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GENERALIZED DERIVATIONS IN RING WITH INVOLUTION INVOLVING SYMMETRIC AND SKEW SYMMETRIC ELEMENTS

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Abstract. In this paper we will demonstrate some results on a prime ring with involution by introducing two generalized derivations acting on symmetric and skew symmetric elements. This approach allows us to generalize some well known results. Furthermore, we provide examples to show that various restrictions imposed in the hypotheses of our theorems are not superfluous.

1. Introduction

Let R be an associative ring with center $Z(R)$ and the extended centroid $\mathcal{C}$ and U be its Utumi ring of quotients. R is said to be prime if $aRb = 0$ implies that $a = 0$ or $b = 0$, and semiprime if $aRa = 0$ implies that $a = 0$, where $a, b \in R$. A prime ring is obviously semiprime. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. R is 2-torsion free if $2x = 0$, with $x \in R$ implies $x = 0$. An additive map $* : R \to R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring R with involution $*$, $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denote by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case $S(R) \cap Z(R) \neq \{0\}$.

An additive mapping $d : R \to R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of derivations defined on R. More recently, several authors have considered a similar situation in the case where the derivation d is replaced by a generalized derivation (see [1], [11] and [14]). More specifically, an additive map $F : R \to R$ is a generalized derivation

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if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Basic examples of generalized derivations are the usual derivations on R and a left R-module mappings from R into itself. Generalized derivations have been primarily studied on operator algebras.

Recall that a mapping $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [4], [15] for references). A mapping $f : R \to R$ is said to be strong commutativity preserving (SCP) on a subset S of R if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [3], Bell and Daif investigated the commutativity of rings admitting a derivation that is SCP on a nonzero right ideal. Indeed, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subset Z(R)$. In particular, R is commutative if $I = R$. Very recently Ali, Dar and Khan [2] studied strong commutativity preserving like mappings in the setting of rings with involution. In fact, they proved the following result: Let R be a prime ring with involution $*$ of the second kind and with $char(R) \neq 2$. Let d be a nonzero derivation of R such that d satisfying $[d(x), d(x^*)] = [x, x^*]$. Later, in 2017 the authors in [7] studied the case when the derivation d is replaced by a generalized derivarion.

The aim of this paper is to generalize the results proved by [7] for symmetric and skew symmetric elements of R.

2. Main results

We will use frequently the following fact which is very crucial for developing the proofs of our main results.

Fact 1. Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If d is a derivation on R such that $d(h) = 0$ for all $h \in H(R) \cap Z(R)$, then $d(Z(R)) = \{0\}$.

Motivated by the notion of the SCP derivations, the authors in [2] initiated the study of a more general concept by considering the identity $[d(x), d(x^*)] = [x, x^*]$. More precisely, they proved in [2, Theorem 1] that a prime ring $(R, *)$ with involution of the second kind must be commutative if it admits a nonzero derivation d satisfying $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$. Inspired by the above result the authors in [7] studied the relation in the setting of generalized derivations. In fact they proved that if R is a noncommutative prime ring with involution of the second kind such that $char(R) \neq 2$ and if R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$ such that $[F(x), F(x^*)] - [x, x^*] = 0$ for all $x \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$. The main purpose of this paper is to study a more general concept. Indeed, we suggest to give generalization in two ways. Firstly, we will extend the above result by considering satisfied identities on subsets of R like the subset of symmetric and skew symmetric elements of the ring.

Secondly, we will investigate a more general differential identities involving two generalized derivations.

Theorem 2.1. *Let $(R,*)$ be a noncommutative 2-torsion free prime ring with involution of the second kind. If R admits two nonzero generalized derivations $F_1, F_2 : R \to R$ satisfying $[F_1(h), F_2(h')] - [h, h'] = 0$ for all $h, h' \in H(R)$, then there exist $\lambda, \beta \in \mathcal{C}$ the extended centroid of R such that*

$$F_1(x) = \beta x, \quad F_2(x) = \lambda\beta x \quad \textit{and} \quad \beta^2\lambda = 1.$$

Proof. Assume that

$$(1) \qquad [F_1(h), F_2(h')] - [h, h'] = 0 \text{ for all } h, h' \in H(R).$$

Substituting $h'h_0$ for h' in (1), where $h_0 \in Z(R) \cap H(R)$, we find that

$$(2) \quad [F_1(h), F_2(h')]h_0 + [F_1(h), h']d_2(h_0) - [h, h']h_0 = 0 \text{ for all } h, h' \in H(R).$$

Invoking (1), the last equation yields

$$(3) \qquad [F_1(h), h']d_2(h_0) = 0 \text{ for all } h, h' \in H(R).$$

Using the primeness of R, we get $[F_1(h), h'] = 0$ for any $h, h' \in H(R)$ or $d_2(h_0) = 0$ for any $h_0 \in Z(R) \cap H(R)$.

If $[F_1(h), h'] = 0$ for all $h, h' \in H(R)$, then for $h' = h$ we find that $[F_1(h), h] = 0$ for any $h \in H(R)$, and thus R is a commutative ring by [6, Theorem 2.5], a contradiction.

Now suppose that $d_2(h_0) = 0$ for any $h_0 \in H(R) \cap Z(R)$, which implies that $d_2(Z(R)) = \{0\}$ by view of Fact 1.

Replacing h' by kk_0, where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$ in (1), we get

$$(4) \qquad \big([F_1(h), F_2(k)] - [h, k]\big)k_0 = 0.$$

Using the primeness of R and the fact that $S(R) \cap Z(R) \neq \{0\}$, we obtain

$$(5) \qquad [F_1(h), F_2(k)] - [h, k] = 0.$$

Taking $h = x + x^*$ and $k = x - x^*$, where $x \in R$, we may write

$$(6) \qquad \begin{aligned} &[F_1(x), F_2(x)] + [F_1(x^*), F_2(x)] - [F_1(x), F_2(x^*)] \\ &\qquad - [F_1(x^*), F_2(x^*)] + 2[x, x^*] = 0. \end{aligned}$$

On the other hand we have

$$(7) \qquad [F_1(h), F_2(h)] = 0 \text{ for all } h \in H(R).$$

Replacing h by $x + x^*$ in (7), where $x \in R$, we find that

$$(8) \quad [F_1(x), F_2(x)] + [F_1(x^*), F_2(x)] + [F_1(x), F_2(x^*)] + [F_1(x^*), F_2(x^*)] = 0.$$

Combining (6) with (8), one can see that

$$(9) \qquad [F_1(x), F_2(x)] + [F_1(x^*), F_2(x)] + [x, x^*] = 0 \text{ for all } x \in R.$$

Linearizing (9) and using it, we obviously get

$$(10) \qquad [F_1(x), F_2(y)] + [F_1(y), F_2(x)] + [F_1(x^*), F_2(y)]$$

$$+ [F_1(y^*), F_2(x)] + [x, y^*] + [y, x^*] = 0.$$

Writing yh_0 instead of y in (10), where $h_0 \in H(R) \cap Z(R)$, and using (10), we obtain

$$[y + y^*, F_2(x)]d_1(h_0) = 0 \text{ for all } x, y \in R. \tag{11}$$

Invoking the primeness of R, it follows that $[y + y^*, F_2(x)] = 0$ for any $x, y \in R$ or $d_1(h_0) = 0$ for any $h_0 \in H(R) \cap Z(R)$.

Assume that

$$[y + y^*, F_2(x)] = 0 \text{ for any } x, y \in R. \tag{12}$$

Replacing y by yk_0 in (12), where $k_0 \in S(R) \cap (Z(R) \setminus \{0\})$ and using the same equation, we get

$$[x, F_2(x)] = 0 \text{ for any } x \in R, \tag{13}$$

therefore R is a commutative integral domain by view of [9, Theorem 1], a contradiction, hence we conclude that $d_1(h_0) = 0$ for any $h_0 \in H(R) \cap Z(R)$. Since R is a 2-torsion free prime ring, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

Thus in view of (1) and (5), we find that

$$[F_1(h), F_2(x)] - [h, x] = 0. \tag{14}$$

Replacing h by kk_0 in (14), where $k \in S(R)$ and $k_0 \in Z(R) \cap (S(R) \setminus \{0\})$, we obtain

$$[F_1(k), F_2(x)]k_0 + [k, F_2(x)]d_1(k_0) - [k, x]k_0 = 0, \tag{15}$$

which leads to

$$[F_1(k), F_2(x)] - [k, x] = 0. \tag{16}$$

Using (14) and (16) and the fact that R is a 2-torsion free prime ring, we have

$$[F_1(x), F_2(y)] - [x, y] = 0 \text{ for all } x, y \in R. \tag{17}$$

Taking $y = x$ in the last equation, one can see that

$$[F_1(x), F_2(x)] = 0 \text{ for any } x \in R. \tag{18}$$

By [8, Theorem 2], there exists $\lambda \in C$ such that $F_1(x) = \lambda F_2(x)$ for any $x \in R$, and thus the equation (17) becomes

$$[F_1(x), \lambda F_1(y)] - [x, y] = 0 \text{ for all } x, y \in R. \tag{19}$$

Replacing y by xy in (19), we find that

$$[x, xy] = \lambda F_1(x)[F_1(x), y] + x[F_1(x), \lambda d(y)] + [F_1(x), x]\lambda d(y). \tag{20}$$

On the other hand we have

$$[x, xy] = x[x, y] = x[F_1(x), \lambda F_1(y)] \text{ for all } x, y \in R. \tag{21}$$

Combining (20) together with (21), we get

$$x[F_1(x), \lambda F_1(y)] = \lambda F_1(x)[F_1(x), y] + x[F_1(x), \lambda d(y)] + [F_1(x), x]\lambda d(y). \tag{22}$$

By view of [10, Theorem 3], the last equation becomes

$$\begin{aligned} (23) \qquad & x[F_1(x), \lambda\beta y] + x[F_1(x), \lambda d(y)] \\ & = \lambda F_1(x)[F_1(x), y] + x[F_1(x), \lambda d(y)] + [F_1(x), x]\lambda d(y), \end{aligned}$$

thereby obtaining

$$(24) \qquad x[F_1(x), \lambda\beta y] = \lambda F_1(x)[F_1(x), y] + [F_1(x), x]\lambda d(y) \text{ for all } x, y \in R.$$

Taking $y = yr$ in the last equation, one can see that

$$\begin{aligned} (25) \qquad & x\lambda\beta y[F_1(x), r] + x[F_1(x), \lambda\beta y]r \\ & = \lambda F_1(x)y[F_1(x), r] + \lambda F_1(x)[F_1(x), y]r + [F_1(x), x]\lambda d(y)r \\ & \quad + [F_1(x), x]\lambda y d(r). \end{aligned}$$

Using (24) and (25) we obtain

$$(26) \quad x\lambda\beta y[F_1(x), r] = \lambda F_1(x)y[F_1(x), r] + [F_1(x), x]\lambda y d(r) \text{ for all } r, x, y \in R.$$

Replacing r by $F_1(x)$ we get

$$(27) \qquad [F_1(x), x]\lambda y d(F_1(x)) = 0 \text{ for all } x, y \in R,$$

which leads to $[F_1(x), x] = 0$ or $d(F_1(x)) = 0$ for all $x \in R$.

We claim that $[F_1(x), x] = 0$ for all $x \in R$, indeed let $x_0 \in R$ such that $[F_1(x_0), x_0] \neq 0$, then $d(F_1(x_0)) = 0$ and thus equation (19) yields

$$[x_0, yF_1(x_0)] = [F_1(x_0), \lambda F_1(y)]F_1(x_0) \text{ for all } y \in R,$$

therefore

$$(28) \qquad [x_0, yF_1(x_0)] = [x_0, y]F_1(x_0) \text{ for all } y \in R.$$

On the other hand, we have

$$(29) \qquad [x_0, yF_1(x_0)] = [x_0, y]F_1(x_0) + y[x_0, F_1(x_0)] \text{ for all } y \in R.$$

Subtracting (28) from (29), we arrive at

$$(30) \qquad y[x_0, F_1(x_0)] = 0 \text{ for all } y \in R.$$

So that

$$(31) \qquad [x_0, F_1(x_0)] = 0.$$

A contradiction, therefore we conclude that

$$(32) \qquad [x, F_1(x)] = 0 \text{ for all } x \in R.$$

In view of [13, Theorem 1], and the fact that R is noncommutative, we get that $d = 0$, so $F_1(x) = \beta x$ for all $x \in R$. Thus equation (17) becomes

$$(33) \qquad \lambda[\beta x, \beta y] - [x, y] = 0 \text{ for all } x, y \in R.$$

By [5, Theorem 2], R and U satisfy the same differential identities, so we can take $y = 1$ in the last equation to get

$$(34) \qquad \lambda[\beta x, \beta] = 0.$$

That is

$$\lambda\beta[x,\beta]=0. \tag{35}$$

Which implies that $\beta=0$ or $\beta\in\mathcal{C}$ but the first case means that $F_1=0$, a contradiction, so we have $\beta\in\mathcal{C}$ and equation (33) becomes

$$(\lambda\beta^2-1)[x,y]=0 \text{ for all } x,y\in R. \tag{36}$$

This completes the proof since R is a noncommutative prime ring. □

Next we turn to a corresponding result in the skew symmetric case.

Theorem 2.2. *Let $(R,*)$ be a noncommutative 2-torsion free prime ring with involution of the second kind. If R admits two nonzero generalized derivations $F_1,F_2:R\to R$ such that $[F_1(k),F_2(k')]-[k,k']=0$ for any $k,k'\in S(R)$, then there exist $\lambda,\beta\in\mathcal{C}$ the extended centroid of R such that*

$$F_1(x)=\beta x,\ \ F_2(x)=\lambda\beta x \ \ \textit{and}\ \ \beta^2\lambda=1.$$

Proof. Suppose that

$$[F_1(k),F_2(k')]-[k,k']=0 \text{ for all } k,k'\in S(R). \tag{37}$$

Substituting $k'h_0$ for k' in (37), where $h_0\in Z(R)\cap H(R)$, we find that

$$[F_1(k),F_2(k')]h_0+[k,F_2(k')]d_1(h_0)-[k,k']h_0=0 \text{ for all } k,k'\in S(R). \tag{38}$$

Combining the last equation with (37), we arrive at

$$[k,F_2(k')]d_1(h_0)=0. \tag{39}$$

In light of the primeness of R, we get $[k,F_2(k')]=0$ for any $k,k'\in S(R)$ or $d_1(h_0)=0$ for any $h_0\in Z(R)\cap H(R)$.

If $[k,F_2(k')]=0$ for any $k,k'\in S(R)$, then for $k'=k$ we find that $[k,F_2(k)]=0$ for any $k\in S(R)$ and R is a commutative ring by [6, Theorem 2.7], a contradiction. Consequently, we are forced to $d_1(h_0)=0$ for any $h_0\in H(R)\cap Z(R)$; replacing k by hk_0, where $h\in H(R)$ and $k_0\in S(R)\cap Z(R)$ in (37), we get

$$\big([F_1(h),F_2(k)]-[h,k]\big)k_0=0. \tag{40}$$

Using the primeness of R and the fact that $S(R)\cap Z(R)\neq\{0\}$, we obtain

$$[F_1(h),F_2(k)]-[h,k]=0. \tag{41}$$

Taking $x+x^*$ and $x-x^*$ for h and k, respectively, in (41), where $x\in R$, we get

$$\begin{aligned}[F_1(x),F_2(x)]+[F_1(x^*),F_2(x)]-[F_1(x),F_2(x^*)]\\ -[F_1(x^*),F_2(x^*)]+2[x,x^*]=0.\end{aligned} \tag{42}$$

On the other hand we have

$$[F_1(k),F_2(k)]=0 \text{ for all } k\in S(R).$$

Replacing k by $x - x^*$, where $x \in R$, in the last equation, one can verify that

$$[F_1(x), F_2(x)] - [F_1(x^*), F_2(x)] - [F_1(x), F_2(x^*)] + [F_1(x^*), F_2(x^*)] = 0. \tag{43}$$

Comparing (42) with (43), it follows that

$$[F_1(x), F_2(x)] - [F_1(x), F_2(x^*)] + [x, x^*] = 0 \text{ for all } x \in R. \tag{44}$$

Linearizing (44), one can see that

$$\begin{aligned}[F_1(x), F_2(y)] + [F_1(y), F_2(x)] - [F_1(x), F_2^*(y)]\\ - [F_1(y), F_2(x^*)] + [x, y^*] - [y, x^*] = 0.\end{aligned} \tag{45}$$

Putting $y = yh_0$ in (45), where $h_0 \in H(R) \cap Z(R)$ and using (45), we obtain

$$[F_1(x), y - y^*]d_2(h_0) = 0 \text{ for all } x, y \in R. \tag{46}$$

Using the primeness of R, we have $[F_1(x), y - y^*] = 0$ for any $x, y \in R$ or $d_2(h_0) = 0$ for any $h_0 \in H(R) \cap Z(R)$.

Assume that

$$[F_1(x), y - y^*] = 0 \text{ for any } x, y \in R. \tag{47}$$

Replacing y by yk_0 in (47), where $k_0 \in S(R) \cap (Z(R) \setminus \{0\})$ and using the same equation, we get

$$[x, F_2(x)] = 0 \text{ for any } x \in R. \tag{48}$$

Therefore R is a commutative integral domain by view of [9, Theorem 1], a contradiction. Consequently, we must have $d_1(h_0) = 0$ for any $h_0 \in H(R) \cap Z(R)$, and so that $d_1(k_0) = 0$ for any $k_0 \in S(R) \cap Z(R)$. Since R is a 2-torsion free prime ring, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

Thus in view of (37) and (41), we find that

$$[F_1(x), F_2(k)] - [x, k] = 0. \tag{49}$$

Replacing k by hk_0 in (49), where $h \in H(R)$ and $k_0 \in S(R) \cap (Z(R) \setminus \{0\})$, we obtain

$$[F_1(x), F_2(h)]k_0 + [F_1(x), h]d_2(k_0) - [x, h]k_0 = 0 \tag{50}$$

in such a way that

$$[F_1(x), F_2(h)] - [x, h] = 0. \tag{51}$$

Combining (50) together with (51) and using the fact that R is a 2-torsion free prime ring, we have

$$[F_1(x), F_2(y)] - [x, y] = 0 \text{ for any } x, y \in R. \tag{52}$$

Taking $y = x$ in the last equation, we get

$$[F_1(x), F_2(x)] = 0 \text{ for any } x \in R. \tag{53}$$

In light of [8, Theorem 2], there exists $\lambda \in \mathcal{C}$ such that $F_1(x) = \lambda F_2(x)$ for any $x \in R$. Reasoning as in the proof of the above theorem, we obtain the required result. □

As an applications of the aforementioned results, we obtain the following corollaries.

Corollary 2.3. *Let $(R, *)$ be a noncommutative 2-torsion free prime ring with involution of the second kind. If R admit two nonzero generalized derivations $F_1, F_2 : R \to R$ satisfying $[F_1(x), F_2(y)] - [x, y] = 0$ for any $x, y \in R$, then there exist $\lambda, \beta \in \mathcal{C}$ the extended centroid of R such that*

$$F_1(x) = \beta x, \quad F_2(x) = \lambda\beta x \quad \text{and} \quad \beta^2\lambda = 1.$$

Corollary 2.4. *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If R admits two nonzero derivations $d_1,\ d_2 : R \to R$ satisfying $[d_1(x), d_2(y)] - [x, y] = 0$ for any $x, y \in R$, then R is a commutative integral domain.*

Corollary 2.5 ([12], Theorem 4)**.** *Let R be a noncommutative prime ring with a nonzero ideal I and a nonzero generalized derivation F. If F is SCP on I, then $F(x) = x$ or $F(x) = -x$.*

3. Examples

The following example proves that the condition $*$ is of the second kind is necessary in Theorem 2.1.

Example 3.1. Let us consider $R = M_2(\mathbb{Z})$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. It is straightforward to check that $(R, *)$ is a prime ring and $*$ is an involution of the first kind. Now if we take the derivation F defined on R by

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix},$$

then for $F_1 = F_2 = F$, the following condition holds:

$$[F(h), F(h')] - [h, h'] = 0 \ \text{ for any } \ h, h' \in H(R).$$

However, R is not commutative and neither $F(x) = x$ nor $F(x) = -x$ for all $x \in R$.

The following example proves that the condition $*$ is of the second kind is necessary in Theorem 2.1.

Example 3.2. Let R be the ring of real quaternions. If we define $* : R \to R$ by $(\alpha + \beta i + \gamma j + \delta k)^* = \alpha - \beta i + \gamma j + \delta k$, then '$*$' is an involution of the first kind and all skew symmetric elements commute. Thus if F is a generalized inner derivation induced by some skew symmetric elements $a, b \in R$ (associated with the inner derivation induced by b), and if we take $F_1 = F_2 = F$, then $[F(k), F(k')] - [k, k'] = 0$ for any $k, k' \in S(R)$.

However, R is not commutative and neither $F(x) = x$ nor $F(x) = -x$ for all $x \in R$.

The following example proves that the primeness hypothesis in Theorem 2.1 is necessary.

Example 3.3. Let R_1 be the ring as in Example 3.1 and $\mathbb{C}$ be the field of complex numbers. Consider $R = R_1 \times \mathbb{C}$. Then R is a non prime ring provided with the involution of the second kind defined by $\sigma(x, z) = (x^*, \bar{z})$. Let G be the generalized derivation of R defined by $G(x, z) = (F(x), 0)$, where F is the generalized derivation defined on R in Example 3.1.

Now, if we take $F_1 = F_2 = G$, then one can see that $[G(h), G(h')] - [h, h'] = 0$ for all $h, h' \in H(R)$. But R is not commutative and neither $G(x) = x$ nor $G(x) = -x$ for all $x \in R$.

The following example proves that the primeness hypothesis in Theorem 2.2 is necessary.

Example 3.4. Consider R_1 to be the ring as in Example 3.2 and G be the generalized derivation of R defined by $G(x, z) = (F(x), 0)$, where F is a generalized inner derivation induced by some skew symmetric elements $a, b \in R$, for $F_1 = F_2 = G$ one can easily find that $[G(k), G(k')] - [k, k'] = 0$ for all $k, k' \in S(R)$. But again R is not commutative and neither $G(x) = x$ nor $G(x) = -x$ for all $x \in R$.

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