Commun. Korean Math. Soc. **39** (2024), No. 1, pp. 187–199
https://doi.org/10.4134/CKMS.c230087
pISSN: 1225-1763 / eISSN: 2234-3024

INTRINSIC THEORY OF C^v-REDUCIBILITY IN FINSLER GEOMETRY

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Abstract. In the present paper, following the pullback approach to Finsler geometry, we study intrinsically the C^v-reducible and generalized C^v-reducible Finsler spaces. Precisely, we introduce a coordinate-free formulation of these manifolds. Then, we prove that a Finsler manifold is C^v-reducible if and only if it is C-reducible and satisfies the $\mathbb{T}$-condition. We study the generalized C^v-reducible Finsler manifold with a scalar π-form $\mathbb{A}$. We show that a Finsler manifold (M, L) is generalized C^v-reducible with $\mathbb{A}$ if and only if it is C-reducible and $\mathbb{T} = \mathbb{A}$. Moreover, we prove that a Landsberg generalized C^v-reducible Finsler manifold with a scalar π-form $\mathbb{A}$ is Berwaldian. Finally, we consider a special C^v-reducible Finsler manifold and conclude that a Finsler manifold is a special C^v-reducible if and only if it is special semi-C-reducible with vanishing $\mathbb{T}$-tensor.

Introduction

Special Finsler manifolds arise not only by imposing extra conditions on the curvature and torsion tensors available in the space but also by considering special formulation of the Finsler structure. The most ideal case of a Finsler metric is that the metric tensor is positive definite on the whole slit tangent bundle. However, we have to pay attention to the recent rapid progress of Finsler geometry; various applications of Finsler geometry to different fields of science [1]. Consequently, the positive-defineteness is too restrictive for the applications and we have to consider weaker cases, for example, pseudo or conic Finsler metrics. As an example, in [2], G. S. Asanov obtained examples of conic Finsler metrics, arising from Finslerian general relativity, of non-Berwaldian Landsberg spaces, of dimension at least 3. In Asanov's examples, the Finsler functions are not defined for all values of the fiber coordinates.

In addition, there is yet no geometric theory that gives a complete characterization of the special Finsler spaces. Moreover, most of these studies are

Received April 19, 2023; Revised November 9, 2023; Accepted November 14, 2023.
2020 *Mathematics Subject Classification.* Primary 53C60, 53B40, 58B20.
Key words and phrases. $\mathbb{T}$-tensor, Cartan connection, C-reducible, C^v-reducible.

obtained from a purely local perspective. Special Finsler spaces are investigated locally by many authors, we refer for example to [8, 10–12]. On the other hand, the global or intrinsic investigation of such spaces is rare in the literature. Some contributions in this direction can be found in [18, 20, 22, 25].

In [16] T. N. Pandey et al. and in [21] B. Tiwari introduced locally a special Finsler space, called special C^v-reducible, in which the Cartan v-covariant derivative of the Cartan tensor is written in a special form. Moreover, a necessary and sufficient condition for a special semi-C-reducible Finsler space to be special C^v-reducible is given. Also, the three dimensional special C^v-reducible Finsler space was considered to illustrate the developed theory. On the other hand, in [15], T. N. Pandey et al. introduced locally a generalization of C^v-reducible Finsler space and called it a generalized C^v-reducible Finsler space. Then they studied some geometric consequences of the C^v-reducible Finsler space and focused on the two and three dimensional Finsler spaces.

In the present paper, following the pullback approach to Finsler geometry, we treat intrinsically the C^v-reducible and generalized C^v-reducible Finsler spaces. Precisely, we introduce the coordinate-free formulation of these Finsler spaces. Then, we prove that a Finsler manifold is C^v-reducible if and only if it is C-reducible and satisfies the $\mathbb{T}$-condition. Hence, by using the observation of Z. Szabo [19], which states that a positive definite Finsler metric satisfying the $\mathbb{T}$-condition is Riemannian, a positive definite C^v-reducible is Riemannian.

Further, we define the generalized C^v-reducible Finsler manifold with scalar form $\mathbb{A}$ by writing the Cartan torsion $\mathbf{T}$ in a special formula. Also, we show that a Finsler manifold (M, L) is generalized C^v-reducible with $\mathbb{A}$ if and only if it is C-reducible and $\mathbb{T} = \mathbb{A}$. Moreover, we prove that a Landsberg generalized C^v-reducible Finsler manifold with a scalar π-form $\mathbb{A}$ is Berwaldian. Finally, we consider a special C^v-reducible Finsler manifold. We conclude that a Finsler manifold is a special C^v-reducible if and only if it is special semi-C-reducible with vanishing $\mathbb{T}$-tensor.

1. Notation and preliminaries

In this section, we go over some of the basics of the pullback approach to intrinsic Finsler geometry that are required for this study. We refer to [14, 17, 23, 24] for further information. Also, we follow the notations of [23].

Let M be a smooth manifold of dimension n, and assume that the tangent bundle $\pi : TM \longrightarrow M$ and its differential $d\pi : TTM \longrightarrow TM$. The vertical bundle $V(TM)$ of TM is defined by $\ker(d\pi)$. We denote the pullback bundle of the tangent bundle by $\pi^{-1}(TM)$. Also, $\mathfrak{F}(TM)$ denotes the algebra of C^∞ functions on TM and $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ are called the π-vector fields and denoted by barred letters $\overline{X}$.

By [5], we recall the short exact sequence of vector bundle morphisms

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M$ is the slit tangent bundle, γ is the natural injection and $\rho := (\pi_{TM}, \pi)$.

The almost tangent structure of TM or the vertical endomorphism is the endomorphism $J : T\mathcal{T}M \mapsto T\mathcal{T}M$ defined by $J = \gamma \circ \rho$. The Liouville vector field is the vector field given by $\mathcal{C} := \gamma \circ \overline{\eta}$, where $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

For a linear connection D on $\pi^{-1}(TM)$, we associate the connection map $K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$, and the horizontal space to M at u, $H_u(\mathcal{T}M) := \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$. The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \;\; \forall\, u \in \mathcal{T}M.$$

For a regular connection D on M, the vector bundle maps γ, $\rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are isomorphisms. In this case, the map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ is called the horizontal map of D.

Let D be a regular connection on $\pi^{-1}(TM)$ with the horizontal map β and the corresponding classical torsion (resp. curvature) tensor field $\mathbf{T}$ (resp. $\mathbf{K}$). Then, we have:

- For a π-tensor field A of type $(0,p)$, the h- and v-covariant derivatives $\overset{h}{D}$ and $\overset{v}{D}$:

$$(\overset{h}{D}\, A)(\o X, \overline{X}_1, \ldots, \overline{X}_p) := (D_{\beta \o X} A)(\overline{X}_1, \ldots, \overline{X}_p),$$
$$(\overset{v}{D}\, A)(\o X, \overline{X}_1, \ldots, \overline{X}_p) := (D_{\gamma \o X} A)(\overline{X}_1, \ldots, \overline{X}_p).$$

- The (h)h-, (h)hv- and (h)v-torsion tensors of D:

$$Q(\overline{X}, \overline{Y}) := \mathbf{T}(\beta\overline{X}, \beta\overline{Y}), \;\; T(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}), \;\; V(\overline{X}, \overline{Y}) := \mathbf{T}(\gamma\overline{X}, \gamma\overline{Y}).$$

- The horizontal, mixed and vertical curvature tensors of D:

$$R(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z},$$

$$S(\overline{X}, \overline{Y})\overline{Z} := \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

- The (v)h-, (v)hv- and (v)v-torsion tensors of D:

$$\widehat{R}(\overline{X}, \overline{Y}) := R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) := P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) := S(\overline{X}, \overline{Y})\overline{\eta}.$$

Definition 1.1. A Finsler structure or function on M is a map $L : TM \longrightarrow [0, \infty)$ such that:

(a) L is C^∞ on $\mathcal{T}M$, C^0 on TM.
(b) L is positively homogeneous of degree 1 in the directional argument y, that is $\mathcal{L}_\mathcal{C} L = L$, where $\mathcal{L}_X$ is the Lie derivative in the direction of X.
(c) The Hilbert 2-form $\Omega := \frac{1}{2}\, dd_J L^2$ has a maximal rank.

The Finsler metric g induced by L on $\pi^{-1}(TM)$ is defined as follows:

$$g(\rho X, \rho Y) := \Omega(JX, Y) \;\forall X, Y \in \mathfrak{X}(TM).$$

In this case, the pair (M, L) is called a Finsler manifold or regular Finsler metric. When the metric tensor g is non-degenerate at each point of $\mathcal{T}M$, the pair (M, L) is called a pseudo-Finsler manifold. When L satisfies the conditions (a)-(c) but only on an open conic subset $\mathcal{U}$ of TM (for every $v \in \mathcal{U}$ and $\mu > 0, \mu v \in \mathcal{U}$), the pair $(\mathcal{U}, L)$ is called a conic Finsler manifold. If, moreover, the metric tensor g is non-degenerate at each point of $\mathcal{U}$, then the pair $(\mathcal{U}, L)$ is called a conic pseudo-Finsler manifold. However, throughout, we use the concept pseudo-Finsler manifold to refer to either pseudo-Finsler manifold or conic Finsler manifold.

2. C^v-reducibility

Let (M, L) be a Finsler or a pseudo-Finsler manifold of dimension n and g is the Finsler metric associated with L. Denote $\ell := L^{-1} i_{\overline{\eta}}\, g$, $\phi := I - L^{-1}\ell \otimes \overline{\eta}$ and $\hbar(\overline{X}, \overline{Y}) := g(\phi(\overline{X}), \overline{Y}) = (g - \ell \otimes \ell)(\overline{X}, \overline{Y})$, the angular metric tensor, also T, $\widehat{P}$ are the $(h)hv$-torsion, $(v)hv$-torsion tensor and the horizontal map associated with Cartan connection ∇, respectively. Moreover, $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ will denote, respectively, the horizontal covariant derivative and the vertical covariant derivative associated with Cartan connection ∇.

We begin with the following definition.

Definition 2.1. Let ∇ be the Cartan connection associated with (M, L). The torsion tensor field T of the connection ∇ induces a π-tensor field of type $(0, 3)$, called the Cartan torsion and denoted $\mathbf{T}$, defined by:

$$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z}).$$

It also induces a π-form C, called the contracted torsion form, defined by:

$$C(\overline{X}) := Tr\{\overline{Y} \longmapsto T(\overline{X}, \overline{Y})\} = \text{Contracting } \overline{Y} \text{ with } \overline{Z} \text{ for } \mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}).$$

Lemma 2.2. *For a Finsler manifold (M, L) we have the following properties:*

(1) $\mathbf{T}$, $\overset{v}{\nabla}\mathbf{T}$ *and* $\hbar$ *are totally symmetric.*
(2) $i_{\overline{\eta}}\mathbf{T} = 0 = i_{\overline{\eta}}\hbar$, $i_{\overline{\eta}}\ell = L$.
(3) $\nabla_{\gamma\overline{\eta}}\mathbf{T} = -\mathbf{T}$.
(4) $\nabla_{\gamma\overline{X}} L = \ell(\overline{X})$, $(\nabla_{\gamma\overline{X}}\ell)(\overline{Y}) = L^{-1}\hbar(\overline{X}, \overline{Y})$.
(5) $(\nabla_{\gamma\overline{X}}\hbar)(\overline{Y}, \overline{Z}) = -L^{-1}\hbar(\overline{X}, \overline{Y})\ell(\overline{Z}) - L^{-1}\hbar(\overline{X}, \overline{Z})\ell(\overline{Y})$.

Proof. The proof is clear and we omit it. □

Proposition 2.3. *Let $A \neq \lambda\mathbf{T}$ be a symmetric π-tensor field of type $(0, 3)$ and B is a scalar one π-form. If the vertical covariant derivative of the Cartan torsion $\mathbf{T}$ (i.e., $\overset{v}{\nabla}\mathbf{T}$) of non-Riemannian Finsler manifold is written as*[1]

$$L\,(\nabla_{\gamma\overline{W}}\boldsymbol{T})(\overline{X}, \overline{Y}, \overline{Z}) = \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\,\{A(\overline{X}, \overline{Y}, \overline{Z})B(\overline{W})\}, \tag{2.1}$$

then $i_{\overline{\eta}}A \neq 0$ and $i_{\overline{\eta}}i_{\overline{\eta}}A = 0 = i_{\overline{\eta}}B$.

[1] $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ means the cyclic sum over $\overline{X}, \overline{Y}, \overline{Z}$.

Proof. Suppose that the vertical covariant derivative $\overset{v}{\nabla}\mathbf{T}$ of non-Riemannian Finsler manifold satisfies the relation (2.1). Hence, by setting $\overline{W}=\overline{\eta}$ taking into account Lemma 2.2, we obtain

$$\begin{aligned}-L\mathbf{T}(\overline{X},\overline{Y},\overline{Z}) &= A(\overline{X},\overline{Y},\overline{Z})B(\overline{\eta})+A(\overline{Y},\overline{Z},\overline{\eta})B(\overline{X})\\ &\quad + A(\overline{Z},\overline{\eta},\overline{X})B(\overline{Y})+A(\overline{\eta},\overline{X},\overline{Y})B(\overline{Z}).\end{aligned} \tag{2.2}$$

From which by setting $\overline{Z}=\overline{\eta}$ and using Lemma 2.2 again, we get

$$2A(\overline{X},\overline{Y},\overline{\eta})B(\overline{\eta})+A(\overline{Y},\overline{\eta},\overline{\eta})B(\overline{X})+A(\overline{\eta},\overline{\eta},\overline{X})B(\overline{Y})=0. \tag{2.3}$$

Again putting $\overline{Y}=\overline{\eta}$ and $\overline{X}=\overline{\eta}$, respectively, we have

$$3A(\overline{X},\overline{\eta},\overline{\eta})B(\overline{\eta})+A(\overline{\eta},\overline{\eta},\overline{\eta})B(\overline{X})=0, \tag{2.4}$$

$$4A(\overline{\eta},\overline{\eta},\overline{\eta})B(\overline{\eta})=0. \tag{2.5}$$

Now, if $B(\overline{\eta})\neq 0$, then using the above equations, we conclude that

$$A(\overline{X},\overline{Y},\overline{Z})=-\frac{L}{B(\overline{\eta})}\mathbf{T}(\overline{X},\overline{Y},\overline{Z}),$$

which contradicts with the given assumption $A\neq\lambda\mathbf{T}$. Therefore, $B(\overline{\eta})=0$ and hence $A(\overline{\eta},\overline{\eta},\overline{\eta})=0$ and

$$\begin{aligned}A(\overline{Y},\overline{\eta},\overline{\eta})B(\overline{X})B(\overline{Z}) &= -A(\overline{\eta},\overline{\eta},\overline{X})B(\overline{Y})B(\overline{Z}),\\ A(\overline{\eta},\overline{\eta},\overline{Z})B(\overline{X})B(\overline{Y}) &= -A(\overline{X},\overline{\eta},\overline{\eta})B(\overline{Z})B(\overline{Y}),\end{aligned}$$

and hence, we obtain

$$-2A(\overline{X},\overline{\eta},\overline{\eta})B(\overline{Z})B(\overline{Y})=0.$$

Consequently, if B vanishes, then using Eq. (2.2) we conclude that $\mathbf{T}=0$, which contradicts with the given assumption that (M,L) is non-Riemannian. Hence, $A(\overline{X},\overline{\eta},\overline{\eta})$ vanishes identically. Again, as $B(\overline{\eta})=0$, using Eq. (2.2), we get

$$-L\mathbf{T}(\overline{X},\overline{Y},\overline{Z})=A(\overline{Y},\overline{Z},\overline{\eta})B(\overline{X})+A(\overline{Z},\overline{\eta},\overline{X})B(\overline{Y})+A(\overline{\eta},\overline{X},\overline{Y})B(\overline{Z}).$$

From which, if $i_{\overline{\eta}}A=0$, then $\mathbf{T}$ vanishes which contradicts with non-Riemannian property. Therefore $i_{\overline{\eta}}A\neq 0$. This completes the proof. □

For a Finsler manifold (M,L), we define the following π-tensor field

$$\mathbb{H}(\overline{X},\overline{Y},\overline{Z}):=-L^2\,\nabla_{\gamma\overline{X}}\nabla_{\gamma\overline{Y}}\nabla_{\gamma\overline{Z}}\,L. \tag{2.6}$$

Lemma 2.4. *The π-tensor field $\mathbb{H}$, defined above by* (2.6)*, satisfies the following properties*:

1. $\mathbb{H}(\overline{X},\overline{Y},\overline{Z})=\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})\ell(\overline{Z})\}$.
2. $i_{\overline{\eta}}\mathbb{H}=L\hbar\neq 0$.
3. $i_{\overline{\eta}}\,i_{\overline{\eta}}\mathbb{H}=0$.
4. $\mathbb{H}$ *is totally symmetric.*

Proof. The proof follows from (2.6) together with Lemma 2.2. □

Proposition 2.5. *If the π-tensor field $A = \mathbb{H}$ in* (2.1)*, then the π-scalar form $B = \frac{-1}{(n+1)} C$.*

Proof. Suppose that $A = \mathbb{H}$ in (2.1), and setting $\overline{W} = \overline{\eta}$ taking into account Lemmas 2.2, 2.4 and the fact that $B(\overline{\eta}) = 0$, we get

$$-\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y}) B(\overline{Z})\}. \tag{2.7}$$

Acting the contracting $\overline{Y}$ with respect to $\overline{Z}$ on both sides of the above relation and using again Lemmas 2.2 and 2.4. Hence, the result follows. □

In view of the above results, we have the following definition.

Definition 2.6. A Finsler manifold (M, L) is called

(1) C-reducible if the Cartan torsion $\mathbf{T}$ has the form

$$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{1}{n+1} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y}) C(\overline{Z})\}.$$

(2) C^v-reducible if the Cartan torsion $\mathbf{T}$ has the form

$$L(\nabla_{\gamma\overline{W}} \mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) = \frac{-1}{n+1} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbb{H}(\overline{X}, \overline{Y}, \overline{Z}) C(\overline{W})\}. \tag{2.8}$$

We know that the $\mathbb{T}$-tensor for a Finsler manifold (M, L) is defined by

$$\mathbb{T}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := L(\nabla_{\gamma\overline{W}} \mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) \ell(\overline{W})\}. \tag{2.9}$$

If the $\mathbb{T}$-tensor vanishes identically, then we say that (M, L) satisfies the $\mathbb{T}$-condition.

In view of the definition of C-reducible and C^v-reducible Finsler manifolds, taking into account Proposition 2.5 and (2.7), we have:

Proposition 2.7. *Every C^v-reducible Finsler manifold is C-reducible.*

It is clear that the converse of the above proposition is not true, however we have:

Theorem 2.8. *A Finsler manifold (M, L) is C^v-reducible if and only if it is C-reducible with the vanishing $\mathbb{T}$-tensor.*

Proof. Firstly, suppose that (M, L) is a C^v-reducible Finsler manifold, then by the above proposition we conclude that it is C-reducible. By the definition of C- and C^v-reducibility, taking into account Lemma 2.4, we obtain

$$\begin{aligned} &\mathbb{T}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) \\ &= L(\nabla_{\gamma\overline{W}} \mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) \ell(\overline{W})\} \\ &= \frac{-1}{n+1} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbb{H}(\overline{X}, \overline{Y}, \overline{Z}) C(\overline{W})\} + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) \ell(\overline{W})\} \\ &= \frac{-1}{n+1} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y}) \ell(\overline{Z})\} C(\overline{W})\} \end{aligned}$$

$$+\frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})\}\ell(\overline{W})\}.$$

After some calculations we deduce that the $\mathbb{T}$-tensor vanishes identically.

Conversely, assume that (M, L) is C-reducible with the vanishing $\mathbb{T}$-tensor. Now, using the definition of C- and C^v-reducibility and taking into account the expression of the $\mathbb{T}$-tensor, we obtain

$$\begin{aligned}
L(\nabla_{\gamma\overline{W}}\mathbf{T})(\overline{X},\overline{Y},\overline{Z}) &= -\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbf{T}(\overline{X},\overline{Y},\overline{Z})\ell(\overline{W})\}\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})C(\overline{Z})\}\ell(\overline{W})\}\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X},\overline{Y})\ell(\overline{Z})\}C(\overline{W})\}\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbb{H}(\overline{X},\overline{Y},\overline{Z})C(\overline{W})\}.
\end{aligned}$$

This means that (M, L) is C^v-reducible. Hence, the proof completes. □

By using the observation of Z. Szabo [19], which states that a positive definite Finsler metric satisfying the $\mathbb{T}$-condition is Riemannian, we have the following corollary.

Corollary 2.9. *A positive definite C^v-reducible Finsler manifold is Riemannian.*

Remark 2.10. Although positive definite Finsler metrics that satisfies the $\mathbb{T}$-condition is Riemannian there exists non-Riemannian pseudo Finsler metrics with vanishing T-tensor, for example, see [3].

By [13], a non-Riemannian Finsler space (M, F) is C-reducible if and only if it is of a Randers type or of a Kropina type. Also, by [3], a necessary and sufficient condition for (α, β)-metrics to satisfy the T-condition is given. Precisely, an (α, β)-metric satisfies the T-condition if and only if it is Riemannian or $\phi(s)$ has the following form

$$F = \alpha\phi(s), \quad \phi(s) = s^{\frac{cb^2-1}{cb^2}}(cb^2 - cs^2)^{\frac{1}{2cb^2}},$$

where c is a constant, α is a Riemannian (or pseudo Riemannian) metric, $\beta = b_i(x)y^i$ is a one form, and $s = \frac{\beta}{\alpha}$, $\beta = b_i(x)y^i$, $b^2 = b_ib^i$. The function $\phi(s)$ of a Randers metric and Kropina metric is given, respectively, by

$$\phi(s) = 1 + s, \quad \phi(s) = \frac{1}{s}.$$

Then, it is clear that Randers metric and Kropina metric do not satisfy the T-condition. Therefore, we confirm the following theorem.

Theorem 2.11. *A pseudo C^v-reducible Finsler space is a Riemannian space.*

3. Generalized C^v-reducibility with Scalar form $\mathbb{A}$

This section is devoted to consider the so-called generalized C^v-reducible Finsler manifold which is defined as follows.

Definition 3.1. A Finsler manifold (M, L) is called generalized C^v-reducible with $\mathbb{A}$, if the Cartan torsion $\mathbf{T}$ has the form

$$L(\nabla_{\gamma\overline{W}}\mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) = \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbb{H}(\overline{X}, \overline{Y}, \overline{Z})C(\overline{W})\} + \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}), \tag{3.1}$$

where $\mathbb{A}$ is a totally symmetric π-scalar form (π-tensor field of type $(0, 4)$) with $i_{\overline{\eta}}\,\mathbb{A} = 0$.

Theorem 3.2. *A Finsler manifold (M, L) is generalized C^v-reducible with $\mathbb{A}$ if and only if it is C-reducible and $\mathbb{T} = \mathbb{A}$.*

Proof. Firstly, if (M, L) is a generalized C^v-reducible Finsler manifold with $\mathbb{A}$, then by setting $\overline{W} = \overline{\eta}$ into (3.1), and taking into account the fact that $i_{\overline{\eta}}\,\mathbb{A} = 0 = i_{\overline{\eta}}\,C$ together with Lemma 2.4, we obtain

$$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})C(\overline{Z})\}. \tag{3.2}$$

This means that (M, L) is a C-reducible Finsler manifold. Hence, one can show that

$$\begin{aligned}
&\mathbb{T}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\\
&= L(\nabla_{\gamma\overline{W}}\mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})\ell(\overline{W})\}\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbb{H}(\overline{X}, \overline{Y}, \overline{Z})C(\overline{W})\} + \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\\
&\quad + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})\ell(\overline{W})\}\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\ell(\overline{Z})\}C(\overline{W})\} + \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\\
&\quad + \frac{1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})C(\overline{Z})\}\ell(\overline{W})\}\\
&= \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}).
\end{aligned}$$

On the other hand, if (M, L) is C-reducible with the $\mathbb{T}$-tensor ($\mathbb{T} = \mathbb{A}$), then, we get

$$\begin{aligned}
&L(\nabla_{\gamma\overline{W}}\mathbf{T})(\overline{X}, \overline{Y}, \overline{Z})\\
&= -\,\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})\ell(\overline{W})\} + \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})C(\overline{Z})\}\ell(\overline{W})\} + \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\\
&= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\hbar(\overline{X}, \overline{Y})\ell(\overline{Z})\}C(\overline{W})\} + \mathbb{A}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})
\end{aligned}$$

$$= \frac{-1}{n+1}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbb{H}(\overline{X},\overline{Y},\overline{Z})C(\overline{W})\} + \mathbb{A}(\overline{X},\overline{Y},\overline{Z},\overline{W}).$$

Hence, (M, L) is generalized C^v-reducible with scalar π-form $\mathbb{A}$. This completes the proof. □

From the above result together with [4], we conclude that:

Theorem 3.3. *A Landsberg generalized C^v-reducible Finsler manifold with a scalar π-form $\mathbb{A}$ is Berwaldian.*

Proof. Let (M, L) be a Landsberg generalized C^v-reducible Finsler metric. Then by Theorem 3.2 the space (M, L) is C-reducible. Now, the space (M, L) is Landsberg and C-reducible. Consequently, by [4], (M, L) is Berwaldian. □

4. Special C^v-reducibility

For a Finsler manifold (M, L) of dimension $n \geq 3$, we define the following π-tensor fields

$$\mathbf{A}(\overline{X},\overline{Y}) := \frac{1}{(n-2)}\{\hbar(\overline{X},\overline{Y}) - \frac{1}{C^2}[C(\overline{X})C(\overline{Y})]\}, \tag{4.1}$$

$$\mathbf{H}(\overline{X},\overline{Y},\overline{Z}) = \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\mathbf{A}(\overline{X},\overline{Y})\ell(\overline{Z})\}, \tag{4.2}$$

where $C(\overline{X}) =: g(\overline{X},\overline{C})$ and $C^2 := g(\overline{C},\overline{C}) = C(\overline{C})$.

Lemma 4.1. *The π-tensor field $\mathbf{H}$, defined above by (4.2), satisfies the following properties*:

(1) $\mathbf{H}$ *is totally symmetric.*
(2) $i_{\overline{\eta}}\mathbf{H} = \frac{L}{(n-2)}\{\hbar - \frac{1}{C^2}[C \otimes C]\} \neq 0$.
(3) $i_{\overline{\eta}}\, i_{\overline{\eta}}\mathbf{H} = 0$.

This result ensure that the π-tensor field $\mathbf{H}$ satisfies the properties mention in Proposition 2.5.

Proposition 4.2. *Let (M, L) be an n-dimensional Finsler manifold with $n \geq 3$. If the vertical covariant derivative of the Cartan torsion $\mathbf{T}$ (i.e., $\overset{v}{\nabla}\mathbf{T}$) of non-Riemannian Finsler manifold has the form*

$$L\,(\nabla_{\gamma\overline{W}}\boldsymbol{T})(\overline{X},\overline{Y},\overline{Z}) = \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}}\{\mathbf{H}(\overline{X},\overline{Y},\overline{Z})B(\overline{W})\}, \tag{4.3}$$

then $B = -C$.

Proof. Assume that the vertical covariant derivative of the Cartan torsion $\mathbf{T}$ satisfies Relation (4.3). By setting $\overline{W} = \overline{\eta}$ into Eq. (4.3), using Lemma 4.1 and the fact that $B(\overline{\eta}) = 0$, we obtain

$$-\mathbf{T}(\overline{X},\overline{Y},\overline{Z}) = \mathbf{A}(\overline{Y},\overline{Z})B(\overline{X}) + \mathbf{A}(\overline{X},\overline{Y})B(\overline{Z}) + \mathbf{A}(\overline{Z},\overline{X})B(\overline{Y}).$$

Applying contracting $\overline{Y}$ with $\overline{Z}$ on both sides, the above equation reduces to

$$-C(\overline{X}) = B(\overline{X}) + \frac{2}{(n-2)}[B(\overline{X}) + C(\overline{X})].$$

Hence, the result follows provided that $n \geq 3$. □

Definition 4.3. Let (M, L) be a Finsler manifold of dimension $n \geq 3$. Then, (M, L) is called

(1) semi-C-reducible if

$$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{\mu}{(n+1)} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y})\, C(\overline{Z})\} + \frac{\tau}{C^2}\, C(\overline{X})\, C(\overline{Y})\, C(\overline{Z});\ \mu + \tau = 1,$$

(2) special semi-C-reducible if

$$\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) = \frac{1}{(n-2)} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\hbar(\overline{X}, \overline{Y})\, C(\overline{Z})\} - \frac{3}{(n-2)C^2}\, C(\overline{X})\, C(\overline{Y})\, C(\overline{Z}), \tag{4.4}$$

(3) special C^v-reducible if

$$L(\nabla_{\gamma \overline{W}} \mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) = -\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{H}(\overline{X}, \overline{Y}, \overline{Z}) C(\overline{W})\}, \tag{4.5}$$

where $\mathbf{H}$ is the π-tensor field defined by (4.2).

Remark 4.4. It is clear that a special semi-C-reducible Finsler manifold is semi-C-reducible with constant coefficients, $\mu = \frac{(n+1)}{(n-2)}$, $\tau = \frac{-3}{(n-2)}$ satisfying $\mu + \tau = 1$. Moreover, Ikeda investigated an (α, β) metric $(L^4 = \alpha^2 \beta^4)$, which satisfies the condition of special semi-C-reducibility (cf. [6, 7]).

According to the above definition, we obtain:

Proposition 4.5. *Every a special C^v-reducible Finsler manifold is special semi-C-reducible.*

Proof. The proof follows from the above definition, by setting $\overline{W} = \overline{\eta}$, taking into account Lemma 4.1. □

It is clear that the converse of the above proposition is not true, however we have:

Theorem 4.6. *A Finsler manifold (M, L) is special C^v-reducible if and only if it is special semi-C-reducible with vanishing $\mathbb{T}$-tensor.*

Proof. Suppose that (M, L) is a special C^v-reducible Finsler manifold. Hence, according to Proposition 4.5, it follows that (M, L) is special semi-C-reducible. Also, from Eqs. (4.1), (4.2), (4.4) and (4.5), one can show that

$$\begin{aligned}
&\mathbb{T}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) \\
&= L(\nabla_{\gamma \overline{W}} \mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) \ell(\overline{W})\} \\
&= -\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{H}(\overline{X}, \overline{Y}, \overline{Z}) C(\overline{W})\} + \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z}) \ell(\overline{W})\} \\
&= -\frac{1}{(n-2)} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\{\hbar(\overline{X}, \overline{Y}) - \frac{1}{C^2} [C(\overline{X}) C(\overline{Y})]\} \ell(\overline{Z})\} C(\overline{W})\} \\
&\quad + \frac{1}{(n-2)} \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\{\hbar(\overline{X}, \overline{Y}) - \frac{1}{C^2} [C(\overline{X})\, C(\overline{Y})]\}\, C(\overline{Z}\} \ell(\overline{W})\}.
\end{aligned}$$

From which, we deduce that $\mathbb{T} = 0$.

On the other hand, assume that (M, L) is a special semi-C-reducible Finsler manifold with $\mathbb{T} = 0$. Hence, we get

$$\begin{aligned}
& L(\nabla_{\gamma\overline{W}}\mathbf{T})(\overline{X}, \overline{Y}, \overline{Z}) \\
= & -\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{T}(\overline{X}, \overline{Y}, \overline{Z})\ell(\overline{W})\} \\
= & -\frac{1}{(n-2)}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\{\hbar(\overline{X}, \overline{Y}) - \frac{1}{C^2}[C(\overline{X})\, C(\overline{Y})]\}\, C(\overline{Z})\}\ell(\overline{W})\} \\
& -\frac{1}{(n-2)}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\{\hbar(\overline{X}, \overline{Y}) - \frac{1}{C^2}[C(\overline{X})\, C(\overline{Y})]\}\, \ell(\overline{Z})\}C(\overline{W})\} \\
= & -\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \{\mathbf{A}(\overline{X}, \overline{Y})\ell(\overline{Z})\}C(\overline{W})\} \\
= & -\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z},\overline{W}} \{\mathbf{H}(\overline{X}, \overline{Y}, \overline{Z})C(\overline{W})\}.
\end{aligned}$$

This means that (M, L) is a special C^v-reducible Finsler manifold. Hence, the result completes. □

Remark 4.7. (1) Ikeda demonstrated, in [7], that the Finsler metric of a special semi-C-reducible Finsler manifold with $\mathbb{T} = 0$ has the form $L^4 = \alpha^2\,\beta^2$, where α is a pseudo-Riemannian metric and β is a 1-form.

(2) According to Kikuchi's condition for the conformal flatness of a Finsler manifold [9], a non-vanishing $\mathbb{T}$-tensor is a necessary condition for a Finsler space to satisfy Kikuchi's condition.

Corollary 4.8. *In view of the above remark, we have*

(1) *The Finsler metric of a special C^v-reducible Finsler manifold is of the form $L^4 = \alpha^2\,\beta^2$, where α is a pseudo-Riemannian metric and β is a 1-form.*

(2) *Kikuchi's condition for conformal flatness is never satisfied by a special C^v-reducible Finsler manifold.*

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