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SOME RESULTS RELATED TO DIFFERENTIAL-DIFFERENCE COUNTERPART OF THE BRÜCK CONJECTURE

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Abstract. In this paper, our focus is on exploring value sharing problems related to a transcendental entire function f and its associated differential-difference polynomials. We aim to establish some results which are related to differential-difference counterpart of the Brück conjecture.

1. Introduction

In this note, we assume that the readers are familiar with the classical Value Distribution theory ([8, 11, 14]). By meromorphic functions, we always mean nonconstant meromorphic functions, unless otherwise specified. Let f and g be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that f and g share the value a CM (respectively, IM) provided that $f - a$ and $g - a$ have the same zeros counting multiplicities (respectively, ignoring multiplicities), and that f and g share the value ∞ CM (respectively, IM) provided that f and g have the same poles counting multiplicities (respectively, ignoring multiplicities).

The classical results in the uniqueness theory of meromorphic functions are the five-value and four-value theorems due to Rolf Nevanlinna. The five-value theorem states that *if two noncostant meromorphic functions f and g share five distinct values in the extended complex plane IM, then $f \equiv g$*. Similarly, the four-value theorem states that *if two nonconstant meromorphic functions f and g share four distinct values in the extended complex plane CM, then $f \equiv T \circ g$, where T is a Möbius transformation.* Later, in the four value theorem, the assumption 4 CM has been improved to 2 CM+ 2 IM by Gundersen [6]. Moreover, Gundersen [5], showed that 4 CM cannot be improved to 4 IM, while 1 CM+ 3 IM remains an open problem.

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For the uniqueness of the entire functions, if we consider a special situation where g is the derivative of f, one usually needs sharing of only two values CM for their uniqueness. In 1977, Rubel and Yang [13] first showed that *if a nonconstant entire function f and its derivative f' share two distinct values a, b CM, then $f \equiv f'$*. In 1979, Mues and Steinmetz [12] observed that in Rubel and Yang's result, the CM sharing can be further relaxed to IM sharing. They proved that *if a non-constant entire function f and its derivative f' share two distinct values a, b IM, then $f \equiv f'$*. It is well known that in Rubel and Yang's result, the two value sharing can not be further relaxed. We recall the following example. Let

$$f(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)dt.$$

Here, one can check that f and f' share 1 CM, but $(f' - 1) = e^z(f - 1)$. In this connection, we recall a famous conjecture proposed by Brück [1].

Conjecture 1.1. *Let f be an entire function and*

$$\rho_2(f) := \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}$$

be the hyper-order of f such that $\rho_2(f) < \infty$ and is not a positive integer. Let $a \in \mathbb{C}$. If f and f' share the value a CM, then

$$\frac{f' - a}{f - a} = c,$$

where c is a non-zero constant.

Brück's himself verified the conjecture for $a = 0$ ([1]) and later Gundersen and Yang proved that the conjecture is true for finite order entire functions ([7]). Recently, many researchers put their attention to consider the complex difference equations and the uniqueness of transcendental entire functions sharing values with their shifts. Using difference analogues of logarithmic derivative lemma, Heittokangas et al. established the following theorems:

Theorem 1.1 ([9])**.** *Let f be a nonconstant meromorphic function such that its order of growth*

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} < 2,$$

and let η be a nonzero complex number and $a \in \mathbb{C}$. If $f(z)$ and $f(z + \eta)$ share the values a CM and ∞ CM, then

$$\frac{f(z + \eta) - a}{f(z) - a} = c,$$

where c is a non-zero constant.

In the same paper ([9]), Heittokangas et al. provided the example $f(z) = e^{z^2} + 1$, which shows that $\sigma(f) < 2$ can't be relaxed to $\sigma(f) \leq 2$.

Let $f(z)$ be a nonconstant meromorphic function and η be a nonzero complex constant. Then $f(z+\eta)$ is called the *shift* of $f(z)$. Also, $\Delta f(z) = f(z+\eta) - f(z)$ is called the *difference operator* of $f(z)$. Moreover,

$\Delta^0_\eta f(z) := f(z)$, $\Delta^1_\eta f(z) := \Delta f(z)$ and $\Delta^k_\eta f(z) = \Delta^{k-1}_\eta (\Delta^1_\eta f(z))$ for $k \in \mathbb{N}, k \geq 2$.

In [2], Chen proved a difference analogue of the Brück conjecture as follows:

Theorem 1.2 ([2]). *Let f be a transcendental entire function of finite order. Also, assume that f has a finite Borel exceptional value $\alpha \in \mathbb{C}$. Let η be a nonzero complex constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f(z)$ and $f(z)$ share a finite value a $(\neq \alpha)$ CM, then*

$$\frac{\Delta f(z) - a}{f(z) - a} = \frac{a}{a-\alpha}.$$

In [10], Huang and Zhang studied a parallel result corresponding to Theorem 1.1 as follows:

Theorem 1.3 ([10]). *Let f be a transcendental entire function of order of growth*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

Let $k \in \mathbb{N}$ and $\eta \in \mathbb{C} \setminus \{0\}$. Assume that $\Delta^k_\eta f(z) \not\equiv 0$. If $f(z)$ and $\Delta^k_\eta f(z)$ share 0 CM, then

$$\Delta^k_c f(z) = cf(z)$$

for some non-zero constant c.

In this paper, we will derive some results related to differential-difference analogues of the Brück conjecture.

2. Main results

A polynomial which includes $f(z)$ and its derivatives or shifts operator is defined to be a differential-difference polynomial. Now, we consider a homogeneous complex differential-difference polynomials of $f(z)$.

$$\Psi(f) = \sum_{i=1}^{m} a_i (f^{(i)}(z+\xi_i))^n + \sum_{j=1}^{k} b_j (f(z+\eta_j))^n,$$

where $k, m, n \in \mathbb{N}$ and a_i, ξ_i, η_j and b_j are complex constants such that $\Psi(f)$ is not identically equal to a constant.

Theorem 2.1. *Let f be a transcendental meromorphic function and the order of f,*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

If f^n and $\Psi(f)$ share 0 and ∞ CM, then $\Psi(f) = cf^n$ for some non-zero constant c.

To prove the above result, we need the help of following lemmas.

Lemma 2.1 ([3]). *Let us choose two complex numbers η_1 and η_2 such that $\eta_1 \neq \eta_2$. Also, let $f(z)$ be a nonconstant meromorphic function of finite order. If σ is the order of $f(z)$, then for each $\varepsilon > 0$,*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.2 ([4]). *Let f be a transcendental meromorphic function. If $\sigma(f) < \infty$, then*

$$m\left(r, \frac{f^{(k)}(z+\eta)}{f(z+\zeta)}\right) = S(r,f)$$

for those z which satisfy $|z| = r \notin E$, where the logarithmic measure of E is finite and ζ and η are constants; k is a non-negative integer.

Proof of Theorem 2.1. Since f^n and $\Psi(f)$ share 0 and ∞ CM, thus we have

$$\frac{\Psi(f)}{f^n} = \exp H(z),$$

where $H(z)$ is a polynomial with deg$H(z) \leq \sigma(f) < 2$. So $H(z)$ is at most one degree polynomial. Now

$$\begin{aligned} T(r, \exp H(z)) &= m(r, \exp H(z)) \\ &= m\left(r, \frac{\Psi(f)}{f^n}\right) \\ &= m\left(r, \frac{\sum_{i=1}^{m} a_i \left(f^{(i)}(z+\xi_i)\right)^n + \sum_{j=1}^{k} b_j \left(f(z+\eta_j)\right)^n}{f^n}\right). \end{aligned}$$

Since $\sigma < 2$, thus for each $\varepsilon > 0$, using Lemma 2.1 and Lemma 2.2, we have

$$T(r, \exp H(z)) \leq O\left(r^{\sigma-1+\varepsilon}\right) + S(r,f).$$

Therefore $H(z)$ must be a constant. This completes the proof. □

Corollary 2.1. *Let f be a transcendental entire function and the order of f,*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

Let η be a non-zero complex constant. Assume that $\Delta_\eta^k(f) \not\equiv 0$. If f and $\Delta_\eta^k f(z)$ share 0 CM, then $\Delta_\eta^k(f) = cf$ for some non-zero constant c.

Theorem 2.2. *Let f be a transcendental entire function and the order of f,*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

If $\Delta_\eta^k(f)\,(\not\equiv 0)$ *and* $\Delta_\eta^{k-1}(f)\,(\not\equiv 0)$ *share* 0 *CM, then*

$$\Delta_\eta^k(f)=c\Delta_\eta^{k-1}(f)\,,\ \text{where } c \text{ is a non-zero constant.}$$

Proof of Theorem 2.2. It is given that $\Delta_\eta^k(f)\,(\not\equiv 0)$ and $\Delta_\eta^{k-1}(f)\,(\not\equiv 0)$ share 0 CM. Thus we can write

$$\frac{\Delta_\eta^k f(z)}{\Delta_\eta^{k-1} f(z)}=e^{H(z)},$$

where $H(z)$ is a polynomial. Now, we consider two cases. If $H(z)$ is a constant polynomial, then our theorem follows. If $H(z)$ is a nonconstant polynomial, then

$$\frac{\Delta_\eta^k f(z)}{\Delta_\eta^{k-1} f(z)}=e^{H(z)}.$$

Thus

$$\frac{\Delta_\eta^{k-1} f(z+\eta)-\Delta_\eta^{k-1} f(z)}{\Delta_\eta^{k-1} f(z)}=e^{H(z)}.$$

Let $F(z)=\Delta_\eta^{k-1} f(z)$. Then the above equation reduces to

$$\frac{F(z+\eta)-F(z)}{F(z)}=e^{H(z)}.$$

Thus

$$\begin{aligned}T\left(r,e^{H(z)}\right)&=m\left(r,e^{H(z)}\right)\\&=m\left(r,\frac{F(z+\eta)-F(z)}{F(z)}\right)\\&\le m\left(r,\frac{F(z+\eta)}{F(z)}\right)+O(1)\,.\end{aligned}$$

Thus for each $\varepsilon>0$, applying Lemma 2.1, we have

$$O\left(r^{\deg H(z)}\right)\le O\left(r^{\sigma-1+\varepsilon}\right)+S(r,f),$$

which implies $\deg H(z)<1$ as $\sigma(f)<2$. Thus $H(z)$ can not be a nonconstant polynomial. Hence our theorem follows. □

Next, we recall the definition of a linear differential polynomial.

Definition 2.1. Let f be a transcendental meromorphic function. Then $L=L\left(f^{(m)}\right)$ denotes a linear differential polynomial of the form

$$L=L\left(f^{(m)}\right)=b_0f^{(m)}+b_1f^{(m+1)}+b_2f^{(m+2)}+\cdots+b_kf^{(m+k)},$$

where $b_0,b_1,b_2,\ldots,b_k\,(\neq 0)$ are complex numbers and $m\,(\geq 1)$ and $k\,(\geq 0)$ are integers such that $k=0$ if $m=1$ and $0\leq k\leq m-2$ if $m\geq 2$.

Theorem 2.3. *Let f be a transcendental meromorphic function and the order of f,*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

If $L\left(f^{(m)}\right) (\not\equiv 0)$ and $f(z+\eta)$ share 0 and ∞ CM, then

$$L\left(f^{(m)}\right) = cf(z+\eta),$$

where c is a non-zero constant.

For proving the above theorem, we need the help of the following lemma.

Lemma 2.3 ([8])**.** *Let l be any positive integer and f be a meromorphic function. Then*

$$m\left(r, \frac{f^{(l)}(z)}{f(z)}\right) = S(r,f).$$

Proof of Theorem 2.3. By the given conditions, we can write

$$\frac{L\left(f^{(m)}\right)}{f(z+\eta)} = e^{H(z)},$$

where $H(z)$ is a polynomial. Then obviously deg$H(z) \le \sigma(f) < 2$. If $H(z)$ is a constant polynomial, then our theorem follows. If $H(z)$ is a polynomial of degree ≥ 1, then

$$\begin{aligned} T(r, \exp H(z)) &= m(r, \exp H(z)) \\ &= m\left(r, \frac{L\left(f^{(m)}\right)}{f(z+\eta)}\right) \\ &= m\left(r, \frac{b_0 f^{(m)} + b_1 f^{(m+1)} + b_2 f^{(m+2)} + \cdots + b_k f^{(m+k)}}{f(z+\eta)}\right) \\ &\le m\left(r, \frac{b_0 f^{(m)} + b_1 f^{(m+1)} + b_2 f^{(m+2)} + \cdots + b_k f^{(m+k)}}{f(z)}\right) \\ &\quad + m\left(r, \frac{f(z)}{f(z+\eta)}\right). \end{aligned}$$

For each $\varepsilon > 0$, using Lemma 2.3 and Lemma 2.1, we have

$$T(r, \exp H(z)) \le S(r,f) + O\left(r^{\sigma-1+\varepsilon}\right).$$

Since $\sigma < 2$, we can conclude that $H(z)$ can not be a polynomial of degree ≥ 1. This completes the proof. □

Theorem 2.4. *Let f be a transcendental entire function and the order of f,*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2.$$

Let $\eta \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{N}$ and $a \in \mathbb{C}$. If $f(z+\eta) - f^{(k)}(z)$ and $f(z)$ share the value a CM, then there exists a constant $c\,(\neq 0)$ such that

$$\frac{f(z+\eta) - f^{(k)}(z) - a}{f(z) - a} = c.$$

Before proving Theorem 2.4, we prove a lemma.

Lemma 2.4. *Let $Q(z)$ be a nonconstant polynomial. If $F(z)$ is a finite order solution of the equation*

$$F(z+\eta) - (1 + \exp Q(z))\, F(z) = F^{(k)}(z),$$

then $\sigma(F) \geq \deg Q(z) + 1$, where η is non zero constant and $\sigma(F)$ is the order of F.

Proof. Let $F(z)$ be a solution of the equation

$$F(z+\eta) - (1 + \exp Q(z))\, F(z) = F^{(k)}(z)$$

and $\sigma(F) < \infty$. Therefore from the above equation, we have

$$\frac{F(z+\eta)}{F(z)} - \frac{F^{(k)}(z)}{F(z)} = (1 + \exp Q(z)).$$

Now

$$\begin{aligned} T(r, \exp Q(z)) &= m(r, \exp Q(z)) \\ &= m\left(r, \frac{F(z+\eta)}{F(z)} - \frac{F^{(k)}(z)}{F(z)} - 1\right) \\ &\leq m\left(r, \frac{F(z+\eta)}{F(z)}\right) + m\left(r, \frac{F^{(k)}(z)}{F(z)}\right) + O(1). \end{aligned}$$

Let $\varepsilon > 0$. Now, using Lemma 2.1 and Lemma 2.3, we have from the above inequality that

$$O\left(r^{\deg Q(z)}\right) \leq O\left(r^{\sigma - 1 + \varepsilon}\right) + S(r, F),$$

which gives the required proof. □

Proof of Theorem 2.4. By the given condition, we can write

$$\frac{f(z+\eta) - f^{(k)}(z) - a}{f(z) - a} = e^{H(z)},$$

where $H(z)$ is a polynomial. If $H(z)$ is a constant, then our theorem follows. If $H(z)$ is a nonconstant polynomial, then substituting $F(z) = f(z) - a$, we get

$$\frac{F(z+\eta) - F^{(k)}(z)}{F(z)} = e^{H(z)}.$$

Proceeding similarly as in Lemma 2.4, and using Lemma 2.1 and Lemma 2.3, for each $\varepsilon > 0$, we have

$$O\left(r^{\deg H(z)}\right) \leq O\left(r^{\sigma-1+\varepsilon}\right) + S(r,f)$$

which implies $\deg H(z) < 1$ as $\sigma(f) < 2$. So $H(z)$ can not be a nonconstant polynomial. Hence our theorem follows. □

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