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AN EXAMPLE FOR THE NON-STABILITY OF MULTI-ADDITIVE-QUADRATIC-CUBIC MAPPINGS

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Abstract. In this paper, we improve Corollary 1 of [4] and then present an example to show that the assertion in the mentioned corollary can not be valid in the singularity case.

1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ are the set of all natural numbers, the set of all rationals and the set of all real numbers, respectively. In addition, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty)$. For each $l \in \mathbb{N}_0, n \in \mathbb{N}$, $t = (t_1, \dots, t_n) \in \{-1, 1\}^n$ and $x = (x_1, \dots, x_n) \in V^n$ we write $lx := (lx_1, \dots, lx_n)$ and $tx := (t_1x_1, \dots, t_nx_n)$, for which lx is the lth power of an element x of the commutative group V.

Let V and W be linear spaces over $\mathbb{Q}$, $n \in \mathbb{N}$ and $k, p \in \{0, \dots, n\}$. Recall from [4] that a multivariable mapping $f : V^n \longrightarrow W$ is called *k-additive*, *p-quadratic* and *$n-k-p$-cubic* or briefly, *multi-additive-quadratic-cubic* if f satisfies $A(x+y) = A(x) + A(y)$ in each of some k components, fulfills $Q(2x+y)+Q(2x-y) = Q(x+y)+Q(x-y)+6Q(x)$ in each of some p components and satisfies equation $C(2x+y)+C(2x-y) = 2C(x+y)+2C(x-y)+12C(x)$ in each of the other components. Let us note that for $k = n$, $p = n$ and $k, p = 0$, this definition leads us to multi-additive, multi-quadratic and multi-cubic mappings, respectively. It is easily verified that the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(v_1, \dots, v_n) = \prod_{j=1}^{k} \prod_{i=k+1}^{k+p} \prod_{r=k+p+1}^{n} v_j v_i^2 v_r^3$ is a multi-additive-quadratic-cubic mapping.

We remember that the celebrated Ulam query [18] about the stability of group homomorphisms has been studied and established for instance in papers and books [8–10, 12, 13, 16, 17] and moreover references therein. In addition, in the two last decades, the Ulam stability challenge has been answered and investigated for some special multivariable mappings such as multi-additive, multi-quadratic and multi-cubic mappings for example in [2, 5–7, 11, 14, 15, 19].

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In [4, Corollary 1], the authors obtained a stability result for the multi-additive-quadratic-cubic functional equation (see equation (2.1)). In this paper, we modify this corollary and then by an example show that the assertion is not true for $\alpha = 3n - 2k - p$.

2. Main results

Let V and W be linear spaces over $\mathbb{Q}$, $n \in \mathbb{N}$ and $k, p \in \{0, \dots, n\}$. It is shown in [4, Proposition 2] that every multi-additive-quadratic-cubic mapping $f : V^n \longrightarrow W$ satisfies the equation

$$\sum_{s\in\{-1,1\}^p} \sum_{t\in\{-1,1\}^{n-k-p}} f\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right)$$

$$= \sum_{l=0}^{p} \sum_{m=0}^{n-k-p} \sum_{i\in\{1,2\}} 6^l \times 2^{n-k-p-m} \times 12^m f\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right) \tag{2.1}$$

for all $x_i^k = (x_{i1}, \dots, x_{ik}) \in V^k$, $x_i^p = (x_{i,k+1}, \dots, x_{i,k+p}) \in V^p$ and $x_i^{n-k-p} = (x_{i,k+p+1}, \dots, x_{in}) \in V^{n-k-p}$, $(i \in \{1, 2\})$, where

$$f\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right) := \sum_{\mathfrak{A}_{k+p}\in\mathbb{A}_l^{k+p}} \sum_{\mathfrak{B}_n\in\mathbb{B}_m^{n-k-p}} f\left(x_i^k, \mathfrak{A}_{k+p}, \mathfrak{B}_n\right)$$

for $i \in \{1, 2\}$, whereas

$$\mathbb{A}_l^{k+p} := \left\{\mathfrak{A}_{k+p} = (A_{k+1}, \dots, A_{k+p}) \in \mathbb{A}^{k+p} \,|\, \mathrm{Card}\{A_j : A_j = x_{1j}\} = l\right\}$$

and

$$\mathbb{B}_m^{n-k-p} := \left\{\mathfrak{B}_n = (B_{k+p+1}, \dots, B_n) \in \mathbb{B}^{n-k-p} \,|\, \mathrm{Card}\{B_j : B_j = x_{1j}\} = m\right\}$$

are the subsets of $\mathbb{A}^{k+p} = \{\mathfrak{A}_{k+p} = (A_{k+1}, \dots, A_{k+p}) \,|\, A_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$ and $\mathbb{B}^{n-k-p} = \{\mathfrak{B}_n = (B_{k+p+1}, \dots, B_n) \,|\, B_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$, respectively.

Recall from [4] that a mapping $f : V^n \longrightarrow W$ has *the s-power condition* in the jth component if

$$f(v_1, \dots, v_{j-1}, 2v_j, v_{j+1}, \dots, v_n) = 2^s f(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_n)$$

for all $v_1, \dots, v_n \in V$. Note that 2-power (resp., 3-power) condition is sometimes called the quadratic (resp., cubic) condition.

For a converse version of the above result, it is proved in Proposition 3 of [4] that each mapping $f : V^n \longrightarrow W$ fulfills equation (2.1) and the cubic condition in the last $n - k - p$ and the quadratic condition in the middle p components, then it is multi-additive-quadratic-cubic. In the next result, we modify Corollary 2 from [4] without the proof.

Corollary 2.1. *Let $\delta > 0$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 3n - 2k - p$. Let also V be a normed space and W be a Banach space. Suppose that $f : V^n \longrightarrow W$ is a* *mapping fulfilling*

$$\|\mathcal{D}_{aqc}f(x_1,x_2)\| \leq \delta \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^\alpha$$

for all $x_1, x_2 \in V^n$*. Then, there exists a solution* $\mathcal{F} : V^n \longrightarrow W$ *of* (2.1) *such that*

$$\|f(x) - \mathcal{F}(x)\| \leq \frac{\delta}{2^{n-k}|2^{3n-2k-p} - 2^\alpha|}\left(2\sum_{j=1}^{k}\|x_j\|^\alpha + \sum_{j=k+1}^{p}\|x_j\|^\alpha\right)$$

for all $x = (x_1, \ldots, x_n) \in V^n$*. Moreover, if* $\mathcal{F}$ *satisfies the cubic condition in the last* $n-k-p$ *components and the quadratic condition in the some* p *components, then it is a unique multi-additive-quadratic-cubic mapping.*

In the sequel, we will indicate an example to show that the condition $\alpha \neq 3n - 2k - p$ in Corollary 2.1 is necessary. For doing this we need some fundamental results as follows. At the first, we bring the upcoming result which was presented in [11, Theorem 13.4.3].

Theorem 2.2. *Let* $h : \mathbb{R}^{d^m} \longrightarrow \mathbb{R}$ *be a continuous* p*-additive function. Then there exist constants* $c_{j_1\cdots j_d} \in \mathbb{R}$*,* $j_1, \ldots, j_d = 1, \ldots, m$*, such that*

$$h(x_1, \ldots, x_d) = \sum_{j_1=1}^{m}\cdots\sum_{j_d=1}^{m} c_{j_1\cdots j_d}x_{1j_1}\cdots x_{dj_d}$$

for all $x_i = (x_{i1}, \ldots, x_{im})$ *and* $i = 1, \ldots, d$*.*

We bring the following results which have been proved in [1] and [3].

Proposition 2.3 ([3, Proposition 14])**.** *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous* n*-quadratic function. Then,* f *has the form*

$$f(r_1, \ldots, r_n) = cr_1^2\cdots r_n^2$$

for all $r_1, \ldots, r_n \in \mathbb{R}$*, where* c *is a constant in* $\mathbb{R}$*.*

Proposition 2.4 ([1, Proposition 2.4])**.** *If* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a continuous* n*-cubic function, then there exists a constant* $c \in \mathbb{R}$ *such that*

$$f(r_1, \ldots, r_n) = cr_1^3\cdots r_n^3, \qquad (r_1, \ldots, r_n \in \mathbb{R}).$$

In the next theorem, we give a representation of the multi-additive-quadratic-cubic mappings on $\mathbb{R}^n$. Indeed, it is a direct consequence of the above results.

Theorem 2.5. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous* k*-additive,* p*-quadratic and* $n-k-p$*-cubic function. Then, there exists a constant* $c \in \mathbb{R}$ *such that*

$$f(x_1, \ldots, x_n) = c\prod_{j=1}^{k}\prod_{i=k+1}^{k+p}\prod_{r=k+p+1}^{n} x_j x_i^2 x_r^3$$

for all $x_1, \ldots, x_n \in \mathbb{R}$*.*

Proof. We firstly identify $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with $(x^k, x^p, x^{n-k}) \in \mathbb{R}^k \times \mathbb{R}^p \times \mathbb{R}^{n-k-p}$, where $x^k := (x_1, \dots, x_k)$, $x^p := (x_{k+1}, \dots, x_{k+p})$ and $x^{n-k-p} := (x_{k+p+1}, \dots, x_n)$. For any $x^p \in \mathbb{R}^p$, $x^{n-k-p} \in \mathbb{R}^{n-k-p}$, consider the mapping $\mathcal{T}_{x^p, x^{n-k-p}} : \mathbb{R}^k \longrightarrow \mathbb{R}$ defined by

$$\mathcal{T}_{x^p, x^{n-k-p}}(x_1, \dots, x_k) := f\left(x_1, \dots, x_k, x^p, x^{n-k-p}\right).$$

By assumption, $\mathcal{T}_{x^p, x^{n-k-p}}$ is k-additive. It follows from Theorem 2.2 for the case $d = 1$ that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\mathcal{T}_{x^p, x^{n-k-p}}(x_1, \dots, x_k) = f\left(x_1, \dots, x_k, x^p, x^{n-k-p}\right) = c_1 \prod_{j=1}^{k} x_j. \tag{2.2}$$

Note that c_1 depends on x^p, x^{n-k-p}. In fact,

$$c_1 = T(x^p, x^{n-k-p}). \tag{2.3}$$

Putting $x_1 = \dots = x_k = 1$ in (2.2) and applying (2.3), we get

$$c_1 = T(x^p, x^{n-k-p}) = f(1, \dots, 1, x^p, x^{n-k-p}). \tag{2.4}$$

Once again, for any $x^{n-k-p} \in \mathbb{R}^{n-k-p}$, define the mapping $\mathcal{S}_{x^{n-k-p}} : \mathbb{R}^p \longrightarrow \mathbb{R}$ through

$$\mathcal{S}_{x^{n-k-p}}(x_{k+1}, \dots, x_{k+p}) := f\left(1, \dots, 1, x_{k+1}, \dots, x_{k+p}, x^{n-k-p}\right).$$

Since $\mathcal{S}_{x^{n-k-p}}$ is p-quadratic, by Proposition 2.3 there exists a constant $c_2 \in \mathbb{R}$ such that

$$\begin{aligned} \mathcal{S}_{x^{n-k-p}}(x_{k+1}, \dots, x_{k+p}) &= f\left(1, \dots, 1, x_{k+1}, \dots, x_{k+p}, x^{n-k-p}\right) \\ &= c_2 \prod_{i=k+1}^{k+p} x_i^2. \end{aligned} \tag{2.5}$$

It is obvious that c_2 depends on x^{n-k-p} and hence

$$c_2 = S(x^{n-k-p}). \tag{2.6}$$

Letting $x_{k+1} = \dots = x_{k+p} = 1$ in (2.5) and using (2.6), we get

$$c_2 = S(x^{n-k-p}) = f(\overbrace{1, \dots, 1}^{k\text{-times}}, \overbrace{1, \dots, 1}^{p\text{-times}}, x^{n-k-p}).$$

On the other hand, S is an $n-k-p$-cubic function, and so by Proposition 2.4, there exists a constant $c_3 \in \mathbb{R}$ such that

$$S(x^{n-k-p}) = f\left(\overbrace{1, \dots, 1}^{k\text{-times}}, \overbrace{1, \dots, 1}^{p\text{-times}}, x_{k+p+1}, \dots, x_n\right) = c_3 \prod_{r=k+p+1}^{n} x_r^3. \tag{2.7}$$

The result now follows from (2.2), (2.4), (2.5), (2.6) and (2.7). □

We remember that in the proofs of Theorem 2.2, Proposition 2.3 and Proposition 2.4 the continuity of f with respect to each variable separately were used, and thus all results again hold if and only if f is assumed separately continuous with respect to each component. On the other hand, in light of the proofs of all mentioned results, if the continuity condition of f is removed, then the results remain valid for a function $f : \mathbb{Q}^p \longrightarrow \mathbb{Q}$. Therefore, the same discussions can be repeated without any gap for Theorem 2.5. We use this fact to make a non-stable example. In other words, we show the hypothesis $\alpha \neq 3n - 2k - p$ can not be eliminated in Corollary 2.1. The argument is taken to what given in [3, Example 1], but we include it completely for the sake of completeness. Before it, we bring a notation as follows.

For a mapping $f : V^n \longrightarrow W$, we have the notation

$$\begin{aligned}&\mathcal{D}_{aqc}f(x_1, x_2)\\ := &\sum_{s\in\{-1,1\}^p} \sum_{t\in\{-1,1\}^{n-k-p}} f\left(x_1^k + x_2^k, 2x_1^p + sx_2^p, 2x_1^{n-k-p} + tx_2^{n-k-p}\right)\\ &-\sum_{l=0}^{p} \sum_{m=0}^{n-k-p} \sum_{i\in\{1,2\}} 6^l \times 2^{n-k-p-m} \times 12^m f\left(x_i^k, \mathbb{A}_l^{k+p}, \mathbb{B}_m^{n-k-p}\right)\end{aligned}$$

for all $x_i = (x_i^k, x_i^p, x_i^{n-k-p})$ in which $x_i^k = (x_{i1}, \dots, x_{ik}) \in V^k$, $x_i^p = (x_{i,k+1}, \dots, x_{i,k+p}) \in V^p$ and $x_i^{n-k-p} = (x_{i,k+p+1}, \dots, x_{in}) \in V^{n-k-p}$, where $i \in \{1, 2\}$ (see also the begging of this section).

Example 2.6. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Consider the function $\mathbf{1} : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ whose range is the constant 1. Set $|\mathcal{D}_{aqc}\mathbf{1}| = M$ and $\lambda = \frac{2^{3n-2k-p}-1}{2^{2(3n-2k-p)}M}\varepsilon$. Define the function $\phi : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ through

$$\phi(r_1, \dots, r_n) := \begin{cases} \lambda \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} r_j r_t^2 r_l^3 & \text{for all } r_u \text{ with } |r_u| < 1,\\ \lambda & \text{otherwise}\end{cases}$$

for $u \in \{1, \dots, n\}$. Moreover, consider the function $f : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ defined via

$$f(r_1, \dots, r_n) = \sum_{s=0}^{\infty} \frac{\phi(2^s r_1, \dots, 2^s r_n)}{2^{(3n-2k-p)s}}, \qquad (r_j \in \mathbb{Q}).$$

It is clear that ϕ is bounded by λ. Indeed, for each $(r_1, \dots, r_n) \in \mathbb{Q}^n$, we have $|f(r_1, \dots, r_n)| \leq \frac{2^{3n-2k-p}}{2^{3n-2k-p}-1}\lambda$. It follows from the last inequality that

$$|\mathcal{D}_{aqc}f(x_1, x_2)| \leq \frac{2^{3n-2k-p}}{2^{3n-2k-p} - 1}\lambda M \tag{2.8}$$

for all $x_i = (x_i^k, x_i^p, x_i^{n-k-p})$ in which $x_i^k = (x_{i1}, \dots, x_{ik}) \in V^k$, $x_i^p = (x_{i,k+1}, \dots, x_{i,k+p}) \in V^p$ and $x_i^{n-k-p} = (x_{i,k+p+1}, \dots, x_{in}) \in V^{n-k-p}$, where $i \in \{1, 2\}$.

We claim that

$$|\mathcal{D}_{aqc}f\,(x_1,x_2)| \leq \varepsilon \sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p} \tag{2.9}$$

for all $x_1, x_2 \in \mathbb{Q}^n$. Here, we discuss three cases as follows:

(i) For the case that $x_1 = x_2 = 0$, obviously, (2.9) is valid.

(ii) Assume that $x_1, x_2 \in \mathbb{Q}^n$ with

$$\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p} < \frac{1}{2^{3n-2k-p}}.$$

Therefore, there exists a positive integer m such that

$$\frac{1}{2^{(m+1)(3n-2k-p)}} < \sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p} < \frac{1}{2^{m(3n-2k-p)}}, \tag{2.10}$$

and so

$$|x_{ij}|^{3n-2k-p} < \sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p} < \frac{1}{2^{m(3n-2k-p)}}. \tag{2.11}$$

We now concludes from (2.11) that $2^m|x_{ij}| < 1$ for all $i \in \{1,2\}$ and $j \in \{1,\ldots,n\}$, and thus $2^{m-1}|x_{ij}| < 1$. Moreover, for any $z_1, z_2 \in \{x_{ij}\,|\, i \in \{1,2\},\ j \in \{1,\ldots,n\}\}$, we have

$$2^{m-1}|\,z_1 \pm z_2| < 1,\ \ 2^{m-1}|\,2z_1 \pm z_2| < 1.$$

The definition of ϕ shows that it is a multi-additive-quadratic-cubic function on $(-1,1)^n$, and hence $\mathcal{D}_{aqc}\phi\,(2^s x_1, 2^s x_2) = 0$ for all $s \in \{0,1,2,\ldots,m-1\}$. Now, (2.10) and the last equality imply that

$$\begin{aligned}
\frac{|\mathcal{D}_{aqc}f\,(2^s x_1, 2^s x_2)|}{\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p}} &\leq \sum_{s=m}^{\infty}\frac{|\mathcal{D}_{aqc}\phi\,(2^s x_1, 2^s x_2)|}{2^{(3n-2k-p)s}\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p}}\\
&\leq \sum_{s=0}^{\infty}\frac{\lambda M}{2^{(3n-2k-p)(s+m)}\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p}}\\
&\leq 2^{3n-2k-p}\lambda M\sum_{s=0}^{\infty}\frac{1}{2^{s(3n-2k-p)}}\\
&= \lambda M\frac{2^{2(3n-2k-p)}}{2^{3n-2k-p}-1} = \varepsilon
\end{aligned}$$

for all $x_1, x_2 \in \mathbb{Q}^n$ and therefore (2.9) holds in this case.

(iii) Let $\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p} \geq \frac{1}{2^{3n-2k-p}}$. Applying (2.8), we obtain

$$\frac{|\mathcal{D}_{aqc}f\,(2^s x_1, 2^s x_2)|}{\sum_{i=1}^{2}\sum_{j=1}^{n}|x_{ij}|^{3n-2k-p}} \leq 2^{3n-2k-p}\frac{2^{3n-2k-p}}{2^{3n-2k-p}-1}\lambda M = \varepsilon.$$

The above arguments necessitate that inequality (2.9) is true for all $x_1, x_2 \in \mathbb{Q}^n$. Suppose contrary to our claim for non-stability, that there exits a multi-additive-quadratic-cubic mapping $\mathcal{F}_{aqc} : \mathbb{Q}^n \longrightarrow \mathbb{Q}$ of (2.1) and $\delta > 0$ such that

$$|f(r_1, \dots, r_n) - \mathcal{F}_{aqc}(r_1, \dots, r_n)| \leq \delta \sum_{j=1}^{k} \sum_{t=k+1}^{k+p} \sum_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3$$

for all $(r_1, \dots, r_n) \in \mathbb{Q}^n$. Without loss of generality, one can take a number $\mu \in [0, \infty)$ so that

$$\delta \sum_{j=1}^{k} \sum_{t=k+1}^{k+p} \sum_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3 \leq \mu \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3.$$

Hence,

$$|f(r_1, \dots, r_n) - \mathcal{F}_{aqc}(r_1, \dots, r_n)| < \mu \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3$$

for all $(r_1, \dots, r_n) \in \mathbb{Q}^n$. A consequence of Theorem 2.5 implies that there is a constant $c \in \mathbb{R}$ such that $\mathcal{F}_{aqc}(r_1, \dots, r_n) = c \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3$ for all $(r_1, \dots, r_n) \in \mathbb{Q}^n$. It follows the discussion above that

$$|f(r_1, \dots, r_n)| \leq (|c| + \mu) \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3 \tag{2.12}$$

for all $(r_1, \dots, r_n) \in \mathbb{Q}^n$. Given $m \in \mathbb{N}$ such that $m\lambda > |c| + \mu$. For $r = (r_1, \dots, r_n) \in \mathbb{Q}^n$ with $r_j \in \left(0, \frac{1}{2^{m-1}}\right)$ for all $j \in \{1, \dots, n\}$, we have $2^s r_j \in (0, 1)$ for all $s = 0, 1, \dots, m-1$. Therefore,

$$\begin{aligned}
|f(r_1, \dots, r_n)| &= \left| \sum_{s=0}^{\infty} \frac{\phi\left(2^s r_1, \dots, 2^s r_n\right)}{2^{s(3n-2k-p)}} \right| \\
&= \left| \lambda \sum_{s=0}^{m-1} \frac{2^{s(3n-2k-p)} \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} r_j r_t^2 r_l^3}{2^{s(3n-2k-p)}} \right| \\
&= m\lambda \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3 \\
&> (|c| + \mu) \prod_{j=1}^{k} \prod_{t=k+1}^{k+p} \prod_{l=k+p+1}^{n} |r_j| r_t^2 |r_l|^3,
\end{aligned}$$

and so we are led to a contradiction with (2.12).

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