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ON STRONGLY QUASI J-IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity. In this paper, we introduce a new class of ideals called the class of strongly quasi J-ideals lying properly between the class of J-ideals and the class of quasi J-ideals. A proper ideal I of R is called a strongly quasi J-ideal if, whenever a, $b \in R$ and $ab \in I$, then $a^2 \in I$ or $b \in \mathrm{Jac}(R)$. Firstly, we investigate some basic properties of strongly quasi J-ideals. Hence, we give the necessary and sufficient conditions for a ring R to contain a strongly quasi J-ideals. Many other results are given to disclose the relations between this new concept and others that already exist. Namely, the primary ideals, the prime ideals and the maximal ideals. Finally, we give an idea about some strongly quasi J-ideals of the quotient rings, the localization of rings, the polynomial rings and the trivial rings extensions.

1. Introduction

Throughout this paper all rings are commutative with $1 \neq 0$. Let R be a ring and I be an ideal of R. Then, $\mathrm{Nil}(R) := \sqrt{0}$ denotes the nil-radical of R, $\mathrm{Jac}(R)$ denotes the Jacobson radical of R, $\mathrm{Z}(R)$ denotes the set of zero divisors of R, and $\sqrt{I}$ denotes the radical of I.

In [16], Tekir et al. introduced the concept of n-ideals. A proper ideal I of a ring R (i.e., $I \neq R$) is called an n-ideal if, whenever a, $b \in R$ with $ab \in I$, then $a \in I$ or $b \in \mathrm{Nil}(R)$. Recently, Khashan and Bani-Ata in [10] introduced the concept of J-ideals as a generalization of n-ideals. A proper ideal I of a ring R is called a J-ideal if, whenever a, $b \in R$ with $ab \in I$, then $a \in I$ or $b \in \mathrm{Jac}(R)$. The concept of J-ideals is generalized many different ways (see [11, 12, 17]). One of the most recent generalizations of J-ideals is the concept of quasi J-ideals introduced by Khashan and Yetkin in [12]. A proper ideal I of a ring R is called a quasi J-ideal if $\sqrt{I}$ is a J-ideal.

In this paper, we investigate a new class of ideals called the class of strongly quasi J-ideals which is a sub-class of quasi-J-ideals and an over-class of J-ideals

$$\{J\text{-ideals}\} \subseteq \{\text{strongly quasi } J\text{-ideals}\} \subseteq \{\text{quasi } J\text{-ideals}\}.$$

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The first goal of this paper is to show that these inclusion may be strict (see Example 2.1 and Example 2.2). Among other results, it is proved that a ring R has a strongly quasi J-ideal if and only if there exits a prime ideal P such that $P \subseteq \mathrm{Jac}(R)$ (Theorem 2.8). In Theorem 2.11, we show that every proper ideal (resp. (nonzero) principal ideal, primary ideal, prime ideal, maximal ideal) of a ring R is a strongly quasi J-ideal if and only if R is a quasi-local ring. After that, we characterize the (Noetherian) rings over which every strongly quasi J-ideal is primary (resp. prime, maximal) ideal (Theorems 2.14, 2.16, 2.17). In the last part of this paper, we give an idea about some strongly quasi J-ideals of the quotient rings, the localization of rings, the polynomial rings and the trivial rings extensions.

Let M be a unitary R-module. Recall that the trivial ring extension of R by M is the ring $R := R \propto M$, where the underlying group is $R \times M$ and the multiplication is defined by $(a, m)(b, m') = (ab, am' + bm)$. It is also called the (Nagata) idealization of M over R and is denoted by $R(+)M$. This construction was first introduced, in 1962, by Nagata [15] with the objective to emphasize the interaction between rings and their modules and, more importantly, to provide numerous families of examples of rings with zero-divisors (for more details see [3]).

2. Strongly quasi *J*-ideals

Definition. Let R be a ring. A proper ideal I of R is called a strongly quasi J-ideal if, whenever a, $b \in R$ and $ab \in I$, then $a^2 \in I$ or $b \in \mathrm{Jac}(R)$.

Clearly, every J-ideal is a strongly quasi J-ideal, and every strongly quasi J-ideal is a quasi J-ideal. However, the next examples show that the converse implications are not true in general.

Example 2.1. Consider the ideal $I = 0 \propto \overline{0}$ of the ring $R = \mathbb{Z} \propto \mathbb{Z}/2\mathbb{Z}$. It is clear that I is a strongly quasi J-ideal of R. However, I is not a J-ideal of R. Indeed, $(2, \overline{0})(0, \overline{1}) = (0, \overline{0}) \in I$, but $(0, \overline{1}) \notin I$ and $(2, \overline{0}) \notin \mathrm{Jac}(R) = 0 \propto \mathbb{Z}/2\mathbb{Z}$.

Example 2.2. Consider the ring $R = \mathbb{Z} + 3X\mathbb{Z}[[X]]$. By [8, Proposition 1.3] and [8, Proposition 1.7], $\mathrm{Jac}(R) = 3X\mathbb{Z}[[X]]$ is a prime ideal of R. Set $I := \mathrm{Jac}(R)^3$. It is clear I is a quasi J-ideal of R. However, $27X^3 \in I$, but neither $(3X^3)^2 \in I$ nor $9 \in \mathrm{Jac}(R)$. Thus, I is not a strongly quasi J-ideal of R.

Recall from [13] that a proper ideal I of a ring R is strongly quasi-primary if, whenever a, $b \in R$ with $ab \in I$, then $a^2 \in I$ or $b \in \sqrt{I}$.

Proposition 2.3. *Let R be a ring and I be a proper ideal of R. If I is a strongly quasi J-ideal, then $I \subseteq \mathrm{Jac}(R)$. The equivalence holds if I is strongly quasi primary.*

Proof. Since every strongly quasi J-ideal is a quasi J-ideal, by [12, Proposition 1], we get $I \subseteq \mathrm{Jac}(R)$.

Let I be a strongly quasi primary ideal with $I \subseteq \mathrm{Jac}(R)$. Let $a, b \in R$ with $ab \in I$ and $b \notin \mathrm{Jac}(R)$. Then, $a^2 \in I$ since $b \notin \sqrt{I}$ and I is a strongly quasi primary ideal. Thus, I is a strongly quasi J-ideal. □

Recall from [7, 14] that a ring R is called a Hilbert ring or a Jacobson ring if every prime ideal of R is an intersection of maximal ideals of R (i.e., in every homomorphic image of R, the Jacobson radical and the nilradical coincide).

Proposition 2.4. *Let R be a Hilbert ring and I be a proper ideal of R. Then, I is a strongly quasi J-ideal of R if and only if $I \subseteq \mathrm{Nil}(R)$ and I is a strongly quasi primary ideal of R.*

Proof. Suppose that I is a strongly quasi J-ideal of R. Then, by Proposition 2.3 $I \subseteq \mathrm{Jac}(R)$. Since R is a Hilbert ring, $\mathrm{Jac}(R) = \mathrm{Nil}(R)$ and so $I \subseteq \mathrm{Nil}(R)$ (equivalent to $\sqrt{I} = \mathrm{Nil}(R)$). On the other hand, let $a, b \in R$ such that $ab \in I$ and $b \notin \sqrt{I}$. As I is a strongly quasi J-ideal and $b \notin \sqrt{I} = \mathrm{Jac}(R)$, we conclude that $a^2 \in I$. Consequently, I is a strongly quasi primary ideal with $I \subseteq \mathrm{Nil}(R)$. The converse follows directly from Proposition 2.3. □

Recall that, for a proper ideal I of a ring R, the ideal generated by squares of elements of I (i.e., $\{x^2 \mid x \in I\}$) is denoted by I_2 [2]. If 2 is a unit in R, then $I_2 = I^2$ (see [2, Theorem 5]). Next, we give some characterizations of strongly quasi J-ideals in terms of ideals.

Proposition 2.5. *Let I be a proper ideal of a ring R. Then, the following are equivalent:*

(1) *I is a strongly quasi J-ideal of R.*
(2) *For every $a \in R$, either $(a) \subseteq (I : a)$ or $(I : a) \subseteq \mathrm{Jac}(R)$.*
(3) *For any ideals A and B of R with $AB \subseteq I$ either $A_2 \subseteq I$ or $B \subseteq \mathrm{Jac}(R)$.*
(4) *For every $b \notin \mathrm{Jac}(R)$, $(I : b)_2 \subseteq I$.*

Proof. (1) ⇒ (2) Suppose that I is a strongly quasi J-ideal and $(a) \not\subseteq (I : a)$. Let $b \in (I : a)$. Then $ab \in I$. Since $a^2 \notin I$ and I is a strongly J-ideal, we conclude that $b \in \mathrm{Jac}(R)$, and so $(I : a) \subseteq \mathrm{Jac}(R)$.

(2) ⇒ (3) Let A and B be two ideals of R such that $AB \subseteq I$ and $B \not\subseteq \mathrm{Jac}(R)$. Let $b \in B \setminus \mathrm{Jac}(R)$. For any $a \in A$, we have $b \in (I : a) \setminus \mathrm{Jac}(R)$. Then, $(a) \subseteq (I : a)$, and so $a^2 \in I$. Thus, $A_2 \subseteq I$.

(3) ⇒ (4) Let $b \notin \mathrm{Jac}(R)$ and set $A = (I : b)$ and $B = (b)$. Then, we have $AB \subseteq I$. Since $B \not\subseteq \mathrm{Jac}(R)$, we get $A_2 = (I : b)_2 \subseteq I$.

(4) ⇒ (1) Let $a, b \in R$ with $ab \in I$ and $b \notin \mathrm{Jac}(R)$. Since $a \in (I : b)$, we get that $a^2 \in (I : b)_2 \subseteq I$. Thus, I is a strongly quasi J-ideal. □

Corollary 2.6. *Let R be a ring in which 2 is a unit and let I be a proper ideal of R. Then, the following are equivalent:*

(1) *I is a strongly quasi J-ideal of R.*
(2) *For every $a \in R$, either $(a) \subseteq (I : a)$ or $(I : a) \subseteq \mathrm{Jac}(R)$.*

(3) *For any ideals A and B of R with $AB \subseteq I$ either $A^2 \subseteq I$ or $B \subseteq \mathrm{Jac}(R)$.*
(4) *For every $b \notin \mathrm{Jac}(R)$, $(I : b)^2 \subseteq I$.*

Next, we give some useful facts concerning strongly quasi J-ideals. These will be used frequently in the sequel without explicit mentions.

Proposition 2.7. *Let R be a ring. Then, the following hold*:

(1) *If I and I' are two J-ideals of R, then II' is a strongly quasi J-ideal of R. In particular, the square of a J-ideal of R is a strongly quasi J-ideal.*
(2) *If I is a J-ideal of R and I' is an ideal of R containing I, then II' is a strongly quasi J-ideal of R.*
(3) *Every maximal strongly quasi J-ideal of R is prime.*
(4) *If I is a strongly quasi J-ideal of R and P is a minimal prime ideal over I, then P is a J-ideal.*

Proof. (1) Let $a, b \in R$ with $ab \in II'$ and $b \notin \mathrm{Jac}(R)$. By hypothesis, we have $a \in I \cap I'$. Hence, $a^2 \in II'$, and so II' is a strongly quasi J-ideal.

(2) Let $a, b \in R$ such that $ab \in II'$ and $b \notin \mathrm{Jac}(R)$. Since $ab \in I$ and I is a J-ideal, we conclude that $a \in I$, and so $a^2 \in II'$. Thus, II' is a strongly quasi J-ideal of R.

(3) Let I is a maximal strongly quasi J-ideal of R and let $a, b \in R$ such that $ab \in I$ and $b \notin I$. Since $\sqrt{I}$ is a J-ideal of R, by maximality of I, we have $I = \sqrt{I}$ and so I is a J-ideal. Moreover, by [10, Lemma 2.11] $(I : b)$ is a J-ideal of R which contains of I. Again, by the maximality of I, we obtain that $I = (I : b)$. Hence, $a \in I$ and so I is a prime ideal, as asserted.

(4) Let $x \in P$. Then, by [9, Theorem 2.1] there exist $y \notin P$ and a positive integer n such that $x^n y \in I$. As I is a strongly quasi J-ideal and $y^2 \notin I$, we conclude that $x^n \in \mathrm{Jac}(R)$. Thus, $x \in \mathrm{Jac}(R)$ and so $P \subseteq \mathrm{Jac}(R)$. Consequently, P is a J-ideal. □

The next result characterizes rings admitting strongly quasi J-ideals.

Theorem 2.8. *A ring R has a strongly quasi J-ideal if and only if there exists a prime ideal P of R such that $P \subseteq \mathrm{Jac}(R)$.*

Proof. If R contains a strongly quasi J-ideal, then R contains a maximal strongly quasi J-ideal P which is a prime ideal. Moreover, $P \subseteq \mathrm{Jac}(R)$ by Proposition 2.3. The converse part is clear. □

Recall that a ring R is said to be semiprimitive if $\mathrm{Jac}(R) = (0)$.
As a consequence of the previous theorem, we have the following result.

Corollary 2.9. (1) *Let R_1 and R_2 be two rings. Then, $R_1 \times R_2$ has no strongly quasi J-ideal.*
(2) *Any semiprimitive ring R, which is not a domain, has no strongly quasi J-ideal.*

Proof. (1) $\mathrm{Jac}(R_1 \times R_2) = \mathrm{Jac}(R_1) \times \mathrm{Jac}(R_2)$ does not contain any prime ideal of $R_1 \times R_2$ since every prime ideal P of $R_1 \times R_2$ has the form $P = P_1 \times R_2$ or $P = R_1 \times P_2$, where P_1 is a prime ideal of R_1 and P_2 is a prime ideal of R_2. Hence, by Theorem 2.8, $R_1 \times R_2$ has no strongly quasi J-ideal.

(2) Let R be a semiprimitive ring which is not a domain. If R admits a strongly quasi J-ideal, then by Theorem 2.8, there exists a prime ideal $P \subseteq \mathrm{Jac}(R) = (0)$. Hence, $P = (0)$ which is a contradiction. □

Proposition 2.10. *Let R be a semiprimitive ring. Then, R is a domain if and only if* (0) *is the only strongly quasi J-ideal of R.*

Proof. (⇒) Suppose that R is a domain. It is clear that (0) is a strongly quasi J-ideal of R. Let I be a strongly quasi J-ideal of R. Then $I \subseteq \mathrm{Jac}(R) = 0$. Hence, $I = (0)$ and so we conclude that (0) is the only strongly quasi J-ideal of the domain R.

(⇐) Let a, $b \in R$ such that $ab = 0$ and $b \neq 0$. Since $b \notin \mathrm{Jac}(R)$ and (0) is a strongly quasi J-ideal, we conclude that $a^2 = 0$. Thus, $a \in \mathrm{Nil}(R) = (0)$ and so R is a domain. □

The next theorem characterizes rings over which every proper ideal (resp. (nonzero) principal ideal, primary ideal, prime ideal, maximal ideal) is a strongly quasi J-ideal.

Theorem 2.11. *Let R be a ring. The following are equivalent*:

(1) *R is a quasi-local ring.*
(2) *Every proper principal ideal of R is a strongly quasi J-ideal.*
(3) *Every nonzero proper principal ideal of R is a strongly quasi J-ideal.*
(4) *Every proper ideal of R is a strongly quasi J-ideal.*
(5) *Every primary ideal of R is a strongly quasi J-ideal.*
(6) *Every prime ideal of R is a strongly quasi J-ideal.*
(7) *Every maximal ideal of R is a strongly quasi J-ideal.*

Proof. (1) ⇒ (2) Let x be a nonunit element of R and consider a, $b \in R$ such that $ab \in (x)$ and $b \notin \mathrm{Jac}(R)$. Since R is quasi-local, b is a unit. Hence, $a \in (x)$ and so $a^2 \in (x)$. Thus, (x) is a strongly quasi J-ideal.

(2) ⇒ (3) Trivial.

(3) ⇒ (4) Let I be a proper ideal of R and consider a, $b \in R$ such that $ab \in I$ and $b \notin \mathrm{Jac}(R)$. Suppose that b is a nonunit. Then, the nonzero proper ideal (b) is a strongly quasi J-ideal. So, by Proposition 2.3, $b \in (b) \subseteq \mathrm{Jac}(R)$, a contradiction. Hence, b is a unit and then $a^2 \in I$. Consequently, I is a strongly quasi J-ideal of R.

(4) ⇒ (5) ⇒ (6) ⇒ (7) Trivial.

(7) ⇒ (1) Let M be a maximal ideal of R. Then, M is a strongly quasi J-ideal. Hence, by Proposition 2.3, $M \subseteq \mathrm{Jac}(R)$, and so $M = \mathrm{Jac}(R)$. Thus, R is a quasi-local ring. □

Corollary 2.12. *Let P be a prime ideal of a ring R. Then, every proper ideal of the ring R_P is a strongly quasi J-ideal.*

Recall from [5], that a ring R is said to be a UN-ring if every nonunit element a of R is a product of a unit and a nilpotent element, or equivalently every element of R is either nilpotent or unit. That is, R is quasi-local with a maximal ideal $\mathrm{Nil}(R)$, or equivalently R is quasi-local with Krull dimension zero. A simple example of UN-rings is $\mathbb{Z}/9\mathbb{Z}$.

Corollary 2.13. *Let R be a zero-dimensional ring. The following are equivalent*:

(1) *R is a UN-ring.*
(2) *Every proper ideal of R is a strongly quasi J-ideal.*
(3) *R has a strongly quasi J-ideal.*

Proof. (1) $\Rightarrow$ (2) Follows from Theorem 2.11 since R is a quasi-local ring.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Suppose that R admits a strongly quasi J-ideal. Then, by Theorem 2.8 there exists a maximal ideal M of R such that $M \subseteq \mathrm{Jac}(R) = \mathrm{Nil}(R)$ (since R is a zero-dimensional ring). Thus, R is a quasi-local ring and so R is a UN-ring, as asserted. □

Note that the set of primary ideals (resp. prime ideals, maximal ideals) in any ring is never empty. However, by Theorem 2.8, strongly quasi J-ideals exist in a ring R only when the Jacobson radical $\mathrm{Jac}(R)$ contains a prime ideal. For that, in the next three theorems we consider this condition, otherwise they have no meaning.

Theorem 2.14. *Let R be a Noetherian ring with the nonzero prime nil-radical* $\mathrm{Nil}(R)$. *The following are equivalent*:

(1) *Every strongly quasi J-ideal of R is primary.*
(2) *R is a UN-ring.*

Proof. (1) $\Rightarrow$ (2) First we will show that (0) is a primary ideal. The ideal $\mathrm{Nil}(R)^2$ is a strongly quasi J-ideal (by Proposition 2.7(1)). Hence, it is a primary, and so a J-ideal. Again by Proposition 2.7(1), $\mathrm{Nil}(R)^3$ is a strongly quasi J-ideal, and so it is primary. So, by induction $\mathrm{Nil}(R)^n$ is primary for all integer $n \geq 1$. Since R is Noetherian, there exits an integer $m \geq 1$ such that $\mathrm{Nil}(R)^m = (0)$. Thus, (0) is primary (i.e., $\mathrm{Nil}(R) = \mathrm{Z}(R)$).

Assume that R is not a quasi-local ring. Then, by hypothesis $\mathrm{Nil}(R)$ is a non-maximal prime ideal, and so $0 \neq \mathrm{Nil}(R) \subseteq \mathrm{Jac}(R)$. Set $\mathrm{Nil}(R) = P$. Let M be a maximal ideal of R. Then, by Proposition 2.7, PM is a strongly quasi J-ideal and so it is primary with $\sqrt{PM} = P$. Let $x \in P$ and $y \in M \setminus P$. Then, $xy \in PM$. Since PM is primary and $y \notin \sqrt{PM}$, we conclude that $x \in PM$. Hence, $P \subseteq PM$ and so $PM = P$. On the other hand, P is finitely generated since R is Noetherian, and so $M + \mathrm{ann}(P) = R$ (by [6, Lemma 1.7]). Hence,

$1 = x + y$ for some $x \in M$ and $y \in \text{ann}(P)$. Let $0 \neq p \in P$. Then $p = px$. Hence, $p(1 - x) = 0$ and so $1 - x \in \text{Z}(R) \subseteq M$, a contradiction. Thus, R is a quasi-local ring. So, by hypothesis and Theorem 2.11, every proper ideal of R is primary. Hence, by [1, Corollary 2.15] R is a UN-ring or R is a domain with a unique nonzero prime ideal. Since R is not a domain, R must be a UN-ring.

(2) ⇒ (1) Clear. □

Remark 2.15. Consider the ring $R = \frac{k[[x,y]]}{(x)(x,y)}$ where k is a field. Then, R is Noetherian with the nonzero prime nil-radical $\text{Nil}(R) = (\overline{x})$ over which the ideal $I = (\overline{0})$ is a strongly quasi J-ideal which is not primary.

Theorem 2.16. *Let R be a Noetherian ring which admits a strongly quasi J-ideal. The following are equivalent*:

1. *Every strongly quasi J-ideal of R is prime.*
2. *R is a domain with the unique strongly quasi J-ideal* (0) *of R.*

Proof. (1) ⇒ (2) Let I be a strongly quasi J-ideal of R. Then, by hypothesis I is a prime ideal. As $I \subseteq \text{Jac}(R)$, then it is clear that I^2 is also a strongly quasi J-ideal of R and so it is prime. Thus, $I^2 = I$. Since R is Noetherian, by the Nakayama's lemma, we get $I = (0)$. Hence, R is a domain and (0) is the unique strongly quasi J-ideal of R.

(2) ⇒ (1) Clear. □

Theorem 2.17. *Let R be a ring which admits a strongly quasi J-ideal. Then, every strongly quasi J-ideal of R is maximal if and only if R is a field.*

Proof. Let I be a strongly quasi J-ideal of R. Since $I \subseteq \text{Jac}(R)$ and I is a maximal ideal, R is a quasi-local ring. Hence, by Theorem 2.11, every proper ideal of R is maximal. Therefore, R is a field. The converse part is clear. □

Proposition 2.18. *Let R be a ring. If $\{I_i\}_{i\in\Omega}$ is a family of strongly quasi J-ideals, then $I := \cap_{i\in\Omega} I_i$ is a strongly quasi J-ideal.*

Proof. Let a, $b \in R$ such that $ab \in I$ and $b \notin \text{Jac}(R)$. Since all I_i are strongly quasi J-ideals of R, we get $a^2 \in I_i$ (for all $i \in \Omega$), and so $a^2 \in I$. Thus, I is a strongly quasi J-ideal. □

3. Change of rings theorems for the strongly quasi *J*-ideals

Proposition 3.1. *Let $f : R_1 \to R_2$ be a ring epimorphism. Then, the following hold*:

1. *If I_1 is a strongly quasi J-ideal of R_1 containing* $\text{Ker}(f)$*, then $f(I_1)$ is a strongly quasi J-ideal of R_2.*
2. *If I_2 is a strongly quasi J-ideal of R_2 and* $\text{Ker}(f) \subseteq \text{Jac}(R_1)$*, then $f^{-1}(I_2)$ is a strongly quasi J-ideal of R_1.*

Proof. (1) Let x, $y \in R_2$ such that $xy \in f(I_1)$. Then, $x = f(a)$, $y = f(b)$ for some $a, b \in R_1$ (f is an epimorphism). Since $\mathrm{Ker}(f) \subseteq I_1$, $ab \in I_1$. Moreover, as I_1 is a strongly quasi J-ideal, we get $a^2 \in I_1$ or $b \in \mathrm{Jac}(R_1)$. Hence, $x^2 \in f(I_1)$ or $y \in f(\mathrm{Jac}(R_1)) \subseteq \mathrm{Jac}(R_2)$. Consequently, $f(I_1)$ is a strongly quasi J-ideal of R_2.

(2) Let a, $b \in R_1$ such that $ab \in f^{-1}(I_2)$ and $b \notin \mathrm{Jac}(R_1)$. First we prove that $f(b) \in \mathrm{Jac}(R_2)$. By contradiction we assume that $f(b) \in \mathrm{Jac}(R_2)$. Let $c \in R_1$. Then $1_{R_2} - f(b)f(c)$ is a unit of R_2, and so there exits $u \in R_1$ such that $(1_{R_2} - f(b)f(c))f(u) = 1_{R_2}$ (since f is an epimorphism). Then, $(1_{R_1} - bc)u - 1_{R_1} \in \mathrm{Ker}(f) \subseteq \mathrm{Jac}(R_1)$. Hence, $(1_{R_1} - bc)$ is a unit of R_1 (which is true for all $c \in R_1$). So, $b \in \mathrm{Jac}(R_1)$, a contradiction. Thus, $f(b) \in \mathrm{Jac}(R_2)$. On the other hand, since $f(ab) = f(a)f(b) \in I_2$ and I_2 is a strongly quasi J-ideal of R_2, we conclude that $f(a)^2 = f(a^2) \in I_2$. Hence, $a^2 \in f^{-1}(I_2)$, and so $f^{-1}(I_2)$ is a strongly quasi J-ideal of R_1. □

Remark 3.2. The assertion (2) of the previous proposition is not valid if one drop the hypothesis that the ring homomorphism $f : R_1 \to R_2$ is surjective or that $\mathrm{Ker}(f) \subseteq \mathrm{Jac}(R_1)$.

- For the first hypothesis, take for example the monomorphism $\iota : \mathbb{Z} \to \mathbb{Z}_5; a \to \frac{a}{1}$ which is not surjective, then $(5)_5$ is a strongly quasi J-ideal of $\mathbb{Z}_5$. However, $\iota^{-1}(5)_5 = (5)$ is not a strongly quasi J-ideal of $\mathbb{Z}$.
- For the second hypothesis, take for example the epimorphism $p : k \times k \to k; (x, y) \to x$, where k is a field with $\mathrm{Ker}(p) \nsubseteq \mathrm{Jac}(R_1)$. Then, (0) is a strongly quasi J-ideal of k. However, $p^{-1}((0)) = (0) \times k = \mathrm{Ker}(p)$ is not a strongly quasi J-ideal of the ring $k \times k$.

Corollary 3.3. *Let $K \subseteq I$ be two ideals of a ring R. Then, I is a strongly quasi J-ideal of R if and only if $K \subseteq \mathrm{Jac}(R)$ and I/K is a strongly quasi J-ideal of R/K.*

Proof. ($\Rightarrow$) Using Proposition 2.3 and applying Proposition 3.1 to the canonical surjection $\pi : R \to R/K$, we conclude that $K \subseteq I \subseteq \mathrm{Jac}(R)$ and I/K is a strongly quasi J-ideal of R/K.

($\Leftarrow$) Applying Proposition 3.1 to the canonical surjection $\pi : R \to R/K$, we conclude that I is a strongly quasi J-ideal of R. □

Proposition 3.4. *Let S be a multiplicative closed subset of a ring R such that $\mathrm{Jac}(S^{-1}R) = S^{-1}\mathrm{Jac}(R)$. If I is a strongly quasi J-ideal of R with $S \cap I = \emptyset$, then $S^{-1}I$ is a strongly quasi J-ideal of $S^{-1}R$.*

Proof. As $S \cap I = \emptyset$, we have that $S^{-1}I \neq S^{-1}R$. Let $\frac{a}{s}$, $\frac{b}{t} \in S^{-1}R$ such that $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ and $\frac{b}{t} \notin \mathrm{Jac}(S^{-1}R)$. Then, $uab \in I$ for some $u \in S$ and $b \notin \mathrm{Jac}(R)$ since $\frac{b}{t} \notin S^{-1}\mathrm{Jac}(R) = \mathrm{Jac}(S^{-1}R)$. Since I is a strongly quasi J-ideal, we get $u^2a^2 \in I$ and this yields $\frac{a^2}{s^2} = \frac{a^2u^2}{s^2u^2} \in S^{-1}I$. Therefore, $S^{-1}I$ is a strongly quasi J-ideal of $S^{-1}R$. □

Proposition 3.5. *Let R be a ring and I be an ideal of R. Then, the following hold*:

(1) *$R[X]$ admits a strongly quasi J-ideal if and only if* $\mathrm{Nil}(R)$ *is a prime ideal of R.*
(2) *If $I[X]$ is a strongly quasi J-ideal of $R[X]$, then I is a strongly quasi J-ideal of R.*
(3) *The ideal (I, X) is never be a strongly quasi J-ideal of $R[X]$.*

Proof. (1) It is well known that $\mathrm{Jac}(R[X]) = \mathrm{Nil}(R[X]) = \mathrm{Nil}(R)[X]$. Hence by Theorem 2.8, $R[X]$ admits a strongly quasi J-ideal if and only if the exists a prime ideal P of $R[X]$ contained in $\mathrm{Jac}(R[X])$ if and only if $\mathrm{Nil}(R)[X]$ is a prime ideal of $R[X]$ if and only if $\mathrm{Nil}(R)$ is a prime ideal of R.

(2) Suppose that $I[X]$ is a strongly quasi J-ideal of $R[X]$ and let a, $b \in R$ such that $ab \in I$ and $b \notin \mathrm{Jac}(R)$. Then, $ab \in I[X]$ and $b \notin \mathrm{Jac}(R[X]) = \mathrm{Nil}(R)[X]$. Hence, $a^2 \in I[X]$ and so $a^2 \in I$. Thus, I is a strongly quasi J-ideal of R.

(3) Since $X \notin \mathrm{Jac}(R[X])$, $(I, X) \nsubseteq \mathrm{Jac}(R[X])$, and so (I, X) cannot be a strongly quasi J-ideal of $R[X]$. □

Theorem 3.6. *Let R be a ring and I be a proper ideal of R. Then, the following are equivalent*:

(1) *$I[X]$ is a strongly quasi J-ideal of $R[X]$.*
(2) *I is a strongly quasi primary ideal of R with $\sqrt{I} = \mathrm{Nil}(R)$.*
(3) *For each ideals A and B of R, $AB \subseteq I$ implies that $A_2 \subseteq I$ or $B \subseteq$* $\mathrm{Nil}(R)$.
(4) *For each finitely generated ideals A and B of R, $AB \subseteq I$ implies that $A_2 \subseteq I$ or $B \subseteq \mathrm{Nil}(R)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $I[X]$ is a strongly quasi J-ideal of $R[X]$. Since $I[X] \subseteq \mathrm{Jac}(R[X]) = \mathrm{Nil}(R[X]) = \mathrm{Nil}(R)[X]$, we get $I \subseteq \mathrm{Nil}(R)$, and so $\sqrt{I} = \mathrm{Nil}(R)$. Let a, $b \in R$ such that $ab \in I$ and $b \notin \sqrt{I}$. Then, $ab \in I[X]$ and $b \notin \mathrm{Nil}(R)[X] = \mathrm{Jac}(R[X])$. Hence, $a^2 \in I[X]$, and so $a^2 \in I$. Thus, I is a strongly quasi primary ideal of R with $\sqrt{I} = \mathrm{Nil}(R)$.

(2) $\Rightarrow$ (3) Follows from [13, Proposition 2.2].

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let f, $g \in R[X]$ with $fg \in I[X]$ and $g \notin \mathrm{Jac}(R[X])$. We claim that $f^2 \in I[X]$. Assume that $f \neq 0$. Following [4, Theorem 1], we have $c(f)c(g)^{n+1} = c(g)^n c(fg)$, where $n = \deg(f)$. Hence, $c(f)c(g)^{n+1} \subseteq I$ since $c(fg) \subseteq I$. Moreover, $c(g) \nsubseteq \mathrm{Nil}(R)$ since $g \notin \mathrm{Jac}(R[X]) = \mathrm{Nil}(R[X]) = \mathrm{Nil}(R)[X]$. By Proposition 3.5, $\mathrm{Nil}(R)$ is a prime ideal of R, hence $c(g)^{n+1} \nsubseteq \mathrm{Nil}(R)$, and so $(c(f))_2 \subseteq I$. Set $f = \sum_{i=0}^{n} a_i X^i$. We have:

$$f^2 = \sum_{k=0}^{n} (a_k)^2 X^{2k} + \sum_{0 \leq i < j \leq n} 2a_i a_j X^{i+j}$$

$$= \sum_{k=0}^{n} (a_k)^2 X^{2k} + \sum_{0 \leq i < j \leq n} \left[(a_i + a_j)^2 - a_i^2 - a_j^2 \right] X^{i+j}.$$

Thus, $f^2 \in (c(f))_2[X] \subseteq I[X]$. Consequently, $I[X]$ is a strongly quasi J-ideal of $R[X]$. □

Corollary 3.7. *Let R be a Hilbert ring and I be an ideal of R. Then, $I[X]$ is a strongly quasi J-ideal of $R[X]$ if and only if I is a strongly quasi J-ideal of R.*

Proof. It follows from Theorem 3.6 and Proposition 2.4. □

Proposition 3.8. *Let R be a ring, I be an ideal of R, and $N \subseteq M$ be R-modules. Then, the following hold:*

(1) *$R \propto M$ admits a strongly quasi J-ideal if and only if R admits a strongly quasi J-ideal.*
(2) *If $I \propto N$ is a strongly quasi J-ideal of $R \propto M$, then I is a strongly quasi J-ideal of R.*
(3) *If I is a strongly quasi J-ideal of R such that $\mathrm{Jac}(R)M \subseteq N$, then $I \propto N$ is a strongly quasi J-ideal of $R \propto M$.*

Proof. (1) Note that $\mathrm{Jac}(R \propto M) = \mathrm{Jac}(R) \propto M$ and that the prime ideals of $R \propto M$ have the form $P \propto M$, where P is a prime ideal of R (by [3, Theorem 3.2]). So, $\mathrm{Jac}(R \propto M)$ contains a prime ideal if and only if $\mathrm{Jac}(R)$ contains a prime ideal. Thus, $R \propto M$ admits a strongly quasi J-ideal if and only if R admits a strongly quasi J-ideal (by Theorem 2.8).

(2) Suppose that $I \propto N$ is a strongly quasi J-ideal of $R \propto M$. Let $a, b \in R$ such that $ab \in I$. Then, $(a, 0)(b, 0) \in I \propto M$. Thus, $(a, 0)^2 = (a^2, 0) \in I \propto M$ or $(b, 0) \in \mathrm{Jac}(R \propto M) = \mathrm{Jac}(R) \propto M$. Then, $a^2 \in I$ or $b \in \mathrm{Jac}(R)$, and so I is a strongly quasi J-ideal of R.

(3) Suppose that I is a strongly quasi J-ideal of R such that $\mathrm{Jac}(R)M \subseteq N$. By [3, Theorem 3.1], $I \propto N$ is an ideal of $R \propto M$ since $IM \subseteq \mathrm{Jac}(R)M \subseteq N$. Now, let $(a, m), (b, m') \in R \propto M$ such that $(a, m)(b, m') = (ab, am' + bm) \in I \propto N$. Then, $ab \in I$ and so $a^2 \in I$ or $b \in \mathrm{Jac}(R)$. If $b \in \mathrm{Jac}(R)$, then $(b, m') \in \mathrm{Jac}(R) \propto M = \mathrm{Jac}(R \propto M)$. If $a^2 \in I$, then $a \in \mathrm{Jac}(R)$. Hence, $am \in N$, and so $(a, m)^2 = (a^2, 2am) \in I \propto N$. Consequently, $I \propto N$ is a strongly quasi J-ideal of $R \propto M$. □

Corollary 3.9. *Let R be a ring, I be an ideal of R, and M be an R-module. Then,*

(1) *$I \propto M$ is a strongly quasi J-ideal of $R \propto M$ if and only if I is a strongly quasi J-ideal of R.*
(2) *$I \propto \mathrm{Jac}(R)M$ is a strongly quasi J-ideal of $R \propto M$ if and only if I is a strongly quasi J-ideal of R.*

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