Commun. Korean Math. Soc. **39** (2024), No. 1, pp. 71–77
https://doi.org/10.4134/CKMS.c230111
pISSN: 1225-1763 / eISSN: 2234-3024

SOME REMARKS ON S-VALUATION DOMAINS

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Abstract. Let A be a commutative integral domain with identity element and S a multiplicatively closed subset of A. In this paper, we introduce the concept of S-valuation domains as follows. The ring A is said to be an S-valuation domain if for every two ideals I and J of A, there exists $s \in S$ such that either $sI \subseteq J$ or $sJ \subseteq I$. We investigate some basic properties of S-valuation domains. Many examples and counterexamples are provided.

1. Introduction

In this article, all rings are assumed to be commutative with identity element $1 \neq 0$, a multiplicative subset S of a ring A is a nonempty subset of A such that $1 \in S$, $0 \notin S$ and for every $a, b \in S$, we have $ab \in S$. An integral domain with quotient field K is called a valuation domain if for every two ideals I and J of A, we have either $I \subseteq J$ or $J \subseteq I$, equivalentely for every $x \in K$, we have either $x \in A$ or $x^{-1} \in A$. Valuation domains have an important role in the ring theory, specially in the construction of examples and counterexamples (see for example [3, 4]).

On the other hand, let A be a ring and $S \subseteq A$ a multiplicative set. In [1], Anderson and Dumitrescu defined an ideal I of A to be S-finite if there exist $s \in S$ and a finitely generated ideal $F \subseteq I$ of A such that $sI \subseteq F$. The ring A is called S-Noetherian if each ideal of A is S-finite. Note that if A is a Noetherian ring, then it is S-Noetherian for each multiplicative subset S of A, but the converse is false (take any non-Noetherian integral domain D and take $S = D \setminus \{0\}$). S-Noetherian rings have been studied by several authors. For more results we invite the reader to visit [1, 2, 5, 6]. Lately, many authors have made their intention to generalize other properties and notions, for example S-Noetherian spectrum and S-prime ideals (see [5]), S-Artinain rings, S-multiplication rings and module (see [2]), etc. Motivated by these generalizations, we would like to make the light on the S-version of valuation domain.

Received May 8, 2023; Revised June 1, 2023; Accepted June 29, 2023.
2020 *Mathematics Subject Classification.* Primary 13B25, 13E05, 13A15.
Key words and phrases. S-valuation domain, valuation domain, S-Noetherian.

An integral domain A is said to be an S-valuation domain, for a given multiplicatively closed subset S of A, if for every two ideals I and J of A, there exists $s \in S$ such that either $sI \subseteq J$ or $sJ \subseteq I$. We start by comparing the class of valuation domains and the class of S-valuation domains. In fact, we give examples of S-valuation domains which are not valuation domains. Note that any valuation domain A is an S-valuation domain for each multiplicative subset S of A. Let A be an integral domain with quotient field K and S a multiplicative subset of A. We show that if A is an S-valuation domain, then for each $x \in K$, there exists $s \in S$ such that either $sx \in A$ or $sx^{-1} \in A$. But the converse is false. It is well known that A is a Noetherian domain if and only if A is an $(A \setminus M)$-Noetherian domain for every maximal ideal M of A (see [2]). It is natural to ask if there is an analogous result for valuation domains. In fact, in the case of valuation domain we have only one implication which is the direct implication. For the other implication, we show by an example that it is not true in general (see Example 2.8). Also, we show that if A is an S-valuation domain, then A_S is a valuation domain, but the converse is not true in general (see Example 2.14).

2. Main results

We start by the following definition, which is the natural generalization of valuation domain as we show in Example 2.1.

Definition. Let A be an integral domain and S a multiplicative subset of A.

(1) Let I and J be two ideals of A. We say that I and J are S-comparable if there exists $s \in S$ such that either $sI \subseteq J$ or $sJ \subseteq I$.
(2) The ring A is said to be an S-valuation domain if all its ideals are S-comparable.

Example 2.1. (1) A valuation domain A is an S-valuation domain for every multiplicative subset S of A.

(2) The converse of (1) is false. Let A be an integral domain and $S = A \setminus \{0\}$. Then A is an S-valuation domain. Indeed, let I and J be two ideals of A. If $I = \{0\}$, then clearly $1.I \subseteq J$. If $I \neq \{0\}$, let $0 \neq s \in I$. Thus $sJ \subseteq sA \subseteq I$, where $s \in S$. Note that A is not necessary a valuation domain.
(3) Let $A = \mathbb{Z}$ and $S = \mathbb{Z} \setminus p\mathbb{Z}$, where p is a prime number. Then A is an S-valuation domain which is not a valuation domain. Indeed, let I and J be two ideals of A. If $I = \{0\}$ or $J = \{0\}$, then the inclusion is clear. Elsewhere, set $I = n\mathbb{Z}$ and $J = m\mathbb{Z}$, with $n, m \in \mathbb{Z} \setminus \{0\}$. Let k and l be the greatest integers such that $p^k \mid n$ and $p^l \mid m$ in $\mathbb{Z}$. Set $n = p^k d$ and $m = p^l r$ with $d, r \in \mathbb{Z}$. It is clear that $d, r \in S$. Without loss of generality, we can assume that $k \leq l$. Hence $dJ = d(m\mathbb{Z}) = dp^k p^{l-k} r\mathbb{Z} = n(p^{l-k} r\mathbb{Z}) \subseteq n\mathbb{Z} = I$.

Proposition 2.2. *Let A be an integral domain with quotient field K and S a multiplicative subset of A. Consider the following statements*:

(1) *The ring A is an S-valuation domain.*
(2) *For every $a, b \in A$, the ideals $\langle a\rangle$ and $\langle b\rangle$ are S-comparable.*
(3) *For each $x \in K$, there exists $s \in S$ such that $sx \in A$ or $sx^{-1} \in A$.*

Then $(1) \Longrightarrow (2) \Longleftrightarrow (3)$.

Proof. "$(1) \Longrightarrow (2)$" Clear. "$(2) \Longrightarrow (3)$" Let $x \in K$. If $x = 0$, it is clear. Otherwise, set $x = \frac{a}{b}$ with $a, b \in A \setminus \{0\}$. By hypothesis, there exists $s \in S$ such that $s\langle a\rangle \subseteq \langle b\rangle$ or $s\langle b\rangle \subseteq \langle a\rangle$. It yields that $sa = b\alpha$ or $sb = a\alpha$, where $\alpha \in A$. Thus $sx = \frac{sa}{b} = \alpha \in A$ or $sx^{-1} = \frac{sb}{a} = \alpha \in A$. "$(3) \Longrightarrow (2)$" Let $a, b \in A \setminus \{0\}$. There exists $s \in S$ such that $s\frac{a}{b} \in A$ or $s\frac{b}{a} \in A$. Hence $sa \in bA$ or $sb \in aA$. □

Example 2.3. (1) Let A be a UFD (unique factorization domain), $a, b \in A$ a non-associated irreducible elements and S a multiplicative subset contained in $A \setminus (aA \cup bA)$. Then A is not an S-valuation domain. Indeed, if A is an S-valuation domain, by Proposition 2.2, there exists $s \in S$ such that $sa \in bA$ or $sb \in aA$. Since A is a UFD and a and b are two non-associated irreducible elements, we have $s \in bA$ or $s \in aA$. Therefore, $s \in aA \bigcup bA$, which is absurd. Hence A is not an S-valuation domain.

(2) Let $A = \mathbb{Z}$, $n \geq 2$ an integer and $S = \{n^k,\ k \geq 0\}$. There are two different prime numbers $p, q \in \mathbb{N}$ such that n is neither divisible by p nor by q. Thus $n \notin pA \cup qA$. It follows that $S \subseteq A \setminus (pA \cup qA)$. By (1), A is not an S-valuation domain.

(3) Let K be a field of cardinality greater than or equal to 3, $A = K[X]$ and $S = \{X^n,\ n \geq 0\}$. Let $a, b \in K$ be two nonzero elements such that $a \neq b$. Then $X - a$ and $X - b$ are two non-associated irreducible elements of A and $S \subseteq A \setminus ((X-a)A \cup (X-b)A)$. By (1), A is not an S-valuation domain.

(4) The implication "$(3) \Longrightarrow (1)$" of Proposition 2.2 is false in general. Indeed, let $A = \mathbb{Z} + X\mathbb{Q}[[X]]$, $S = \mathbb{Z} \setminus \{0\}$, $I = \langle \frac{X}{2^n},\ n \geq 1\rangle$ and $J = \langle \frac{X}{3^n},\ n \geq 1\rangle$. Assume that A is an S-valuation domain. Then there exists $k \in S$ such that $kI \subseteq J$ or $kJ \subseteq I$. If $kI \subseteq J$, thus for every $n \geq 1$, $k\frac{X}{2^n} \in J$. Let $n \geq 1$. We have $k\frac{X}{2^n} = \sum_{i=1}^{l} \frac{X}{3^i} f_i(X)$, where $f_1(X), \dots, f_l(X) \in A$. Hence $k\frac{1}{2^n} = \sum_{i=1}^{l} \frac{1}{3^i} f_i(X)$. Therefore, $k\frac{3^l}{2^n} = \sum_{i=1}^{l} 3^{l-i} f_i(0) \in \mathbb{Z}$, which is absurd. By the same way, we show that $kJ \nsubseteq I$. Consequently, the ring A is not an S-valuation domain.

On the other hand, it is clear that $qf(A) = \mathbb{Q}((X))$. Let $0 \neq f \in qf(A)$. There exist $n \in \mathbb{Z}$ and $g \in \mathbb{Q}[[X]]$ with $g(0) \neq 0$ such that $f = X^n g$. If $n \geq 0$, there exists $0 \neq k \in \mathbb{Z}$ such that $kg(0) \in \mathbb{Z}$. Therefore, $kf = X^n(kg) \in A$. If $n \leq 1$, then $f^{-1} = X^{-n}g^{-1} \in X\mathbb{Q}[[X]] \subseteq A$.

Proposition 2.4. *Let A be an integral domain and S a multiplicative subset of A. If A is an S-valuation domain, then the set of prime ideals of A disjoint with S has a greatest element.*

Proof. Let $\mathcal{F}$ be the set of all ideals of A disjoint with S. We have $\mathcal{F} \neq \emptyset$ because $\{0\} \in \mathcal{F}$. It is well-known that $(\mathcal{F}, \subseteq)$ has a maximal element P which is also a prime ideal of A.

Now, we are going to prove that P is the greatest prime ideal of A disjoint with S. Let Q be a prime ideal of A disjoint with S. Since A is an S-valuation domain, there exists $s \in S$ such that $sP \subseteq Q$ or $sQ \subseteq P$. Thus $P \subseteq Q$ or $Q \subseteq P$. In the case $P \subseteq Q$, we have $P = Q$ because P is maximal in $\mathcal{F}$ and $Q \in \mathcal{F}$, which ends the proof. □

Proposition 2.5. *Let A be a domain and S a multiplicative subset of A. If A is an S-valuation domain, then for every $P \in spec(A)$ such that $P \cap S = \emptyset$, the ring A/P is an $\bar{S}$-valuation domain, where $\bar{S} = \{\bar{s} \in A/P,\ s \in S\}$.*

Proof. The ideals of A/P are of the form I/P with I is an ideal of A containing P. Let I and J be two ideals of A containing P. Then there exists $s \in S$ such that $sI \subseteq J$ or $sJ \subseteq I$. Therefore, $\bar{s}(I/P) \subseteq J/P$ or $\bar{s}(J/P) \subseteq I/P$. □

Definition. Let A be a domain and S a multiplicative subset of A. The saturation of S is defined as follow:

$$S' = \{a \in A : \text{there exists } s \in S \text{ such that } a \mid s \text{ in } A\}.$$

Proposition 2.6. *Let A be an integral domain and S a multiplicative subset of A. Then A is an S-valuation domain if and only if A is an S'-valuation domain.*

Proof. "$\Longrightarrow$" Clear since $S \subseteq S'$. "$\Longleftarrow$" Let I and J be two arbitrary ideals of A. By hypothesis, there exists $a \in S'$ such that $aI \subseteq J$ or $aJ \subseteq I$. Let $s \in S$ such that $a \mid s$ in A. Set $s = ab$ with $b \in A$. Hence $sI = abI \subseteq aI \subseteq J$ or $sJ = abJ \subseteq aJ \subseteq I$. □

Let S be a multiplicative subset of a domain A. In Proposition 2.4, it is shown that if A is an S-valuation domain, then the set of prime ideals of A disjoint with S has a greatest element. Now, we are going to give a characterization of this ideal.

Proposition 2.7. *Let S be a multiplicative subset of a domain A. If A is an S-valuation domain, then $M = A \setminus S'$ is the greatest prime ideal of A disjoint with S.*

Proof. Since $S \subseteq S'$, we have $M \cap S = \emptyset$. We are going to show that M is an ideal of A.

Let $a, b \in M$ and $x \in A$. If $ax \notin M$, then $ax \in S'$. It yields that there exists $s \in S$ such that $ax \mid s$ in A. Consequently, $a \mid s$ in A. Hence $a \in S'$, which is absurd. Thus $ax \in M$.

It yields that there exists $s \in S$ such that $s\langle a\rangle \subseteq \langle b\rangle$ or $s\langle b\rangle \subseteq \langle a\rangle$. Without loss of generality we can assume that $s\langle a\rangle \subseteq \langle b\rangle$. Set $sa = br$ with $r \in A$. It follows that $s(a+b) = sa + sb = br + sb = (r+s)b \in M$. If $a+b \notin M$, then $a+b \in S'$: a multiplicative set. Thus $s(a+b) \in S' \cap M$, which is absurd. Thus $a+b \in M$.

Since S' is a multiplicative subset of A, M is a prime ideal of A.

Now, we show that M is the greatest prime ideal of A disjoint with S. Let P be the greatest prime ideal of A disjoint with S. Then $M \subseteq P$. If $P \neq M$, then there exists $a \in P \setminus M$. Consequently, $a \in S'$. It implies that there exists $s \in S$ such that $a \mid s$ in A. Thus $s = ab$ with $b \in A$. Therefore, $s \in P \bigcap S$: absurd. Consequently, $P = M$. $\square$

Let S be a multiplicative subset of a domain A. Recall that an ideal I of A is said to be S-finite if there exist $s \in S$ and a finitely generated ideal F of A such that $sI \subseteq F \subseteq I$. The ring A is called S-Noetherian if all its ideals are S-finite. In [1], the authors have shown the following equivalence: The ring A is Noetherian if and only if it is $(A \setminus M)$-Noetherian for every $M \in Max(A)$. Now the question is, is the equivalence between "A is a valuation domain" and "A is $(A \setminus M)$-valuation" for every $M \in Max(A)$ true? In the next example, we give a negative answer of this question.

Example 2.8. Let $A = \mathbb{Z}$, p be a prime number and $S = \mathbb{Z} \setminus p\mathbb{Z}$. By Example 2.1, A is an S-valuation domain. Since the maximal ideals of $\mathbb{Z}$ are of the form $p\mathbb{Z}$ with p a prime number, A is an $(A \setminus M)$-valuation domain for every $M \in Max(A)$. But A is not a valuation domain.

Example 2.9. Let A be a principal ideal domain. Then for each $M \in Max(A)$, the ring A is $(A \setminus M)$-valuation. Indeed, let M be a maximal ideal of A. Set $M = pA$ with p an irreducible element of A. Let $I = xA$ and $J = yA$ be ideals of A. Then $x = p^k x'$ and $y = p^l y'$ with p does not divide x' and y' and $k, l \in \mathbb{N}$. It is clear that $x', y' \in A \setminus M$. If $k \leq l$, then $x'J = x'(yA) = (x'p^k)(p^{l-k}y'A) \subseteq xA = I$. Again, if $l \leq k$, then $y'I \subseteq J$. Thus A is an $(A \setminus M)$-valuation domain. In particular, if A is a PID which is not a valuation (for example $\mathbb{Z}$), it is an $(A \setminus M)$-valuation domain for every maximal ideal M of A.

Let A be an integral domain and S a multiplicative subset of A. Recall from [1] that an ideal I of A is said to be S-principal if there exist $s \in S$ and $a \in I$ such that $sI \subseteq aA$. The ring A is called S-principal if each ideal of A is S-principal.

Lemma 2.10. *Let A be an integral domain and S a multiplicative subset of A. If A is an S-valuation domain, then each S-finite ideal I of A is S-principal.*

Proof. Let I be an S-finite ideal of A. There exist $s \in S$ and $a_1, \dots, a_n \in I$ such that $sI \subseteq \langle a_1, \dots, a_n\rangle$. By induction on n, assume that $n = 2$. Since A is an S-valuation domain, there exists $t \in S$ such that $ta_1 \in a_2A$ or $ta_2 \in a_1A$. It yields that $t\langle a_1, a_2\rangle \subseteq a_2A$ or $t\langle a_1, a_2\rangle \subseteq a_1A$. Now, let $n \geq 2$ and $2 \leq k \leq n-1$, assume that there exist $t \in S$ and $1 \leq i \leq k$ such that $t\langle a_1, \ldots, a_k\rangle \subseteq a_iA$. As A is an S-valuation domain, there exists $r \in S$ such that $r\langle a_i, a_{k+1}\rangle \subseteq a_iA$ or $r\langle a_i, a_{k+1}\rangle \subseteq a_{k+1}A$. Thus $(rt)\langle a_1, \ldots, a_{k+1}\rangle \subseteq r\langle a_i, a_k\rangle \subseteq a_iA$ or $(rt)\langle a_1, \ldots, a_{k+1}\rangle \subseteq r\langle a_i, a_k\rangle \subseteq a_{k+1}A$. Therefore, there exist $t \in S$ and $1 \leq i \leq n$ such that $t\langle a_1, \ldots, a_n\rangle \subseteq a_iA$. Hence $(ts)I \subseteq t\langle a_1, \ldots, a_n\rangle \subseteq a_iA$ with $ts \in S$. □

Theorem 2.11. *Let A be an integral domain with quotient field K and S a multiplicative subset of A. If A is an S-Noetherian ring, then A is an S-valuation domain if and only if it is S-principal and for each $x \in K$, there exists $s \in S$ such that either $sx \in A$ or $sx^{-1} \in A$.*

Proof. "$\Longrightarrow$" By the previous lemma, A is an S-principal domain. By Proposition 2.2, for each $x \in K$ there exists $s \in S$ such that either $sx \in A$ or $sx^{-1} \in A$.

"$\Longleftarrow$" Let I and J be two nonzero ideals of A. Since A is an S-principal domain, there exist $s, t \in S$, $a \in I$ and $b \in J$ such that $sI \subseteq aA$ and $tJ \subseteq bA$. By hypothesis, there exists $r \in S$ such that either $r\frac{a}{b} \in A$ or $r\frac{b}{a} \in A$ (we have $a \neq 0$ and $b \neq 0$ because I and J are two nonzero ideals and A is integral). Therefore, $ra \in bA$ or $rb \in aA$. Thus $(rs)I \subseteq r(aA) \subseteq bA \subseteq J$ or $(rt)J \subseteq r(bA) \subseteq aA \subseteq I$. Consequently, A is an S-valuation domain. □

Example 2.12. Let $A = \mathbb{Z} + X\mathbb{Q}[[X]]$, $S = \{X^n,\ n \geq 0\}$ and I an ideal of A. There exists $f \in I$ such that $I\mathbb{Q}[[X]] = f\mathbb{Q}[[X]]$. Then $XI = XIA \subseteq XI\mathbb{Q}[[X]] = Xf\mathbb{Q}[[X]] = fX\mathbb{Q}[[X]] \subseteq fA \subseteq I$. Thus A is an S-principal ideal domain. Let $0 \neq f \in A$. Set $f = \sum_{i=k}^{+\infty} a_iX^i$ with $k \geq 0$ and $a_k \neq 0$. Then $f = X^kg$, where $g = \sum_{i=k}^{+\infty} a_iX^{i-k}$. As $g(0) = a_k \neq 0$. Thus $\frac{1}{g} \in \mathbb{Q}[[X]]$. Therefore, $X\frac{1}{g} \in A$. It shows that $X^{k+1}\frac{1}{f} = X\frac{1}{g} \in A$. Now, let $f, g \in A \setminus \{0\}$. By the previous part, there exists $k \geq 1$ such that $X^k\frac{1}{g} \in A$. Thus $X^k\frac{f}{g} \in A$ with $X^k \in S$. By the previous theorem, A is an S-valuation domain.

Proposition 2.13. *Let A be an integral domain and S a multiplicative subset of A. If A is an S-valuation domain, then A_S is a valuation domain.*

Proof. Set $K = qf(A)$. As A is an S-valuation domain, by Proposition 2.2, for every $x \in K$, there exists $s \in S$ such that $sx \in A$ or $sx^{-1} \in A$. Since $S \subseteq U(A_S)$, for each $x \in K$, we have $x \in A_S$ or $x^{-1} \in A_S$. It follows that A_S is a valuation domain. □

Example 2.14. The converse of Proposition 2.13 is false. Indeed, let $A = \mathbb{Z} + X\mathbb{Q}[[X]]$ and $S = \mathbb{Z} \setminus \{0\}$. By (4) of Example 2.3, for every $f \in qf(A)$, there exists $s \in S$ such that $sf \in A$ or $s\frac{1}{f} \in A$. Thus for each $f \in qf(A_S) = qf(A)$, either $f \in A_S$ or $\frac{1}{f} \in A_S$. Therefore, A_S is a valuation domain. By (4) of Example 2.3, A is not an S-valuation domain.

Acknowledgements. The authors would like to thank the referee for his/her valuable comments which helps us improve the presentation of our paper.

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