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ON A UNIQUENESS QUESTION OF MEROMORPHIC FUNCTIONS AND PARTIAL SHARED VALUES

Imrul Kaish and Rana Mondal

Abstract. In this paper, we prove a uniqueness theorem of non-constant meromorphic functions of hyper-order less than 1 sharing two values CM and two partial shared values IM with their shifts. Our result in this paper improves and extends the corresponding results from Chen-Lin [2], Charak-Korhonen-Kumar [1], Heittokangas-Korhonen-Laine-Rieppo-Zhang [9] and Li-Yi [12]. Some examples are provided to show that some assumptions of the main result of the paper are necessary.

1. Introduction, definitions and results

Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. Throughout the paper, it is acceded that the reader is well known with the fundamental standard notations and terminology of Nevanlinna's value distribution theory of meromorphic functions [7, 10, 13]. It will be convenient to let E denote any set of positive real numbers of finite logarithmic measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic function of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h))$, as r runs to infinity outside of a set of finite logarithmic measure. In particular, we also denote by $S_1(r,h)$ any quantity satisfying $S_1(r,h) = o(T(r,h))$, as r runs to infinity on a set of logarithmic density 1.

In addition, for a meromorphic function f, the order $\rho(f)$ and the hyper-order $\rho_2(f)$ are introduced as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},\ \ \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In this note, we denote by $S(f)$ the collection of all meromorphic functions a satisfied $T(r,a) = S(r,f)$. Furthermore, we include all constant functions in $S(f)$ and put $S_\infty(f) = S(f) \cup \{\infty\}$. We say two meromorphic functions f and

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g share $a \in S_\infty(f)$ IM when f and g have same a-points. If f and g have same a-points with same multiplicities, then we also say that f and g share a CM. As usual, the summary IM means "ignoring multiplicity", while CM means "counting multiplicity".

On the other hand, we denote by $E(a, f)$ the set of zeros of $f - a$ and by $\overline{E}(a, f)$ the reduced set of zeros of $f - a$, where each zeros of $f - a$ appears only once in the set. We say that two meromorphic functions f and g share a CM when the two sets $E(a, f)$ and $E(a, g)$ coincide. Moreover, if the sets $\overline{E}(a, f)$ and $\overline{E}(a, g)$ coincide, then we say that f and g share a IM. Also, we denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicity less or equal to k and by $\overline{E}_{k)}(a, f)$ the reduced set of zeros of $f - a$ with multiplicity less or equal to k, where a zero of $f - a$ is counted only once in the set.

In recent ten years, the difference variant of the Nevanlinna theory has been established in [3, 4] and in particular, in [5], by Halburd-Korhonen and by Chiang-Feng, independently. Later on, the difference variant of the Nevanlinna theory was improved by Halburd-Korhonen-Tohge [6]. Using these theories, some mathematicians began to consider questions of uniqueness of meromorphic functions sharing values with their shifts and produced many fine works, for example, see [8, 9, 12]. In 2011, Heittokangas-Korhonen-Laine-Rieppo [8] presented an uniqueness theorem for periodicity of a meromorphic function of finite order as follows:

Theorem 1.1 ([8], Thm 2.1(a))**.** *Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}\backslash\{0\}$ and let $a_1, a_2, a_3 \in S_\infty(f)$ be three distinct periodic functions with period c. If $f(z)$, $f(z + c)$ share a_1, a_2, a_3 CM, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

Later on, Heittokangas-Korhonen-Laine-Rieppo-Zhang [9] proved the following result that improved Theorem 1.1 by replacing the assumption "3 CM" in Theorem 1.1 with the assumption "2 CM+ 1 IM":

Theorem 1.2 ([9], Thm 2)**.** *Let f be a non-constant meromorphic function of finite order, $c \in \mathbb{C}\backslash\{0\}$ and let $a_1, a_2, a_3 \in S_\infty(f)$ be three distinct periodic functions with period c. If $f(z)$, $f(z + c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

In 2017, Chen-Lin [2] proved the following result that improved Theorem 1.2 by replacing the assumption $\rho(f) < \infty$ with the assumption $\rho_2(f) < \infty$.

Theorem 1.3 ([2], Thm 1.3)**.** *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$, $c \in \mathbb{C}\backslash\{0\}$ and let $a_1, a_2, a_3 \in S_\infty(f)$ be three distinct periodic functions with period c. If $f(z)$, $f(z + c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z + c)$ for all $z \in \mathbb{C}$.*

On the other hand, Li-Yi [12] proved the following result of meromorphic functions of hyper-order $\rho_2(f) < 1$ sharing three distinct finite values IM and ∞ CM with their shifts:

Theorem 1.4 ([12], Thm 1.3)**.** *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C}\backslash\{0\}$. Suppose that $f(z), f(z+c)$ share $0, 1, a$ IM and ∞ CM, where a is a finite value such that $a \neq 0, 1$. Then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

In 2016, Charak-Korhonen-Kumar [1] considered the notions of the partial shared value and the partial shared small function and proved the following result:

Theorem 1.5 ([1], Thm 1.6)**.** *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C}\backslash\{0\}$. Let $a_1, a_2, a_3, a_4 \in S_\infty(f)$ be four distinct periodic functions with period c. If $\delta(a, f) > 0$ for some $a \in S_\infty(f)$ and*

$$\overline{E}(a_i, f(z)) \subseteq \overline{E}(a_i, f(z+c)),\ i = 1, 2, 3, 4,$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

In this paper, we will prove the following result of meromorphic functions of hyper-order $\rho_2(f) < 1$ sharing two values CM and two partially shared values IM:

Theorem 1.6. *Let f be a non-constant meromorphic function of hyper-order $\rho_2(f) < 1$, $c \in \mathbb{C}\backslash\{0\}$ and let a be a non-zero finite complex number. If $f(z)$ and $f(z+c)$ share $0, \infty$ CM and satisfy*

$$\overline{E}(a, f(z)) \subseteq \overline{E}(a, f(z+c)),\ \overline{E}(-a, f(z)) \subseteq \overline{E}(-a, f(z+c)), \tag{1.1}$$

then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

The following examples are available to confirm that our result is on of the best possible.

Example 1.1. Let $f(z) = \frac{3e^{2z}}{e^{3z}+3e^z-1}$ and $c = 2\pi i$. It is easy to see that $f(z)$ and $f(z+c)$ share $0, \infty$ CM and satisfy the condition (1.1), then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Example 1.2. Let $f(z) = \frac{2e^z}{e^{2z}+1}$ and $c = 2\pi i$. Then $f(z)$ and $f(z+c)$ share $0, \infty$ CM and satisfy the condition (1.1) and then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Example 1.3. Let $f(z) = \sin z$ and $c = 2\pi$. Then $f(z)$ and $f(z+c)$ share $0, \infty$ CM and satisfy the condition (1.1) and then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

Remark 1.4. Naturally, one question arise in our mind that what happens if the condition for $a_1 = a$, $a_2 = -a$ "partially shared values $\overline{E}(a_j, f(z)) \subseteq \overline{E}(a_j, f(z+c))$ for $j \in \{1, 2\}$" is replaced by "weakly partially shared values $\overline{E}_{k)}(a_j, f(z)) \subseteq \overline{E}_{k)}(a_j, f(z+c))$ for $j \in \{1, 2\}$ of order k", where k is a positive integer. Moreover, the next example suggests that Theorem 1.6 does not hold for each positive integer k.

Example 1.5. Let $f(z)=\frac{e^{3z}+3e^{z}}{3e^{2z}+1}$ and $c=\pi i$. Then $f(z)$ and $f(z+c)$ share $0,\infty$ CM and $a_1=1,\ a_2=-1,\ k=1,2$ satisfy $\overline{E}_{k)}(a_j,f(z))=\overline{E}_{k)}(a_j,f(z+c))=\emptyset,\ j=1,2$ but $f(z)\not\equiv f(z+c)$ in $z\in\mathbb{C}$.

Remark 1.6. The following example is to see that the condition "2 CM"and "2 partially IM"can not be weakened to the assumption "2 CM "in Theorem 1.6.

Example 1.7. If $f(z)=\frac{2e^z}{(e^z+1)^2}$ and $c=\pi i$, then $f(z+c)=\frac{-2e^z}{(e^z-1)^2}$. Then we can see that $f(z)$ and $f(z+c)$ share $0,\infty$ CM but $f(z)\not\equiv f(z+c)$ in $z\in\mathbb{C}$.

2. Preliminary lemmas

Firstly, we introduce the following results by Halburd-Korhonen-Tohge [6]:

Lemma 2.1 ([6], Lem 8.3)**.** *Let $T:[0,+\infty)\to[0,+\infty)$ be a non-decreasing continuous function and let $s\in(0,\infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty}\frac{\log\log T(r)}{\log r}=\rho_2<1,$$

and $\epsilon\in(0,1-\rho_2)$, then

$$T(r+s)=T(r)+o\left(\frac{T(r)}{r^{\epsilon}}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

Lemma 2.2 ([6], Thm 5.1)**.** *Let f be a meromorphic function of hyper-order $\rho_2(f)=:\rho_2<1$ and let $c\in\mathbb{C}$. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=o\left(\frac{T(r,f)}{r^{1-\rho_2-\epsilon}}\right),$$

where $\epsilon\in(0,1-\rho_2)$ and r runs to infinity outside of a set of finite logarithmic measure.

The lemma of a difference analogue of the second main theorem was first proved in [5] and the next following was extended to the case of hyper-order less than 1 in [6].

Lemma 2.3 ([6], Thm 2.1)**.** *Let f be a meromorphic function of hyper-order $\rho_2(f)=:\rho_2<1$ and let $c\in\mathbb{C}$ be such that $\triangle_c f(z)=:f(z+c)-f(z)\not\equiv 0$. Let $p\geq 2$ and let $a_1,a_2,\ldots,a_p$ be p distinct finite complex numbers. Then we have*

$$m(r,f)+\sum_{q=1}^{p}m\left(r,\frac{1}{f-a_q}\right)\leq 2T(r,f)-N_{pair}(r,f)+S(r,f),$$

where

$$N_{pair}(r,f)=:2N(r,f)-N(r,\triangle_c f)+N\left(r,\frac{1}{\triangle_c f}\right).$$

The following lemma is called the three small function theorem:

Lemma 2.4 ([13], Thm 1.38)**.** *Let f be a meromorphic function in the complex plane and a_1, a_2, a_3 be three distinct small functions of f. Then*

$$T(r,f) \leq \sum_{p=1}^{3} \overline{N}\left(r, \frac{1}{f-a_p}\right) + S(r,f).$$

We also need the following lemmas:

Lemma 2.5 ([13], Thm 1.13)**.** *Let f be a meromorphic function. If*

$$g = \frac{af+b}{cf+d},$$

where a, b, c, d are small functions with respect to f such that $ad \not\equiv bc$, then

$$T(r,g) = T(r,f) + S(r,f).$$

Lemma 2.6 ([11], Lem 5)**.** *Let f_1 and f_2 be two non-constant meromorphic functions satisfying*

$$\overline{N}(r,f_j) + \overline{N}\left(r, \frac{1}{f_j}\right) = S(r), \ j = 1, 2.$$

If $f_1^u f_2^v - 1$ is not identically zero for all integers u and v ($|u|+|v|>0$), then for any positive number ϵ, we have

$$N_0(r, 1; f_1, f_2) \leq T(r) + S(r),$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ as $r \to \infty$, except for a set of r of finite linear measure.

3. Proof of Theorem 1.6

First of all, we prove that f is a transcendental meromorphic function on account of the assumptions of Theorem 1.6. On contrary, assume that f is a non-constant rational function and we set

$$f(z) = \frac{P(z)}{Q(z)}, \ f(z+c) = \frac{P(z+c)}{Q(z+c)}, \ z \in \mathbb{C},$$

where P and Q are two relatively prime polynomials such that $PQ \not\equiv 0$. Now, we assume that Q is not a constant. Since $f(z)$ and $f(z+c)$ share ∞ CM, it is easy to see that if $z_0 \in \{z : |z| < r\}$ is a pole of $f(z)$, then z_0 also is a pole of $f(z+c)$. Thus, if Q has a zero z_0, then we have $Q(z_0+c) = 0$. Keep going, for all positive integers $Q(z_0+nc) = 0$ holds. Thus, Q is a non-zero constant. However, f is a non-constant polynomial. Under the assumption that $f(z)$ and $f(z+c)$ share 0 CM, it follows that

$$\frac{f(z+c)}{f(z)} = A,$$

where A is a non-zero constant. We obtain that $f(z+c) = f(z)$, which is due to the assumption (1.1) and which leads to the contradiction that f is not a constant. Thus, f is transcendental.

We assume contrary to our desired result that $f(z) = f(z+c)$ were true. Since, the assumption that $f(z), f(z+c)$ share 0 CM, it deduces that

$$\frac{f(z+c)}{f(z)} = e^{\nu(z)}, \tag{3.1}$$

where ν is an entire function with $\rho(\nu) < 1$. According to Lemma 2.2, it follows that $m(r, e^{\nu}) = S(r, f)$ and so that

$$T(r, e^{\nu}) = S(r, f). \tag{3.2}$$

Together this with Lemma 2.5, it shows that

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f).$$

Thus, we have

$$S(r, f(z+c)) = S(r, f(z)),$$

and we denote, for benefit

$$S(r) = S(r, f(z+c)) = S(r, f(z)).$$

Then the assumption $f(z) \not\equiv f(z+c)$ implies $e^{\nu(z)} \not\equiv 1$. Then, from the estimates it follows

$$\overline{N}\left(r, \frac{1}{f(z)-a}\right) \leq \overline{N}\left(r, \frac{1}{e^{\nu}-1}\right) = S(r), \tag{3.3}$$

and also

$$\overline{N}\left(r, \frac{1}{f(z)+a}\right) \leq \overline{N}\left(r, \frac{1}{e^{\nu}-1}\right) = S(r). \tag{3.4}$$

Let us define the non-constant meromorphic function λ by

$$\lambda(z) = \frac{f(z+c)-a}{f(z)-a}, \tag{3.5}$$

we claim that

$$\overline{N}(r, \lambda) + \overline{N}\left(r, \frac{1}{\lambda}\right) = S(r). \tag{3.6}$$

In the following discussion, we introduce some notations as below.

We denote $\deg_a(f, z)$ by the multiplicity of a-point of f at z, by N_{Γ_k} the counting function with respect to the set Γ_k $(k = 1, 2, 3, 4)$ and by $\overline{N}_{\Gamma_k}$ the reduced counting function as well.

Let

$\Gamma_1 := \{z \,|f(z) = \infty, f(z+c) \neq \infty\}$,

$\Gamma_2 := \{z \,|\, f(z) \neq \infty, f(z+c) = \infty\}$,

$\Gamma_3 := \{z \,|\, f(z) = \infty, f(z+c) = \infty, \deg_\infty(f, z) = \deg_\infty(f(z+c), z)\}$,

$$\Gamma_4 := \{z \,|\, f(z) = \infty, f(z+c) = \infty, \deg_\infty(f, z) \neq \deg_\infty(f(z+c), z)\}.$$

It is clear that Γ_1 is the set of the poles of $f(z)$, that is not the poles of $f(z+c)$; Γ_2 is the set of the poles of $f(z+c)$, that is not the poles of $f(z)$; Γ_3 is the set of the common poles of $f(z)$ and $f(z+c)$ with same multiplicity; Γ_4 is the set of the common poles of $f(z)$ and $f(z+c)$ with different multiplicity.

If $z_1 \in \Gamma_2$, then z_1 is a pole of $e^{\nu(z)}$ according to (3.1). Thus, $\Gamma_2 = \emptyset$, i.e.,

$$N_{\Gamma_2} = S(r). \tag{3.7}$$

Again, if $z_1 \in \Gamma_1$, then z_1 is also a zero of $e^{\nu(z)}$ according to (3.1). Thus, $\Gamma_1 = \emptyset$, i.e.,

$$N_{\Gamma_1} = S(r). \tag{3.8}$$

Similar to the proof as above, it yields

$$N_{\Gamma_4} = S(r). \tag{3.9}$$

Hence, from (3.5) we have

$$\overline{N}(r, \lambda) \leq \overline{N}\left(r, \frac{1}{f(z) - a}\right) + \overline{N}_{\Gamma_2} \tag{3.10}$$

and also using Lemma 2.1, we have from (3.5) that

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{\lambda}\right) &\leq \overline{N}\left(r, \frac{1}{f(z+c) - a}\right) + \overline{N}_{\Gamma_1} + \overline{N}_{\Gamma_4} \\
&\leq \overline{N}\left(r + |c|, \frac{1}{f(z) - a}\right) + \overline{N}_{\Gamma_1} + \overline{N}_{\Gamma_4} \\
&= \overline{N}\left(r, \frac{1}{f(z) - a}\right) + \overline{N}_{\Gamma_1} + \overline{N}_{\Gamma_4}. \qquad (3.11)
\end{aligned}$$

Combining (3.3) and (3.7)-(3.11), we have the desired assertion (3.6) as we claimed. Set the non-constant meromorphic function μ by

$$\mu(z) = \frac{f(z) + a}{f(z+c) + a}. \tag{3.12}$$

We claim that

$$\overline{N}(r, \mu) + \overline{N}\left(r, \frac{1}{\mu}\right) = S(r). \tag{3.13}$$

In the same sense as (3.4) and (3.7)-(3.11), the desired conclusion (3.13) as we claimed is true. Submitting (3.1) into (3.5) and (3.12), it follows that

$$\lambda(z) = \frac{f(z)e^{\nu(z)} - a}{f(z) - a}, \ \lambda(z) - 1 = \frac{f(z)(e^{\nu(z)} - 1)}{f(z) - a} \tag{3.14}$$

and

$$\mu(z) = \frac{f(z) + a}{f(z)e^{\nu(z)} + a}, \ \mu(z) - 1 = \frac{f(z)(1 - e^{\nu(z)})}{f(z)e^{\nu(z)} + a}, \tag{3.15}$$

which using Lemma 2.5 and from (3.2) give

$$T(r,\lambda)=T(r,f)+S(r) \tag{3.16}$$

and

$$T(r,\mu)=T(r,f)+S(r) \tag{3.17}$$

and that

$$N_0(r,1;\lambda,\mu)=\overline{N}\left(r,\frac{1}{f(z)}\right)+S(r), \tag{3.18}$$

where $N_0(r,1;\lambda,\mu)$ is defined as in Lemma 2.6. Now by Lemma 2.6, we have for some integers p and q such that $|p|+|q|>0$ either

$$\lambda^p(z)\mu^q(z)\equiv 1 \tag{3.19}$$

holds or for any positive number ϵ

$$N_0(r,1;\lambda,\mu)\le \epsilon T(r) \tag{3.20}$$

holds, where $T(r)=T(r,\lambda)+T(r,\mu)$. Next, we consider the following two cases concerning (3.19) and (3.20) hold, separately.

Case 1. First we assume that the result (3.20) holds.

By choosing $\epsilon=\frac{1}{4}$, it follows from (3.16) and (3.17) that

$$N_0(r,1;\lambda,\mu)\le \frac{1}{2}T(r,f)+S(r). \tag{3.21}$$

Now we rewrite the result (3.1), as on the another hand

$$f(z+c)-a=e^{\nu(z)}(f(z)-ae^{-\nu(z)}),$$

then it deduces from Lemma 2.1 and (3.3) that

$$\overline{N}\left(r,\frac{1}{f-ae^{-\nu}}\right)=\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)\le \overline{N}\left(r,\frac{1}{f(z)-a}\right)\le S(r). \tag{3.22}$$

Noting $e^{\nu(z)}\not\equiv 1$, using Lemma 2.4, we deduce from (3.3), (3.18), (3.21) and (3.22) that

$$T(r,f)\le \overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-ae^{-\nu}}\right)\le \frac{1}{2}T(r,f)+S(r),$$

which gives to a contradiction.

Case 2. Secondly we assume that the result (3.19) holds, i.e.,

$$\left(\frac{f(z+c)-a}{f(z)-a}\right)^p\left(\frac{f(z)+a}{f(z+c)+a}\right)^q\equiv 1. \tag{3.23}$$

We discuss the following four subcases for further examination.

Subcase 2.1. Suppose that $p=0$, and $q\ne 0$. Then, it follows from (3.23) that

$$\frac{f(z)+a}{f(z+c)+a}\equiv B, \tag{3.24}$$

where B is a non-zero constant. 0 is not a Picard value of f, otherwise it follows from (3.1) and (3.4) that

$$T(r,f) \le \overline{N}\left(r, \frac{1}{f+a}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f+ae^{-\nu}}\right) \le S(r),$$

which leads to a contradiction. Since 0 is not a Picard value of f, then there exists one point $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$, then we have $f(z_0 + c) = 0$ due to the assuming condition that $f(z)$ and $f(z+c)$ share 0 CM. Hence, by (3.24) we can obtain $B = 1$, which leads to a contradiction of our present assumption.

Subcase 2.2. Suppose that $q = 0$, and $p \neq 0$. Then, it deduces from (3.23) that

$$\frac{f(z+c)-a}{f(z)-a} \equiv C,$$

where C is a non-zero constant. If 0 is a Picard value of f, we have a contradiction from (3.3) and (3.22) that

$$T(r,f) \le \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-ae^{-\nu}}\right)$$

can not hold. Again, if 0 is not a Picard value of f, then similar to the proof of Subcase 2.1, we can get a contradiction.

Subcase 2.3. Suppose that $pq \neq 0$, and $pq > 0$. If a is a Picard value of f, then we can get a contradiction use the difference version of the second fundamental theorem of Lemma 2.3 and the assuming conditions that $f(z)$, $f(z+c)$ share $0, \infty$ CM as follows. That is

$$\begin{aligned} & m(r,f) + m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-a}\right) \\ & \le 2T(r,f) - 2N(r,f) + N(r, \Delta_c f) - N\left(r, \frac{1}{\Delta_c f}\right) + S_1(r,f) \\ & \le 2T(r,f) - N(r,f) - N\left(r, \frac{1}{f}\right) + S_1(r,f). \end{aligned}$$

The definition of characteristic function implies from the above estimate that

$$T(r,f) \le N\left(r, \frac{1}{f-a}\right) + S_1(r,f) \le S_1(r,f).$$

So, let $z_0 \in \{z : |z| < r\}$ be one of the common zero of $f(z) - a$ and $f(z+c) - a$ with multiplicity u and v, respectively. It deduces from (3.23) that

$$u = v.$$

Moreover, we can get the expression from (3.1) as follows:

$$\nu'(z) = \frac{f'(z+c)}{f(z+c)} - \frac{f'(z)}{f(z)}. \tag{3.25}$$

Obviously, $\nu' \not\equiv 0$. Otherwise, it reduces to $\frac{f(z+c)}{f(z)} = D$, where D is a non-zero constant. Since a is not a Picard value of f, instantly we obtain that $D = 1$ and we get again a contradiction. However, it follows from (3.25) that z_0 must be a zero of $\nu'(z)$ with multiplicity at least $u - 1$. Thus, we have

$$N_{(2}\left(r, \frac{1}{f-a}\right) \leq 2N\left(r, \frac{1}{\nu'}\right) \leq 4T(r, \nu) \leq 4T(r, e^{\nu}) = S(r).$$

Together this with (3.3), it yields that

$$N\left(r, \frac{1}{f-a}\right) \leq S(r).$$

In the same manner before we can use the difference version of the second fundamental theorem of Lemma 2.3 and our given assumption to get a contradiction again.

Subcase 2.4. Suppose that $pq \neq 0$, and $pq < 0$. Then, without loss of generality, we may assume that $p > 0$ and $q < 0$. Letting $s = -q$, (3.23) can be written as

$$\left(\frac{f(z+c)-a}{f(z)-a}\right)^p \equiv \left(\frac{f(z)+a}{f(z+c)+a}\right)^s.$$

This implies that $p = s$. Otherwise, we can write

$$pT(r, \lambda) = sT(r, \mu) + S(r),$$

which violets (3.16) and (3.17) together. Then, there exists a non-zero constant W with $W^{2p} = 1$ such that

$$\frac{f(z+c)-a}{f(z)-a} \equiv W^2\left(\frac{f(z)+a}{f(z+c)+a}\right). \tag{3.26}$$

We see from the above equality that $W^2 \neq 1$. Otherwise, we obtain that $f(z+c) = f(z)$ or $f(z+c) = -f(z)$. But $f(z+c) = f(z)$, contradicts our present assumption. So, $f(z+c) = -f(z)$. If a is a Picard value of f, then we can get a contradiction as we did in Subcase 2.3. So, a is not a Picard value of f. Then, there exists $z_0 \in \mathbb{C}$ such that $f(z_0) = a = f(z_0+c)$, due to the assuming condition (1.1). Hence, we can get from the relation $f(z+c) = -f(z)$ that $a = 0$. This is impossible, since a is a non-zero finite value. Then, by (3.26) it yields

$$(f(z+c) - Wf(z))(f(z+c) + Wf(z)) = a^2(1 - W^2). \tag{3.27}$$

Let us assume that

$$\gamma_1(z) = f(z+c) - Wf(z),\ \gamma_2(z) = f(z+c) + Wf(z).$$

Then each of the above estimates imply

$$f(z+c) = \frac{\gamma_1(z) + \gamma_2(z)}{2},\ f(z) = \frac{\gamma_2(z) - \gamma_1(z)}{2W}. \tag{3.28}$$

Next, (3.27) can be written in the form

$$\gamma_1(z)\gamma_2(z) = a^2(1 - W^2). \tag{3.29}$$

Now from the results of $\gamma_1(z)$ and $\gamma_2(z)$, we get

$$T(r, \gamma_1) \le 2T(r, f) + S(r),\ T(r, \gamma_2) \le 2T(r, f) + S(r).$$

From the above estimates, we can write for it convenience

$$S(r, \gamma_1) = S(r, \gamma_2) = S(r).$$

Let

$$\phi(z) = \frac{\gamma_1(z + c)}{\gamma_1(z)},\ \psi(z) = \frac{\gamma_2(z + c)}{\gamma_2(z)}.$$

Now, taking into account of (3.1) in $\phi(z)$, we have

$$\phi(z) = \frac{e^{\nu(z+c)} - W}{1 - We^{-\nu(z)}}.$$

It follows from (3.2) that

$$N(r, \phi) \le N\left(r, \frac{1}{1 - We^{-\nu}}\right) \le T(r, e^{\nu}) = S(r).$$

In the same manner, we get

$$N(r, \psi) \le S(r).$$

This together with $m(r, \phi) = S(r, \gamma_1)$ and $m(r, \psi) = S(r, \gamma_2)$ follows simply by Lemma 2.2, that

$$T(r, \phi) = S(r),\ T(r, \psi) = S(r). \tag{3.30}$$

From (3.28), we deduce that

$$W\gamma_1(z) + W\gamma_2(z) = \gamma_2(z + c) - \gamma_1(z + c).$$

On dividing $\gamma_1(z)\gamma_2(z)$ both sides, we obtain that

$$(W + \phi(z))\gamma_1(z) = (\psi(z) - W)\gamma_2(z). \tag{3.31}$$

From (3.29) and (3.31), it deduces that

$$\gamma_1^2(z) = a^2(1 - W^2)\frac{\psi(z) - W}{\phi(z) + W}.$$

Together with (3.30) we get a contradiction that

$$2T(r, \gamma_1) = T(r, \gamma_1^2) \le T(r, \phi) + T(r, \psi) + O(1)$$

can not hold. Therefore, this completes the proof of Theorem 1.6.

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Imrul Kaish
Department of Mathematics and Statistics
Aliah University
Kolkata 700160, West Bengal, India
Email address: imrulksh3@gmail.com

Rana Mondal
Department of Mathematics and Statistics
Aliah University
Kolkata 700160, West Bengal, India
Email address: ranku.mondalrana@gmail.com