^e}$ is $2^{e+1}g = 2\ell$. Hence

$$D_{\mathfrak{P}/\mathbb{Q}} = \left\langle (\mathfrak{c}\sigma)^g \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{2^e} \right\rangle.$$

(iii) By the fact $[\mathrm{Gal}(K_{(N)}/\mathbb{Q}) : \mathrm{Gal}(K_{(N)}/K)] = 2$ and the definition of Artin symbol, we have

$$\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)^2 = \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right). \tag{6}$$

Since $\mathfrak{c}\sigma$ is in $D_{\mathfrak{P}/\mathbb{Q}}$, we find by (5) that

$$\left\{(\mathfrak{c}\sigma)^{\ell}\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{\frac{\ell+1}{2}}\right\}^2 = (\mathfrak{c}\sigma)^{2\ell}\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{\ell+1} = \left(\frac{K_{(N)}/K}{\mathfrak{p}}\right). \tag{7}$$

Here, $\frac{\ell+1}{2}$ is an integer because ℓ is odd. If we let $\phi : D_{\mathfrak{P}/\mathbb{Q}} \to \mathbb{Z}/2\ell\mathbb{Z}$ be the isomorphism sending $\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)$ to $1 + 2\ell\mathbb{Z}$, then we establish that

$$\phi(D_{\mathfrak{P}/K}) = \left\langle \phi\left(\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)\right)\right\rangle = \left\langle 2\phi\left(\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)\right)\right\rangle = \langle 2 + 2\ell\mathbb{Z}\rangle \tag{8}$$

by (6), and

$$2\phi\left((\mathfrak{c}\sigma)^\ell\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{\frac{\ell+1}{2}}\right)=2+2\ell\mathbb{Z} \tag{9}$$

by (6) and (7). On the other hand, since $(\mathfrak{c}\sigma)^\ell\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{\frac{\ell+1}{2}}$ does not belong to $\mathrm{Gal}(K_{(N)}/K)$ due to the fact that ℓ is odd, we derive from (8) and (9) that

$$\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)=(\mathfrak{c}\sigma)^\ell\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{\frac{\ell+1}{2}}$$

as desired.

□

Remark 5. By Theorem 6.2, we have

$$\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)=(\mathfrak{c}\sigma)^u\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^v \quad \text{for some integers } u \text{ and } v. \tag{10}$$

Now, consider the case where p is inert and ℓ is even. If $\mathfrak{c}\in D_{\mathfrak{P}/\mathbb{Q}}$, then we see that

$$\begin{aligned}\left(\frac{K_{(N)}/\mathbb{Q}}{\mathfrak{P}}\right)^\ell &= \mathfrak{c}^{u\ell}\sigma^{u\ell}\left(\frac{K_{(N)}/K}{\mathfrak{p}}\right)^{v\ell} && \text{by (10) and the fact that } D_{\mathfrak{P}/\mathbb{Q}} \text{ is abelian}\\ &= \mathrm{id}_{K_{(N)}} && \text{because } \mathfrak{c} \text{ is of order 2 and } |D_{\mathfrak{P}/K}|=\ell \text{ is even.}\end{aligned}$$

But this contradicts the fact $|D_{\mathfrak{P}/\mathbb{Q}}|=2\ell$. Therefore, in this case, $\mathfrak{c}$ does not belong to $D_{\mathfrak{P}/\mathbb{Q}}$.

Remark 6. We observe that $\mathrm{Gal}(K_{(N)}/\mathbb{Q})=\mathrm{Gal}(K_{(N)}/K)\rtimes\langle\mathfrak{c}\rangle$. The action of the group $\Gamma_1(N)$ on the set $\mathcal{Q}_N(d_K)$ can be extended to the set of definite quadratic forms

$$\mathcal{Q}_N^\pm(d_K)=\{ax^2+bxy+cy^2\in\mathbb{Z}[x,y]\mid b^2-4ac=d_K,\ \gcd(a,b,c)=\gcd(a,N)=1\},$$

which induces the equivalence denote by $\sim_N^\pm$. Recently, Jung et al. showed that the set $\mathcal{C}_N^\pm(d_K)=\mathcal{Q}_N^\pm(d_K)/\sim_N^\pm$ can be regarded as a group isomorphic to $\mathrm{Gal}(K_{(N)}/\mathbb{Q})$ ([9, Theorem 9.2]), and further defined the $\mathcal{C}_N^\pm(d_K)$-class invariants as special values of modular functions of level N ([9, Definition 9.4 and Theorem 9.6]).

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