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GENERALIZED η-RICCI SOLITONS ON PARA-KENMOTSU MANIFOLDS ASSOCIATED TO THE ZAMKOVOY CONNECTION

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Abstract. In this paper, we study para-Kenmotsu manifolds admitting generalized η-Ricci solitons associated to the Zamkovoy connection. We provide an example of generalized η-Ricci soliton on a para-Kenmotsu manifold to prove our results.

1. Introduction

The almost para contact Riemannian manifold was introduced by Sato [33] in 1976. Then, the notion of a para-Sasakian manifold has been defined and studied by Adati and Matsumoto [1] as a class of almost contact Riemannian manifolds. The Kenmotsu manifold was introduced by Kenmotsu [16] in 1972 as a new class of almost contact metric manifolds. Kenmotsu manifolds are very closely related to the warped product manifolds. Sinha and Prasad [37] studied para-Kenmotsu manifolds as a class of almost para contact metric manifolds. For further reading on Kenmotsu manifolds and their generalizations, see [12, 19, 21, 32].

In 1982, Hamilton [14] introduced the notion of Ricci soliton as a special solution to Ricci flow and as a generalization of Einstein metrics on a Riemannian manifold. A Ricci soliton [6] is a triplet (g, V, λ) on a pseudo-Riemannian manifold M such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\mathcal{L}_V$ is the Lie derivative along the potential vector field V, S is the Ricci tensor, and λ is a real constant. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively. If the vector field V is the gradient of a potential function ψ, then g is called a gradient Ricci soliton.

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In 2016, Nurowski and Randall [28] introduced the notion of generalized Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat - 2\alpha S - 2\lambda g = 0, \tag{1.2}$$

where $V^\flat$ is the canonical 1-form associated to V. Also, as a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [10] which it is a 4-tuple (g, V, λ, ρ), where V is a vector field on M, λ and ρ are constants, and g is a pseudo-Riemannian metric satisfying the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. Many authors studied the η-Ricci solitons [5, 11, 13, 17, 22–27, 29, 30, 39]. In particular, if $\rho = 0$, then the η-Ricci soliton equation reduces to the Ricci soliton equation. Motivated by the above studies M. D. Siddiqi [36] introduced the notion of generalized η-Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0. \tag{1.4}$$

Motivated by [3, 7, 20] and the above works, we study generalized η-Ricci solitons on para-Kenmotsu manifolds associated the Zamkovoy connection. We give an example of generalized η-Ricci soliton on a para-Kenmotsu manifold associated the Zamkovoy connection.

The paper is organized as follows. In Section 2, we recall some necessary and fundamental concepts and formulas on para-Kenmotsu manifolds which be used throughout the paper. In Section 3, we give the main results and their proofs. In Section 4, we give an example of a para-Kenmotsu manifold admit in generalized η-Ricci soliton with respect to the Zamkovoy connection.

2. Preliminaries

An n-dimensional pseudo-Riemannian manifold (M, g) is said to be an almost para-contact manifold [2] with an almost contact structure (ϕ, ξ, η, g) if there exist a $(1, 1)$-tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2(X) = X - \eta(X)\xi, \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y on M. In this case, we have $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\eta(X) = g(X, \xi)$. From (2.2) it can be easily deduce that

$$g(X, \phi Y) = -g(\phi X, Y)$$

for all vector fields X, Y on M. An almost para-contact manifold M is called a para-Kenmotsu manifold [15] if

$$(\nabla_X \phi)Y = g(X, \phi Y)\xi - \eta(Y)\phi X \tag{2.3}$$

for all vector fields X, Y on M, where ∇ is the Levi-Civita connection with respect to the metric g. In a para-Kenmotsu manifold, we have

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y). \tag{2.5}$$

Using (2.4) and (2.5), we find

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.7}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{2.8}$$

for all vector fields X, Y, Z, where R is the Riemannian curvature tensor. The Ricci tensor S of a para-Kenmotsu manifold M is defined by $S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$ and we have

$$S(X, \xi) = -(n-1)\eta(X) \tag{2.9}$$

for any vector field X on M.

Let M be an almost contact metric manifold and TM be the tangent bundle of M. We have two naturally defined distribution on tangent bundle TM [38] as follows

$$H = \ker\eta, \qquad \hat{H} = \mathrm{span}\{\xi\},$$

thus we get $TM = H \oplus \hat{H}$. Therefore, by this composition we can define the Zamkovoy connection $\bar{\nabla}$ [4, 18, 31, 43] on M with respect to Levi-Civita connection ∇ as follows

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + ((\nabla_X \eta)(Y))\xi + \eta(X)\phi(Y) \tag{2.10}$$

for all vector fields X, Y on M. On para-Kenmotsu manifolds, using (2.4), (2.5), and (2.10) we obtain

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)X + g(X, Y)\xi + \eta(X)\phi(Y) \tag{2.11}$$

for all vector fields X, Y on M. Let $\bar{R}$ and $\bar{S}$ be the curvature tensor and the Ricci tensor of the connection $\bar{\nabla}$, respectively, that is,

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z,$$

$$\bar{S}(X, Y) = \sum_{i=1}^{n} \epsilon_i g(\bar{R}(e_i, X)Y, e_i).$$

On para-Kenmotsu manifolds, applying (2.11) and the above relation we have

$$\bar{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y \tag{2.12}$$

and

$$\bar{S}(X, Y) = S(X, Y) + (n-1)g(X, Y) \tag{2.13}$$

for all vector fields X, Y, Z on M, where S denotes the Ricci tensor of the connection ∇. Using (2.13), the Ricci operator $\bar{Q}$ of the connection $\bar{\nabla}$ is determined by

$$\bar{Q}X = QX + (n-1)X. \tag{2.14}$$

Let r and $\bar{r}$ be the scalar curvature of the Levi-Civita connection ∇ and the Zamkovoy connection $\bar{\nabla}$. The equation (2.13) yields

$$\bar{r} = r + n^2 - n. \tag{2.15}$$

The generalized η-Ricci soliton associated to the Zamkovoy connection is defined by

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta\otimes\eta + \lambda g = 0, \tag{2.16}$$

where $\bar{S}$ denotes the Ricci tensor of the connection $\bar{\nabla}$,

$$(\overline{\mathcal{L}}_V g)(Y, Z) := g(\tilde{\nabla}_Y V, Z) + g(Y, \tilde{\nabla}_Z V),$$

$V^\flat$ is the canonical 1-form associated to V, that is, $V^\flat(X) = g(V, X)$ for any vector field X, λ is a smooth function on M, and α, β, μ, ρ are real constants such that $(\alpha, \beta, \mu) \neq (0, 0, 0)$.

The generalized η-Ricci soliton equation reduces to

(1) the η-Ricci soliton equation when $\alpha = 1$ and $\mu = 0$,
(2) the Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$,
(3) the generalized Ricci soliton equation when $\rho = 0$.

Note that

$$\begin{aligned}
&(\overline{\mathcal{L}}_V g)(X, Y) \\
&= g(\tilde{\nabla}_X V, Y) + g(X, \tilde{\nabla}_Y V) \\
&= g(\nabla_X V - \eta(V)X + g(X, V)\xi + \eta(X)\phi V, Y) \\
&\quad + g(X, \nabla_Y V - \eta(V)Y + g(Y, V)\xi + \eta(Y)\phi V) \\
&= \mathcal{L}_V g(X, Y) - 2\eta(V)g(X, Y) + g(X, V)\eta(Y) + g(Y, V)\eta(X) \\
&\quad + \eta(X)g(\phi V, Y) + \eta(Y)g(X, \phi V).
\end{aligned} \tag{2.17}$$

3. Main results and their proofs

A para-Kenmotsu manifold is called η-Einstein with respect to the Zamkovoy connection if its Ricci tensor $\bar{S}$ satisfies in the following equation

$$\bar{S} = ag + b\eta\otimes\eta,$$

where a and b are smooth functions on manifold. Let M be a para-Kenmotsu manifold. Now, we consider M satisfies the generalized η-Ricci soliton (2.16) associated to the Zamkovoy connection and the potential vector field V is a pointwise collinear vector field with the structure vector field ξ, that is, $V = f\xi$ for some function f on M. Using (2.14) we get

$$\begin{aligned}\mathcal{L}_{f\xi}g(X,Y) &= g(\nabla_X f\xi, Y) + g(X, \nabla_Y f\xi)\\ &= (Xf)\eta(Y) + fg(X-\eta(X)\xi, Y) + (Yf)\eta(X) + fg(X, Y-\eta(Y)\xi)\\ &= (Xf)\eta(Y) + (Yf)\eta(X) + 2f(g(X,Y) - \eta(X)\eta(Y)),\end{aligned}$$

hence

$$\overline{\mathcal{L}}_{f\xi}g(X,Y) = (Xf)\eta(Y) + (Yf)\eta(X)$$

for all vector fields X, Y on M. Also, we obtain

$$\xi^\flat \otimes \xi^\flat(X,Y) = \eta(X)\eta(Y) \tag{3.1}$$

for all vector fields X, Y. Applying $V = f\xi$, (2.13) and (3.1) in the equation (2.16) we infer

$$\alpha\bar{S}(X,Y) + \frac{\beta}{2}\left[(Xf)\eta(Y) + (Yf)\eta(X)\right] + (\mu f^2 + \rho)\eta(X)\eta(Y) + \lambda g(X,Y) = 0 \tag{3.2}$$

for all vector fields X, Y on M. We plug $Y = \xi$ in the above equation and using (2.9) and (2.13) to yield

$$\frac{\beta}{2}Xf + \frac{\beta}{2}(\xi f)\eta(X) + (\mu f^2 + \rho + \lambda)\eta(X) = 0 \tag{3.3}$$

for any vector fields X on M. Taking $X = \xi$ in (3.3) gives

$$\beta\xi f = -(\mu f^2 + \rho + \lambda). \tag{3.4}$$

Inserting (3.4) in (3.3), we conclude

$$\beta Xf = -(\mu f^2 + \rho + \lambda)\eta(X), \tag{3.5}$$

which yields

$$\beta df = -(\mu f^2 + \rho + \lambda)\eta. \tag{3.6}$$

Applying (3.6) in (3.2) we obtain

$$\alpha\bar{S}(X,Y) = 0, \tag{3.7}$$

which implies that $\alpha\bar{r} = 0$.

Therefore, this leads to the following theorem:

Theorem 3.1. *Let (M, g, ϕ, ξ, η) be a para-Kenmotsu manifold. If M admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Zamkovoy connection such that $\alpha \neq 0$ and $V = f\xi$ for some smooth function f on M, then M is a flat manifold with respect to the Zamkovoy connection.*

Now, let M be an η-Einstein para-Kenmotsu manifold with respect to the Zamkovoy connection and $V = \xi$. Then we get $\bar{S} = ag + b\eta \otimes \eta$ for some constants a and b on M. From (2.17) we have

$$\overline{\mathcal{L}}_\xi g(X,Y) = 0$$

for all vector fields X, Y. Therefore,

$$\begin{aligned}&\alpha\bar{S}+\frac{\beta}{2}\bar{\mathcal{L}}_{\xi}g+\mu\xi^{\flat}\otimes\xi^{\flat}+\rho\eta\otimes\eta+\lambda g\\&=a\alpha g+b\alpha\eta\otimes\eta+\mu\eta\otimes\eta+\rho\eta\otimes\eta+\lambda g\\&=(a\alpha+\lambda)g+(b\alpha+\mu+\rho)\eta\otimes\eta.\end{aligned}$$

From the above equation M admits a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Zamkovoy connection if $\lambda = -a\alpha$ and $\rho = -b\alpha - \mu$.

Hence, we can state the following theorem:

Theorem 3.2. *Suppose that M is an η-Einstein para-Kenmotsu manifold with respect to the Zamkovoy connection such that $\bar{S} = ag+b\eta\otimes\eta$ for some constants a and b. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -b\alpha - \mu, -a\alpha)$ with respect to the Zamkovoy connection.*

Definition. Let M be a para-Kenmotsu manifold with the Zamkovoy connection $\bar{\nabla}$. The M-projective curvature tensor $\bar{M}$ [35] with respect to the Zamkovoy connection on M is defined by

$$\begin{aligned}&\bar{M}(X,Y)Z \quad (3.8)\\&=\bar{R}(X,Y)Z-\frac{\bar{1}}{2(n-1)}\left(\bar{S}(Y,Z)X-\bar{S}(X,Z)Y+g(X,Z)\bar{Q}Y-g(Y,Z)\bar{Q}X\right)\end{aligned}$$

for all vector fields X, Y, Z on M. A para-Kenmotsu manifold M is called quasi-M-projectively flat with respect to the Zamkovoy connection if

$$g(\bar{M}(\phi X,Y)Z,\phi W)=0$$

for all vector fields X, Y, Z and W on M.

Now consider a para-Kenmotsu manifold M is quasi-M-projectively flat with respect to the Zamkovoy connection. From [35] we have

$$S(X,Z)=\frac{-2n^2+4n-2-r}{n-1}g(X,Z) \tag{3.9}$$

and

$$\bar{S}(X,Z)=\frac{-n^2+2n-1-r}{n-1}g(X,Z) \tag{3.10}$$

for all vector fields X, Z on M. Therefore, we have the following corollary.

Corollary 3.3. *Let M be a quasi-M-projectively flat para-Kenmotsu manifold with respect to the Zamkovoy connection. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -\mu, -\frac{-n^2+2n-1-r}{n-1}\alpha)$ with respect to the Zamkovoy connection.*

Let M be an M-projectively flat para-Kenmotsu manifold with the Zamkovoy connection, that is, $\bar{M}(X,Y)Z = 0$ for all vector fields X, Y, Z on M. In this case, from [35] we get

$$\bar{S}(X,Z) = \frac{\bar{r}}{n} g(X,Z) \tag{3.11}$$

for all vector fields X, Z on M. Therefore, we have the following corollary.

Corollary 3.4. *Let M be an M-projectively flat para-Kenmotsu manifold with respect to the Zamkovoy connection. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -\mu, -\frac{\bar{r}}{n}\alpha)$ with respect to the Zamkovoy connection.*

Suppose that M is a ξ-M-projectively flat para-Kenmotsu manifold with the Zamkovoy connection, that is, $\bar{M}(X,Y)\xi = 0$ for all vector fields X, Y on M. In this case, from [35] we obtain

$$\bar{S}(X,Z) = 0 \tag{3.12}$$

for all vector fields X, Z on M. Therefore, we have the following corollary.

Corollary 3.5. *Let M be a ξ-M-projectively flat para-Kenmotsu manifold with respect to the Zamkovoy connection. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -\mu, 0)$ with respect to the Zamkovoy connection.*

Now assume that M is a ϕ-M-projectively flat para-Kenmotsu manifold with the Zamkovoy connection, that is, $g(\bar{M}(\phi X, \phi Y)\phi Z, \phi W) = 0$ for all vector fields X, Y, Z, W on M. In this case, from [35] we can write $\bar{S}(X,Y) = -(n-1)\eta(X)\eta(Y) + (n-1)g(X,Y)$ for all vector fields X, Y on M. Therefore, we have the following corollary.

Corollary 3.6. *Let M be a ϕ-M-projectively flat para-Kenmotsu manifold with respect to the Zamkovoy connection. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, (n-1)\alpha - \mu, -(n-1)\alpha)$ with respect to the Zamkovoy connection.*

Let that M be a para-Kenmotsu manifold with the Zamkovoy connection satisfying the condition $\bar{M}(\xi, X) \cdot \bar{S} = 0$ for any vector fields X on M. In this case, from [35] we have $\bar{S}(X,Y) = 0$ for all vector fields X, Y on M. Therefore, we have the following corollary.

Corollary 3.7. *Let M be a para-Kenmotsu manifold with respect to the Zamkovoy connection satisfying the condition $\bar{M}(\xi, X) \cdot \bar{S} = 0$. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -\mu, 0)$ with respect to the Zamkovoy connection.*

Now assume that M is a para-Kenmotsu manifold with the Zamkovoy connection satisfying the condition $\bar{M}(\xi, X) \cdot \bar{R} = 0$ for any vector fields X on M. In this case, from [35] we have $\bar{S}(X,Y) = \frac{4}{3}(n-1)g(X,Y)$ for all vector fields X, Y on M. Therefore, we have the following corollary.

Corollary 3.8. *Let M be a para-Kenmotsu manifold with respect to the Zamkovoy connection satisfying the condition $\bar{M}(\xi, X) \cdot \bar{R} = 0$. Then M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -\mu, -\frac{4}{3}(n-1)\alpha)$ with respect to the Zamkovoy connection.*

Definition. A vector field V is said to a conformal Killing vector field if

$$(\mathcal{L}_V g)(X,Y) = 2hg(X,Y) \tag{3.13}$$

for all vector fields X, Y, where h is some function on M. The conformal Killing vector field V is called

- proper when h is not constant,
- homothetic vector field when h is a constant,
- Killing vector field when $h = 0$.

Let vector field V be a conformal Killing vector field with respect to the Zamkovoy connection and satisfies in $(\overline{\mathcal{L}}_V g)(X,Y) = 2hg(X,Y)$. By (2.13) and (2.16) we have

$$\alpha\bar{S}(X,Y) + \beta hg(X,Y) + \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0 \tag{3.14}$$

for all vector fields X, Y. By inserting $Y = \xi$ in the above equation we get

$$g(-(n-1)\alpha\xi + \beta h\xi + \mu\eta(V)V + \rho\xi + \lambda\xi, X) = 0. \tag{3.15}$$

Since X is an arbitrary vector field we have the following theorem.

Theorem 3.9. *If the metric g of a para-Kenmotsu manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$, where V is a conformal Killing vector field with respect to the Zamkovoy connection, that is $\overline{\mathcal{L}}_V g = 2hg$, then*

$$(-(n-1)\alpha + \beta h + \rho + \lambda)\xi + \mu\eta(V)V = 0. \tag{3.16}$$

Definition. A nonvanishing vector field V on a pseudo-Riemannian manifold (M, g) is called torse-forming [41] if

$$\nabla_X V = fX + \omega(X)V \tag{3.17}$$

for all vector field X, where ∇ is the Levi-Civita connection of g, f is a smooth function and ω is a 1-form. The vector field V is called

- concircular [8,40] whenever in the equation (3.17) the 1-form ω vanishes identically,
- concurrent [34,42] if in equation (3.17) the 1-form ω vanishes identically and $f = 1$,
- parallel vector field if in equation (3.17) $f = \omega = 0$,
- torqued vector field [9] if in equation (3.17) $\omega(V) = 0$.

Let $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ be a generalized η-Ricci soliton on a para-Kenmotsu manifold, where V is a torse-forming vector filed and satisfied in (3.17). Then

$$\begin{aligned}\alpha\bar{S}(X,Y) + \frac{\beta}{2}&[(\mathcal{L}_V g)(X,Y) - 2\eta(V)g(X,Y) + g(X,V)\eta(Y) \\ &+ g(Y,V)\eta(X) + \eta(X)g(\phi V, Y) + \eta(Y)g(X, \phi V)] \\ &+ \mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0\end{aligned} \tag{3.18}$$

for all vector fields X, Y. On the other hand,

$$(\mathcal{L}_V g)(X,Y) = 2fg(X,Y) + \omega(X)g(V,Y) + \omega(Y)g(Y,X) \tag{3.19}$$

for all vector fields X, Y. Applying (3.19) into (3.18) we arrive at

$$\begin{aligned}\alpha\bar{S}(X,Y)+\frac{\beta}{2}\,&[2fg(X,Y)+\omega(X)g(V,Y)+\omega(Y)g(Y,X)-2\eta(V)g(X,Y)\\&+g(X,V)\eta(Y)+g(Y,V)\eta(X)+\eta(X)g(\phi V,Y)+\eta(Y)g(X,\phi V)]\\&+\mu V^{\flat}(X)V^{\flat}(Y)+\rho\eta(X)\eta(Y)+\lambda g(X,Y)=0.\end{aligned} \tag{3.20}$$

We take contraction of the above equation over X and Y to obtain

$$\alpha\bar{r}+n\,[\beta f+\lambda]+\rho+\beta\omega(V)-(n-1)\beta\eta(V)+\mu|V|^2=0. \tag{3.21}$$

Therefore we have the following theorem.

Theorem 3.10. *If the metric g of a para-Kenmotsu manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$, where V is the torse-forming vector filed and satisfied in* (3.17)*, then*

$$\lambda=-\frac{1}{n}\left[\alpha(r+3n^2-9n)+\rho+\beta\omega(V)-(n-1)\beta\eta(V)+\mu|V|^2\right]-\beta f. \tag{3.22}$$

4. Example

In this section, we give an example of a para-Kenmotsu manifold with respect to the Zamkovoy connection.

Example 4.1. Let (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and $M = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. We consider the linearly independent vector fields

$$e_1=\frac{\partial}{\partial x},\qquad e_2=\frac{\partial}{\partial y},\qquad e_3=(x+2y)\frac{\partial}{\partial x}+(2x+y)\frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$

We define the metric g by

$$g(e_i,e_j)=\begin{cases}1, & \text{if } i=j \text{ and } i,j\in\{1,3\},\\ -1, & \text{if } i=j=2,\\ 0, & \text{otherwise,}\end{cases}$$

and an almost contact structure (ϕ, ξ, η) on M by

$$\xi=e_3,\quad \eta(X)=g(X,e_3),\quad \phi=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$$

for all vector field X. Note the relations $\phi^2(X) = X - \eta(X)\xi$, $\eta(\xi) = 1$, and $g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)$ hold. Thus (M, ϕ, ξ, η, g) defines an almost para-contact structure on M. We obtain the following:

$[,]$	e_1	e_2	e_3
e_1	0	0	e_1+2e_2
e_2	0	0	$2e_1+e_2$
e_3	$-e_1-2e_2$	$-2e_1-e_2$	0

The Levi-Civita connection ∇ of M is give by

$$\nabla_{e_i}e_j = \begin{pmatrix} -e_3 & 0 & e_1 \\ 0 & e_3 & e_2 \\ -2e_2 & -2e_1 & 0 \end{pmatrix}.$$

Hence the structure (ϕ, ξ, η) satisfies the formula $\nabla_X\xi = X - \eta(X)\xi$ and $(\nabla_X\phi)Y = g(\phi X, Y) - \eta(Y)\phi X$, thus (M, ϕ, ξ, η, g) becomes a para-Kenmotsu manifold. Now, using (2.12) we get the Zamkovoy connection on M as follows

$$\bar{\nabla}_{e_i}e_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -e_2 & -e_1 & 0 \end{pmatrix}.$$

The all components of curvature tensor with respect to the Zamkovoy connection are zero, that is, $\bar{R}(e_i, e_j)e_k = 0$ for all $1 \leq i, j, k \leq 3$. Thus, we get $\bar{S} = 0$. If we assume that $V = \xi$, then $\overline{\mathcal{L}}_V g = 0$. Thus $(g, \xi, \alpha, \beta, \mu, \rho = -\mu, \lambda = 0)$ is a generalized η-Ricci soliton on manifold M with respect to the Zamkovoy connection.

References

[1] T. Adati and K. Matsumoto, *On conformally recurrent and conformally symmetric P-Sasakian manifolds*, TRU Math. **13** (1977), no. 1, 25–32.

[2] P. Alegre, *Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds*, Taiwanese J. Math. **17** (2013), no. 3, 897–910. https://doi.org/10.11650/tjm.17.2013.2427

[3] S. Azami, *Generalized Ricci solitons of three-dimensional Lorentzian Lie groups associated canonical connections and Kobayashi-Nomizu connections*, J. Nonlinear Math. Phys. **30** (2023), no. 1, 1–33. https://doi.org/10.1007/s44198-022-00069-2

[4] A. M. Blaga, *Canonical connections on para-Kenmotsu manifolds*, Novi Sad J. Math. **45** (2015), no. 2, 131–142. https://doi.org/10.30755/nsjom.2014.050

[5] A. M. Blaga and M. Crâşmăreanu, *Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry*, Filomat **31** (2017), no. 2, 499–504. https://doi.org/10.2298/FIL1702499B

[6] C. Călin and M. Crâşmăreanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 3, 361–368.

[7] G. Calvaruso, *Three-dimensional homogeneous generalized Ricci solitons*, Mediterr. J. Math. **14** (2017), no. 5, Paper No. 216, 21 pp. https://doi.org/10.1007/s00009-017-1019-2

[8] B.-Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **46** (2014), no. 12, Art. 1833, 5 pp. https://doi.org/10.1007/s10714-014-1833-9

[9] B.-Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math. **41** (2017), no. 2, 239–250. https://doi.org/10.5937/kgjmath1702239c

[10] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. (2) **61** (2009), no. 2, 205–212. https://doi.org/10.2748/tmj/1245849443

[11] K. De and U. C. De, *η-Ricci solitons on Kenmotsu 3-manifolds*, An. Univ. Vest Timiş. Ser. Mat.-Inform. **56** (2018), no. 1, 51–63. https://doi.org/10.2478/awutm-2018-0004

[12] S. S. Devi, K. L. S. Prasad, and T. Satyanarayana, *Certain curvature connections on Lorentzian para-Kenmotsu manifolds*, RT & A **17** (2022), no. 2, 413–421.

[13] A. Ghosh, *Ricci almost solitons and contact geometry*, Adv. Geom. **21** (2021), no. 2, 169–178. https://doi.org/10.1515/advgeom-2019-0026

[14] R. S. Hamilton, *The Ricci flow on surfaces*, in Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988. https://doi.org/10.1090/conm/071/954419

[15] A. Haseeb and R. Prasad, *Certain results on Lorentzian para-Kenmotsu manifolds*, Bulletin of Parana's Mathematical Society, 2018.

[16] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. (2) **24** (1972), 93–103. https://doi.org/10.2748/tmj/1178241594

[17] P. Majhi, U. C. De, and D. Kar, *η-Ricci solitons on Sasakian 3-manifolds*, An. Univ. Vest Timiş. Ser. Mat.-Inform. **55** (2017), no. 2, 143–156. https://doi.org/10.1515/awutm-2017-0019

[18] A. Mandai, A. H. Sarkar, A. Biswas, and A. Das, *Some curvature properties of para-Kenmotsu manifolds with respect to Zamkovoy connection*, J. Hyperstructure **11** (2022), no. 1, 166–182.

[19] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12** (1989), no. 2, 151–156.

[20] M. E. A. Mekki and A. M. Cherif, *Generalised Ricci solitons on Sasakian manifolds*, Kyungpook Math. J. **57** (2017), no. 4, 677–682. https://doi.org/10.5666/KMJ.2017.57.4.677

[21] I. Mihai and R. M. Roşca, *On Lorentzian P-Sasakian manifolds*, in Classical analysis (Kazimierz Dolny, 1991), 155–169, World Sci. Publ., River Edge, NJ, 1992.

[22] D. M. Naik, *Ricci solitons on Riemannian manifolds admitting certain vector field*, Ricerche Mat. (2021). https://doi.org/10.1007/s11587-021-00622-z

[23] D. M. Naik, H. A. Kumara, and V. Venkatesha, *Generalized Ricci solitons on Riemannian manifolds admitting concurrent-recurrent vector field*, J. Anal. **30** (2022), no. 3, 1023–1031. https://doi.org/10.1007/s41478-022-00387-0

[24] D. M. Naik and V. Venkatesha, *η-Ricci solitons and almost η-Ricci solitons on para-Sasakian manifolds*, Int. J. Geom. Methods Mod. Phys. **16** (2019), no. 9, 1950134, 18 pp. https://doi.org/10.1142/S0219887819501342

[25] D. M. Naik, V. Venkatesha, and H. A. Kumara, *Ricci solitons and certain related metrics on almost co-Kaehler manifolds*, J. Math. Phys. Anal. Geom. **16** (2020), no. 4, 402–417. https://doi.org/10.15407/mag16.04.402

[26] D. M. Naik, V. Venkatesha, and H. A. Kumara, *Generalized Ricci soliton and paracontact geometry*, São Paulo J. Math. Sci. **15** (2021), no. 2, 916–927. https://doi.org/10.1007/s40863-021-00260-1

[27] D. M. Naik, V. Venkatesha, and D. G. Prakasha, *Certain results on Kenmotsu pseudo-metric manifolds*, Miskolc Math. Notes **20** (2019), no. 2, 1083–1099.

[28] P. Nurowski and M. Randall, *Generalized Ricci solitons*, J. Geom. Anal. **26** (2016), no. 2, 1280–1345. https://doi.org/10.1007/s12220-015-9592-8

[29] S. Pandey, A. Singh, and V. N. Mishra, *η-Ricci solutions on Lorentzian para-Kenmotsu manifolds*, Facta Univ. Ser. Math. Inform. **36** (2021), no. 2, 419–434.

[30] D. G. Prakasha and B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds*, J. Geom. **108** (2017), no. 2, 383–392. https://doi.org/10.1007/s00022-016-0345-z

[31] D. G. Prakasha, H. Harish, P. Veeresha, and Venkatesha, *The Zamkovoy canonical paracontact connection on a para-Kenmotsu manifold*, Cubo A Math. J. **23** (2021), no. 2, 191–206. http://dx.doi.org/10.4067/S0719-06462021000200191

[32] R. Prasad, A. Haseeb, and U. K. Gautam, *On $\check{\phi}$-semisymmetric LP-Kenmotsu manifolds with a QSNM-connection admitting Ricci solitons*, Kragujevac J. Math. **45** (2021), no. 5, 815–827. https://doi.org/10.46793/kgjmat2105.815p

[33] I. Sato, *On a structure similar to the almost contact structure*, Tensor (N.S.) **30** (1976), no. 3, 219–224.

[34] J. A. Schouten, *Ricci Calculus*, Springer-Verlag, Berlin, 1954.

[35] S. Sharma, M. Pandey, G. Pandey, and R. N. Singh, *On M-projective curvature tensor of para-Kenmotsu manifolds admitting Zamkovoy connection*, South East Asian J. Math. Math. Sci. **18** (2022), no. 2, 259–274.

[36] M. D. Siddiqi, *Generalized η-Ricci solitons in trans Sasakian manifolds*, Eurasian Bulltein of Mathematics **1** (2018), no. 3, 107–116.

[37] B. B. Sinha and K. L. Sai Prasad, *A class of almost para contact metric manifold*, Bull. Calcutta Math. Soc. **87** (1995), no. 4, 307–312.

[38] A. F. Solovév, *On the curvature of the connection induced on a hyperdistribution in a Riemannian space*, Geom. Sb. **19** (1978), 12–23.

[39] M. Turan, C. Yetim, and S. K. Chaubey, *On quasi-Sasakian* 3*-manifolds admitting η-Ricci solitons*, Filomat **33** (2019), no. 15, 4923–4930.

[40] K. Yano, *Concircular geometry. I. Concircular transformations*, Proc. Imp. Acad. Tokyo **16** (1940), 195–200. http://projecteuclid.org/euclid.pja/1195579139

[41] K. Yano, *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 340–345. http://projecteuclid.org/euclid.pja/1195572958

[42] K. Yano and B.-Y. Chen, *On the concurrent vector fields of immersed manifolds*, Kodai Math. Sem. Rep. **23** (1971), 343–350. http://projecteuclid.org/euclid.kmj/1138846372

[43] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom. **36** (2009), no. 1, 37–60. https://doi.org/10.1007/s10455-008-9147-3

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