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THE HOMOLOGICAL PROPERTIES OF REGULAR INJECTIVE MODULES

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Abstract. Let R be a commutative ring. An R-module E is said to be regular injective provided that $\mathrm{Ext}^1_R(R/I,E)=0$ for any regular ideal I of R. We first show that the class of regular injective modules have the hereditary property, and then introduce and study the regular injective dimension of modules and regular global dimension of rings. Finally, we give some homological characterizations of total rings of quotients and Dedekind rings.

1. Introduction

Throughout this paper, all rings are commutative rings with identity and all modules are unital. Let R be a ring. An element $r\in R$ is said to be regular if $ra=0$ with $a\in R$ implies $a=0$. Otherwise, r is said to be a zero-divisor. Let I be an ideal of R. If I contains a regular element, then I is said to be a regular ideal. Denote by $\mathrm{Z}(R)$ the set of all zero-divisors of R. Denote by $\mathrm{T}(R)$ the total ring of R, i.e., $\mathrm{T}(R)=R_{R-\mathrm{Z}(R)}$. If every regular element in R is a unit, then R is said to be a total ring of quotients.

The class of injective modules, as one of the most classical classes of modules in homological algebra, plays a crucial role in the development of rings and categories of modules. For examples, injective modules can be used to characterize semi-simple rings, hereditary rings and Noetherian rings and so on. For generalizing injective modules, Maddox [7] introduced the notion of absolutely pure modules in 1967. Subsequently, Megibben [8] characterized Noetherian rings and semi-hereditary rings in terms of absolutely pure modules. It is well-known that the Baer's criterion states that an R-module E is injective if and only if $\mathrm{Ext}^1_R(R/I,E)=0$ for any ideal I of R. If we replace "any ideal I of R" with some special classes of ideals, then it will produce some meaningful classes of generalized injective modules. In recent decades these generalized injective

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modules have attracted many algebraists. For example, Damiano [3] introduced the class of coflat modules by using finitely generated ideals; Wang [12] introduced the class of maximally injective modules by using maximal ideals; Yang [13] introduced ϕ-injective modules by using nonnil ideals; and Wang [11] introduced the notion of regular injective modules by using regular ideals.

It is well-known that the class of injective modules is a coresolving class. The coresolving properties of a given class of R-modules are very crucial to study the homological dimensions. Previous research has shown that the classes of coflat modules, maximally injective modules and ϕ-injective modules are generally not coresolving classes. So it is natural and worth asking that:

Is the class of regular injective modules coresolving?

We will study this question in this paper. Actually, we show that the class of regular injective modules is coresolving (see Theorem 2.5). We will also introduce the regular injective dimension of modules and global regular dimension of rings. In particular, we give a homological characterization of total rings of quotients and Dedekind rings.

2. The coresolving property of regular injective modules

We begin with the definition of regular injective modules introduced by Wang et al. [11] in 2011.

Definition 2.1. Let R be a ring and E an R-module. If $\mathrm{Ext}^1_R(R/I, E) = 0$ for every regular ideal I, then E is said to be a reg-injective module (abbreviates regular injective module).

Let R be a ring and M an R-module. If, for any $m \in M$, there exists a regular element $r \in R$ such that $rm = 0$, then M is said to be a torsion module; if $rm = 0$ with $r \in R$ regular and $m \in M$ implies that $m = 0$, then M is said to be a torsion-free module.

Theorem 2.2. *Let R be a ring and E an R-module. Then the following statements are equivalent.*

1. *E is a reg-injective module.*
2. $\mathrm{Ext}^1_R(T, E) = 0$ *for any torsion R-module T.*
3. *Any short exact sequence $0 \to E \to B \to T \to 0$, where T is a torsion R-module, splits.*
4. *For any short exact sequence $0 \to A \xrightarrow{g} B \to T \to 0$, where T is a torsion R-module and any R-homomorphism $f : A \to E$, there exists an R-homomorphism $h : B \to E$ such that $f = h \circ g$.*

Proof. (1) $\Rightarrow$ (4): The proof is similar to that of Baer's criterion for injective modules (see [10, Theorem 2.4.4]), and we show it for completeness. Let E be an R-module satisfying (1). Let B be an R-module and A a submodule of B

such that B/A is torsion. Let $f : A \to E$ be an R-homomorphism. Let

$$\Gamma = \{(C, d) \mid C \text{ is a submodule of } B \text{ that contains } A,\\ d : C \to E \text{ an } R\text{-homomorphism such that } d|_A = f\}.$$

Since $(A, f) \in \Gamma$, we have Γ is non-empty. Define $(C_1, d_1) \le (C_2, d_2)$ if and only if $C_1 \subseteq C_2$ and $d_2|_{C_1} = d_1$. Consequently, Γ is a partial order. For any chain $\{(C_j, d_j)\}$, let $C_0 = \bigcup_j C_j$, and if $c \in C_j$, then $d_0(c) = d_j(c)$. It is easy to verify that (C_0, d_0) is an upper bound of $\{(C_j, d_j)\}$. By Zorn's lemma, there is a maximal element in Γ which is assumed to be (C, d). We claim that $C = B$. On the contrary, let $x \in B - C$. Denote by $I = \{r \in R \mid rx \in C\}$. Since B/A is torsion, so is its quotient B/C. Hence the submodule $(Rx + C)/C \cong R/I$ is also a torsion module. It follows that I is a regular ideal. Let $h : I \to E$ be the R-homomorphism satisfying $h(r) = d(rx)$. By (1), there exists an R-homomorphism $g : R \to E$ satisfying $g(r) = h(r) = d(rx)$ $(r \in I)$. Set $C_1 = C + Rx$ and $d_1(c + rx) = d(c) + g(r)$, where $c \in C$ and $r \in R$. If $c+rx = 0$, then $r \in I$. Consequently, $d(c)+g(r) = d(c)+h(r) = d(c)+d(rx) = d(c + rx) = 0$. Hence, d_1 is well-defined and $d_1|_A = f$. So $(C_1, d_1) \in \Gamma$. However, $(C_1, d_1) > (C, d)$, which is a contradiction to the maximality of (C, d).

(2) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1): Obvious.

(4) $\Rightarrow$ (3): It follows by setting $A = E$ and $f = \mathrm{Id}_E$. $\square$

Corollary 2.3. *Let R be a ring and E an R-module. Then E is a reg-injective torsion-free R-module if and only if E is a $\mathrm{T}(R)$-module.*

Proof. Suppose E is a $\mathrm{T}(R)$-module. Suppose $rm = 0$ with $r \in R$ regular and $m \in M$. Then $\frac{r}{1}m = 0$ and so $m = \frac{1}{r}(\frac{r}{1}m) = 0$, and hence M is a torsion-free R-module. Let I be a regular ideal containing a regular element r, and $f : I \to E$ be an R-homomorphism. Write $g : R \to E$ to be an R-homomorphism satisfying $g(a) = \frac{a}{r}f(r)$ for any $a \in R$. Then for any $b \in I$, we have $f(b) = \frac{r}{r}f(b) = \frac{1}{r}f(rb) = \frac{b}{r}f(r) = g(b)$. So g is a lift of f, and hence E is a reg-injective R-module.

On the other hand, we need to show: for any $r \in R$, the multiplication $m_r : E \xrightarrow{\times r} E$ is an isomorphism. In fact, suppose $r \in R$ is a regular element. Since E is torsion-free, so $re = 0 \in E$ implies $e = 0$. Hence m_r is a monomorphism. Since E is reg-injective, we have $E/rE \cong \mathrm{Ext}^1_R(R/Rr, E) = 0$. Hence, m_r is an epimorphism. Consequently, E is a $\mathrm{T}(R)$-module. $\square$

Trivially, if D is an integral domain, then every reg-injective D-module is injective. Furthermore, we have the following result.

Proposition 2.4. *Let $R = D_1 \times D_2 \times \cdots \times D_n$ be a finite product of integral domains. Then every reg-injective R-module is injective.*

Proof. Let E be a reg-injective R-module. Then $E \cong E_1 \times E_2 \times \cdots \times E_n$, where E_i is a D_i-module $(i = 1, 2, \ldots, n)$. Claim that each E_i is a reg-injective

D_i-module. In fact, if I_i is a regular ideal of D_i, then

$$I = D_1 \times \cdots \times D_{i-1} \times I_i \times D_{i+1} \times \cdots \times D_n$$

is a regular ideal of R. Hence

$$\mathrm{Ext}^1_{D_i}(D_i/I_i, E_i) \cong \mathrm{Ext}^1_R(R/I, E) = 0.$$

It follows that each E_i is a reg-injective D_i-module. So each E_i is an injective D_i-module. Consequently, E is an injective R-module. □

Recall that a class $\mathscr{C}$ of R-modules is said to be coresolving if it contains all injective modules, and is closed under direct summands, extensions and cokernels of monomorphisms. It is trivial that the class of injective modules is coresolving. The following result shows that the class of reg-injective modules is also coresolving.

Theorem 2.5. *Let R be a ring and E an R-module. Then the following statements are equivalent.*

(1) *E is a reg-injective module.*
(2) $\mathrm{Ext}^n_R(R/I, E) = 0$ *for any regular ideal I of R and any $n \geq 1$.*
(3) *The $(n-1)$-cosyzygy $\Omega_{n-1}(E)$ of E is a reg-injective module for any $n \geq 1$.*
(4) $\mathrm{Ext}^n_R(T, E) = 0$ *for any torsion module T and any $n \geq 1$.*

Consequently, the class of reg-injective modules is coresolving.

Proof. (1) ⇒ (2): Let E be a reg-injective R-module, I a regular ideal of R and $n \geq 1$. Let $a \in I$ be a regular element. Then $\mathrm{pd}_R R/Ra \leq 1$. Hence, $\mathrm{Ext}^n_R(R/Ra, E) = 0$ for any $n \geq 1$. According to [2, Proposition 4.1.4] we have

$$\mathrm{Ext}^1_{R/Ra}(R/I, \mathrm{Hom}_R(R/Ra, E)) \cong \mathrm{Ext}^1_R(R/I, E) = 0.$$

So $\mathrm{Hom}_R(R/Ra, E)$ is an injective R/Ra-module by Baer's criterion. It follows by [2, Proposition 4.1.4] again that, for any $n \geq 1$, we have

$$\mathrm{Ext}^n_R(R/I, E) \cong \mathrm{Ext}^n_{R/Ra}(R/I, \mathrm{Hom}_R(R/Ra, E)) = 0.$$

(2) ⇒ (1): Obvious.

(2) ⇔ (3): It follows by $\mathrm{Ext}^1_R(R/I, \Omega_{n-1}(E)) \cong \mathrm{Ext}^n_R(R/I, E)$.

(3) ⇔ (4): It follows by $\mathrm{Ext}^1_R(R/I, \Omega_{n-1}(E)) \cong \mathrm{Ext}^n_R(R/I, E)$ and Theorem 2.2.

Next, we will show the class of reg-injective modules is coresolving. Obviously, the class of reg-injective modules contains all injective modules, and is closed under direct summands, extensions. It remains to show that it is closed under cokernels of monomorphisms. Let I be a regular ideal of R and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules, where A and B are reg-injective. Then there exists an exact sequence $\mathrm{Ext}^1_R(R/I, B) \to \mathrm{Ext}^1_R(R/I, C) \to \mathrm{Ext}^2_R(R/I, A)$. Since A and B are reg-injective, we have $\mathrm{Ext}^1_R(R/I, B) = \mathrm{Ext}^2_R(R/I, A) = 0$. So $\mathrm{Ext}^1_R(R/I, C) = 0$. Consequently, C is also reg-injective. □

3. The reg-injective dimension of modules

Let R be a ring and M an R-module. The injective dimension $\mathrm{id}_R(M)$ of an R-module M is defined as the length of the shortest injective resolutions of M. Next, we will introduce the reg-injective dimension of modules.

Definition 3.1. Let R be a ring and M an R-module. The reg-injective dimension of M, denote by $r\text{-}\mathrm{id}_R(M) = n$, is the length of the shortest long exact sequences

$$(\diamondsuit) \qquad 0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$$

of R-modules, where each E_i is a reg-injective R-module ($i = 0, \dots, n$). We call ($\diamondsuit$) a reg-injective resolution of M of length n. If there is no such finite reg-injective resolution of M, then we set $r\text{-}\mathrm{id}_R(M) = \infty$.

Let R be a ring and M an R-module. Then trivially $r\text{-}\mathrm{id}_R(M) \leq \mathrm{id}_R(M)$. Moreover, if R is an integral domain, then $r\text{-}\mathrm{id}_R(M) = \mathrm{id}_R(M)$.

Theorem 3.2. *Let R be a ring and M an R-module. Then the following statements are equivalent.*

(1) $r\text{-}\mathrm{id}_R(M) \leq n$.
(2) $\mathrm{Ext}_R^{n+k}(T, M) = 0$ *for any torsion R-module T and any integer $k \geq 1$.*
(3) $\mathrm{Ext}_R^{n+k}(R/I, M) = 0$ *for any regular ideal I of R and any integer $k \geq 1$.*
(4) $\mathrm{Ext}_R^{n+1}(T, M) = 0$ *for any torsion R-module T.*
(5) $\mathrm{Ext}_R^{n+1}(R/I, M) = 0$ *for any regular ideal I of R.*
(6) *If $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ is an exact sequence, where $E_0, E_1, \dots, E_{n-1}$ are reg-injective R-modules, then E_n is also reg-injective.*
(7) *If $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ is an exact sequence, where $E_0, E_1, \dots, E_{n-1}$ are injective R-modules, then E_n is a reg-injective R-module.*
(8) *There exists an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$, where $E_0, E_1, \dots, E_{n-1}$ are injective R-modules, and E_n is a reg-injective R-module.*

Proof. (1) $\Rightarrow$ (2): We prove (2) by induction on n. First, we consider $n = 0$, (2) follows by Theorem 2.5. If $n > 0$, then there exists an exact sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0,$$

where each E_i is a reg-injective R-module ($i = 0, \dots, n$). Write

$$K_0 = \mathrm{Coker}(M \to E_0).$$

Then we have exact sequences $0 \to M \to E_0 \to K_0 \to 0$ and $0 \to K_0 \to E_1 \to \cdots \to E_n \to 0$. So $r\text{-}\mathrm{id}_R(K_0) \leq n - 1$. By induction, we have $\mathrm{Ext}_R^{n-1+k}(T, K_0) = 0$ for any torsion R-module T and any integer $k \geq 1$.

It follows by the exact sequence

$$\begin{aligned}0 = \mathrm{Ext}_R^{n+k-1}(T, E_0) &\to \mathrm{Ext}_R^{n+k-1}(T, K_0)\\ &\to \mathrm{Ext}_R^{n+k}(T, M) \to \mathrm{Ext}_R^{n+k}(T, E_0) = 0\end{aligned}$$

that $\mathrm{Ext}_R^{n+k}(M, T) \cong \mathrm{Ext}_R^{n-1+k}(T, K_0) = 0$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5), (4) $\Rightarrow$ (5) and (6) $\Rightarrow$ (7): Trivial.

(5) $\Rightarrow$ (6): Set $K_0 = \mathrm{Coker}(M \to E_0)$ and $K_i = \mathrm{Coker}(E_{i-1} \to E_i)$, where $i = 1, \dots, n-1$. Then $K_{n-1} \cong E_n$. Since $E_0, E_1, \dots, E_{n-1}$ are reg-injective R-modules, we have

$$\mathrm{Ext}_R^1(R/I, E_n) \cong \mathrm{Ext}_R^2(R/I, K_{n-2}) \cong \cdots \cong \mathrm{Ext}_R^{n+1}(R/I, M) = 0$$

for any regular ideal I of R. Hence E_n is a reg-injective R-module.

(7) $\Rightarrow$ (8): Consider the injective resolution of M:

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-2} \xrightarrow{d_{n-2}} E_{n-1} \to \cdots,$$

where $E_0, E_1, \dots, E_{n-1}$ are injective R-modules. Then $E_n := \mathrm{Coker}(d_{n-2})$ is a reg-injective R-module by (7).

(8) $\Rightarrow$ (4): Consider the exact sequence:

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0,$$

where $E_0, E_1, \dots, E_{n-1}$ are injective R-modules, and E_n is a reg-injective R-module. Then by Theorem 2.2, we have $\mathrm{Ext}_R^{n+1}(T, M) = 0 \cong \mathrm{Ext}_R^1(T, E_n) = 0$.

(8) $\Rightarrow$ (1): Obvious. □

Corollary 3.3. *Let R be a ring and N an R-module. If $r\text{-id}_R(N) = n > 0$, then there exists a reg-injective R-module E such that $\mathrm{Ext}_R^n(E, N) \neq 0$.*

Proof. Since $r\text{-id}_R(N) = n > 0$, then there is a torsion R-module T satisfying $\mathrm{Ext}_R^n(T, N) \neq 0$. It follows by [11, Proposition 5.6] that there exists a reg-injective envelope E of T such that $T \subseteq E$ and E is a torsion module. So E/T is also a torsion module. Consequently, there exists an exact sequence

$$\mathrm{Ext}_R^n(E, N) \to \mathrm{Ext}_R^n(T, N) \to \mathrm{Ext}_R^{n+1}(E/T, N).$$

Since $\mathrm{Ext}_R^{n+1}(E/T, N) = 0$ and $\mathrm{Ext}_R^n(T, N) \neq 0$, we have $\mathrm{Ext}_R^n(E, N) \neq 0$. □

The following three propositions are standard homological algebra, and so we omit their proofs.

Proposition 3.4. *Let R be a ring and $\{M_i \mid i \in \Gamma\}$ a family of R-modules. Then*

$$r\text{-id}_R(\prod_{i \in \Gamma} M_i) = \sup\{r\text{-id}_R(M_i)\}.$$

Proposition 3.5. *Let R be a ring and $0 \to N \to E \to C \to 0$ be an exact sequence of R-modules, where E is a reg-injective R-module. Then the following results hold*:

(1) *If $r\text{-id}_R(N) = 0$, then $r\text{-id}_R(C) = 0$;*

(2) *If* $r\text{-id}_R(N) = n > 0$, *then* $r\text{-id}_R(C) = n - 1$.

Proposition 3.6. *Let R be a ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then the following results hold*:

(1) $r\text{-id}_R(A) \leq 1 + \max\{r\text{-id}_R(B), r\text{-id}_R(C)\}$;

(2) *If* $r\text{-id}_R(B) < r\text{-id}_R(A)$, *then* $r\text{-id}_R(C) = r\text{-id}_R(A) - 1 \geq r\text{-id}_R(B)$.

4. The regular global dimensions of rings

The global dimension gl.dim(R) of a ring R is defined as the supremum of injective dimensions of all R-modules. In this section, we introduce and study the regular global dimensions of rings:

Definition 4.1. Let R be a ring. The regular global dimension of R is defined as

$$r\text{-gl.dim}(R) = \sup\{r\text{-id}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Trivially, we have $r\text{-gl.dim}(R) \leq \text{gl.dim}(R)$ by their definitions. If R is an integral domain, then $r\text{-gl.dim}(R) = \text{gl.dim}(R)$. Let M be an R-module. We denote by $\text{pd}_R(M)$ the projective dimension of R-module M.

Theorem 4.2. *Let R be a ring. Then the following statements are equivalent.*

(1) $r\text{-gl.dim}(R) \leq n$.

(2) $r\text{-id}_R(M) \leq n$ *for any R-module M.*

(3) $\text{Ext}_R^{n+k}(T, M) = 0$ *for any R-module M, any torsion R-module T, and any integer $k \geq 1$.*

(4) $\text{Ext}_R^{n+1}(T, M) = 0$ *for any R-module M and any torsion R-module T.*

(5) $\text{Ext}_R^{n+k}(R/I, M) = 0$ *for any R-module M, any regular ideal I of R, and any integer $k \geq 1$.*

(6) $\text{Ext}_R^{n+1}(R/I, M) = 0$ *for any R-module M and any regular ideal I of R.*

(7) $\text{pd}_R T \leq n$ *for any torsion R-module T.*

(8) $\text{pd}_R R/I \leq n$ *for any regular ideal I of R.*

Consequently,

$$r\text{-gl.dim}(R) = \sup\{\text{pd}_R R/I \mid I \textit{ is a regular ideal of } R\}.$$

Proof. The equivalence of (1)-(6) follows by Theorem 3.2.

(4) $\Leftrightarrow$ (7) and (6) $\Leftrightarrow$ (8): Trivially hold. $\square$

Proposition 4.3. *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be a direct product of rings. Then* $r\text{-gl.dim}(R) = \max\limits_{1 \leq i \leq n}\{r\text{-gl.dim}(R_i)\}$.

Proof. Let I be a regular ideal of R. Then $I \cong I_1 \times I_2 \times \cdots \times I_n$, where each I_i is a regular ideal of R_i. And the converse is also true. Consequently, the result follows by $\text{pd}_R R/I = \max\limits_{1 \leq i \leq n}\{\text{pd}_{R_i} R_i/I_i\}$ and Theorem 4.2. $\square$

The following result easily follows by Proposition 4.3 and Proposition 2.4.

Corollary 4.4. *Let $R \cong D_1 \times D_2 \times \cdots \times D_n$ be a direct product of integral domains. Then r-gl.dim$(R) =$ gl.dim(R).*

Let R be a ring and M an R-module. Some examples of non-integral domains can be constructed by the idealization $R(+)M$ (see [1]). Let $R(+)M$ be isomorphic to $R \oplus M$ as R-modules. Define

(1) $(r, m) + (s, n) = (r + s, m + n)$,
(2) $(r, m)(s, n) = (rs, sm + rn)$.

Then $R(+)M$ becomes a commutative ring with identity $(1, 0)$ under this definition.

Proposition 4.5. *Let D be an integral domain, Q its quotient field and V a Q-vector space. Then r-gl.dim$(D(+)V) =$ gl.dim(D).*

Proof. Let $R = D(+)V$. Suppose gl.dim$(D) \leq n$. Let M be an R-module. Then M is naturally a D-module. Suppose J is a regular ideal of R. Then it follows by [1, Corollary 3.4] that $J = I(+)V$, where I is a nonzero ideal of D. Since R is a flat D-module, it follows by [2, Proposition 4.1.3] that

$$\mathrm{Ext}_R^{n+1}(R/J, M) \cong \mathrm{Ext}_R^{n+1}(D/I \otimes_D R, M) \cong \mathrm{Ext}_D^{n+1}(D/I, M) = 0.$$

Hence, r-gl.dim$(D(+)V) \leq$ gl.dim(D).

On the other hand, suppose r-gl.dim$(R) \leq m$. Let N be a D-module. Then for any $(a, b) \in R$ and $t \in N$, define $(a, b)t = at$. Then N is naturally an R-module. Let I be a nonzero ideal of D. Then $J = I(+)V$ is a regular ideal of R. So, by [2, Proposition 4.1.3] again, we have

$$\mathrm{Ext}_D^{m+1}(D/I, N) \cong \mathrm{Ext}_R^{m+1}(D/I \otimes_D R, N) \cong \mathrm{Ext}_R^{m+1}(R/J, N) = 0.$$

Consequently, r-gl.dim$(D(+)V) \geq$ gl.dim(D). Thus

$$r\text{-gl.dim}(D(+)V) = \text{gl.dim}(D). \qquad \square$$

It is well-known that a ring R is a semi-simple ring if and only if its global dimension gl.dim$(R) = 0$. Next we characterize the rings R with r-gl.dim$(R) = 0$.

Theorem 4.6. *Let R be a ring. Then the following statements are equivalent.*

(1) *r-gl.dim$(R) = 0$.*
(2) *Every R-module is reg-injective.*
(3) *R/I is a projective R-module for any regular ideal I of R.*
(4) *If T is a torsion module, then $T = 0$.*
(5) *R is a total ring of quotients (equivalent, $\mathrm{T}(R) = R$).*
(6) *$\mathrm{T}(R)$ is a projective R-module.*

Proof. The equivalence (1)-(3) follows by Theorem 4.2.

(3) $\Rightarrow$ (5): Let I be a regular ideal of R and $a \in I$ a regular element. Considering the short exact sequence $0 \to Ra \to R \to R/Ra \to 0$, we have

R/Ra is projective. It follows by [5, Chapter I, Proposition 1.10] that Ra is an idempotent ideal. Since a is regular, we have a is a unit. Hence $I = R$.

(5) $\Rightarrow$ (4): Let T be a torsion module. Then for any element $t \in T$, there exists a regular element r satisfying $rt = 0$. So $t = 0$ by (5).

(4) $\Rightarrow$ (5): Let I be a regular ideal of R. Then R/I is a torsion module. So $I = R$ by (4).

(5) $\Rightarrow$ (3): Let I be a regular ideal of R. Then $I = R$. So $R/I = 0$ is a projective module.

(5) $\Rightarrow$ (6): Trivial.

(6) $\Rightarrow$ (5): Suppose $\mathrm{T}(R)$ is a projective R-module. It follows by the projective basis lemma (see [10, Theorem 2.3.6]) that there exist $\{\frac{r_i}{s_i}\} \subseteq \mathrm{T}(R)$ and $\{f_i\} \subseteq \mathrm{Hom}_R(\mathrm{T}(R), R)$ such that for any $\frac{r}{s} \in \mathrm{T}(R)$ we have

$$\frac{r}{s} = \sum_i f_i(\frac{r}{s})\frac{r_i}{s_i} = \sum_i f_i(\frac{rs_i}{ss_i})\frac{r_i}{s_i} = \sum_i s_i f_i(\frac{r}{ss_i})\frac{r_i}{s_i} = \sum_i f_i(\frac{r}{ss_i})r_i \in R.$$

So $\mathrm{T}(R) = R$ and hence R is a total ring of quotients. □

Remark 4.7. Let D be an integral domain and Q the quotient field of D. Set $R = D(+)Q/D$. Then the set of all zero-divisors of R

$$\mathrm{Z}(R) = \{(r, m) \mid r \in \mathrm{Z}(D) \cup \mathrm{Z}(K/D)\} = R - \mathrm{U}(D)(+)K/D = R - \mathrm{U}(R),$$

where $\mathrm{U}(-)$ represents its set of all units. So R is a total ring of quotients. Hence r-gl.dim$(R) = 0$ by Theorem 4.6.

Let R be a ring and I a regular ideal of R. Set $I^{-1} = \{q \in \mathrm{T}(R) \mid Iq \subseteq R\}$. If $II^{-1} = R$, then I is said to be an invertible ideal. Trivially, invertible ideals are finitely generated. Recall from [6], a ring R is said to be a Dedekind ring if every regular ideal of R is invertible. The following result is well-known and we give a proof for completeness.

Proposition 4.8. *Let I be a regular ideal of R. Then I is invertible if and only if I is projective. Consequently, any regular projective ideal is finitely generated.*

Proof. Let I be a regular ideal of R. Suppose I is invertible. Then there exist $a_i \in I$ and $b_i \in \mathrm{T}(R)$ with $i = 1, \ldots, n$ such that $\sum_{i=1}^n a_i b_i = 1$. We also denote by $b_i \in \mathrm{Hom}_R(I, R)$ the multiplication by b_i. Then $\sum_{i=1}^n a_i b_i(r) = r\sum_{i=1}^n a_i b_i = r$ for any $r \in R$. It follows by the projective basis lemma that I is a finitely generated projective ideal. On the other hand, suppose I is a projective regular ideal of R. Then, by the projective basis lemma, there exist elements $\{a_i \in I\}$ and $\{f_i\} \subseteq \mathrm{Hom}_R(I, R)$ such that

(1) if $x \in I$, then almost all $f_i(x) = 0$;
(2) if $x \in I$, then $x = \sum f_i(x)$.

Since I is regular, then there is a regular element a in I. Let $I_0 = Ra$. Set $g_i = f_i \circ \delta$, where $\delta : I_0 \hookrightarrow I$ is the natural embedding map. Set $g_i(a) = y_i$. Then there are finite elements $i = 1, \dots, m$ such that $y_i \neq 0$. Thus $g_i = \frac{y_i}{a} \in \mathrm{T}(R)$ for each $i = 1, \dots, m$. So

$$a = \sum_{i=1}^{m} a_i g_i(a) = \sum_{i=1}^{m} a_i y_i = \sum_{i=1}^{m} a_i g_i a.$$

It follows that $a(1 - \sum_{i=1}^{m} a_i g_i) = 0$. Since a is regular, we have $\sum_{i=1}^{m} a_i g_i = 1$. Consequently, I is invertible. □

Remark 4.9. It follows from Proposition 4.8 that regular projective ideals are always finitely generated. Recall that an ideal I of a ring R is said to be dense if $Ir = 0$ with $r \in R$ implies $r = 0$. Note that dense projective ideals are not always finitely generated in general. Indeed, let $R = \prod_{i=1}^{\infty} \mathbb{F}_2$ be the countably infinite direct product of copies of the finite field $\mathbb{F}_2$, and $e_i = (1, \dots, 1, 0, \dots)$, where the sequence of 1's has length i. Let I be the ideal generated by all e_i's. It follows by [4, 05WH] that I is a dense projective ideal, which is obviously not finitely generated.

Next we give a homological characterization of Dedekind rings in terms of regular global dimensions.

Theorem 4.10. *Let R be a ring. Then the following statements are equivalent.*

(1) r-gl.dim$(R) \leq 1$.
(2) *every quotient of reg-injective R-module is reg-injective.*
(3) *every quotient of injective R-module is reg-injective.*
(4) *every regular ideal of R is projective.*
(5) *R is a Dedekind ring.*

Proof. The equivalence of (1)-(4) follows by Theorem 4.2.

(4) $\Leftrightarrow$ (5): This follows by Proposition 4.8. □

Remark 4.11. Let R be a ring. Then trivially we have r-gl.dim$(R) \leq$ gl.dim(R). In fact, r-gl.dim(R) and gl.dim(R) can vary arbitrarily large. Let $R = \mathbb{Z}_{p^2}$ be the residue class ring, where p is a prime. Then R is a total ring of quotients. So r-gl.dim$(R) = 0$. However, gl.dim$(R) = \infty$. On the other hand, suppose $S = k^{\aleph_n}$ is a direct product of $\aleph_n$ copies of a field k. Then S is a von Neumann regular ring, so is a total ring of quotients. Hence r-gl.dim$(S) = 0$. However, it follows by [9, Theorem 2.51] that if the cardinal axiom $2^{\aleph_n} = \aleph_m$ holds, then gl.dim$(S) = m + 1$.

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