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SPACETIMES ADMITTING DIVERGENCE FREE m-PROJECTIVE CURVATURE TENSOR

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Abstract. This paper is concerned with the study of spacetimes satisfying div$\mathcal{M} = 0$, where "div" denotes the divergence and $\mathcal{M}$ is the m-projective curvature tensor. We establish that a perfect fluid spacetime with div$\mathcal{M} = 0$ is a generalized Robertson-Walker spacetime and vorticity free; whereas a four-dimensional perfect fluid spacetime becomes a Robertson-Walker spacetime. Moreover, we establish that a Ricci recurrent spacetime with div$\mathcal{M} = 0$ represents a generalized Robertson-Walker spacetime.

1. Introduction

Lorentzian manifold is the subclass of a semi-Riemannian manifold. The index of the Lorentzian metric g is 1. A spacetime is a Lorentzian manifold M^n ($n \geq 4$) which admits a globally timelike vector. Various types of spacetimes have been investigated in several ways, such as [2,3,6,7,11,12,14,21,23,24] and also numerous others.

Lorentzian manifolds with the Ricci tensor

$$\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = \gamma g(\mathcal{U}_1, \mathcal{V}_1) + \delta \Pi(\mathcal{U}_1) \Pi(\mathcal{V}_1), \tag{1.1}$$

where γ, δ are scalars and μ is a unit timelike vector field corresponding to the non-vanishing one-form Π, that is, $\Pi(\mu) = g(\mu, \mu) = -1$, are called perfect fluid spacetimes. If in particular γ and δ are constants, then the geometers called quasi-Einstein spacetimes.

The energy momentum tensor $\mathcal{T}$ recounts the matter content of the spacetimes in general relativity theory. In general relativity theory, the fluid is termed perfect fluid since it does not have the heat conduction terms [13]. The energy momentum tensor for a perfect fluid spacetime resembles the shape [18]

$$\mathcal{T}(\mathcal{U}_1, \mathcal{V}_1) = pg(\mathcal{U}_1, \mathcal{V}_1) + (p + \sigma) \Pi(\mathcal{U}_1) \Pi(\mathcal{V}_1), \tag{1.2}$$

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where σ stands for energy density and p stands for isotropic pressure. The velocity vector field μ is the metrically analogous unit timelike vector field to the non-vanishing one-form Π.

The Einstein's field equations is as follows:

$$\mathcal{S}\left(\mathcal{U}_1,\mathcal{V}_1\right)-\frac{\rho}{2}g\left(\mathcal{U}_1,\mathcal{V}_1\right)=\kappa\mathcal{T}\left(\mathcal{U}_1,\mathcal{V}_1\right), \tag{1.3}$$

where $\mathcal{S}$ stands for the Ricci tensor and ρ stands for the scalar curvature, κ is the gravitational constant. According to Einstein's field equations, the geometry of spacetime is determined by matter, whereas the non-flat metric of spacetime governs the motion of matter. The above form (1.1) of the Ricci tensor is determined from Einstein's field equations using (1.2).

Using (1.1) and (1.2) from (1.3) we infer that

$$\gamma=\kappa\left(\frac{p-\sigma}{2-n}\right)\quad\text{and}\quad\delta=\kappa\left(p+\sigma\right). \tag{1.4}$$

An n-dimensional ($n>2$) Lorentzian manifold is referred to be a generalized Robertson-Walker spacetime if the metric adopts the following local structure:

$$ds^2=-\left(d\zeta\right)^2+q^2\left(\zeta\right)g^*_{u_1u_2}dx^{u_1}dx^{u_2}, \tag{1.5}$$

where q is a ζ-dependent function and $g^*_{u_1u_2}=g^*_{u_1u_2}\left(x^{u_3}\right)$ are only functions of x^{u_3} $(u_1,u_2,u_3=2,3,\ldots,n)$. Thus a generalized Robertson-Walker spacetime can be represented as $-\mathcal{I}\times q^2\bar{M}$, where $\bar{M}$ is a Riemannian manifold of dimension $(n-1)$. If the dimension of $\bar{M}$ is three and of constant sectional curvature, then the spacetime becomes a Robertson-Walker spacetime.

In a Lorentzian manifold (M^n,g) $(n\geq 4)$, the conformal curvature tensor $\mathcal{C}$ is stated as

$$\begin{aligned}\mathcal{C}\left(\mathcal{U}_1,\mathcal{V}_1\right)\mathcal{W}_1=\mathcal{R}\left(\mathcal{U}_1,\mathcal{V}_1\right)\mathcal{W}_1-\frac{1}{(n-2)}&\left[g\left(\mathcal{V}_1,\mathcal{W}_1\right)\mathcal{Q}\mathcal{U}_1-g\left(\mathcal{U}_1,\mathcal{W}_1\right)\mathcal{Q}\mathcal{V}_1\right.\\ &\left.+\mathcal{S}\left(\mathcal{V}_1,\mathcal{W}_1\right)\mathcal{U}_1-\mathcal{S}\left(\mathcal{U}_1,\mathcal{W}_1\right)\mathcal{V}_1\right]\\ +\frac{\rho}{(n-1)(n-2)}&\left[g\left(\mathcal{V}_1,\mathcal{W}_1\right)\mathcal{U}_1-g\left(\mathcal{U}_1,\mathcal{W}_1\right)\mathcal{V}_1\right],\end{aligned} \tag{1.6}$$

where $\mathcal{Q}$ is the Ricci operator satisfying the relation $\mathcal{S}\left(\mathcal{U}_1,\mathcal{V}_1\right)=g\left(\mathcal{Q}\mathcal{U}_1,\mathcal{V}_1\right)$ and ρ being the scalar curvature.

From the above definition, it can be seen that

$$\begin{aligned}\left(\operatorname{div}\mathcal{C}\right)\left(\mathcal{U}_1,\mathcal{V}_1\right)\mathcal{W}_1=\left(\frac{n-3}{n-2}\right)\Big[&\left\{\left(\nabla_{\mathcal{U}_1}\mathcal{S}\right)\left(\mathcal{V}_1,\mathcal{W}_1\right)-\left(\nabla_{\mathcal{V}_1}\mathcal{S}\right)\left(\mathcal{U}_1,\mathcal{W}_1\right)\right\}\\ &-\frac{1}{2(n-1)}\left\{g\left(\mathcal{V}_1,\mathcal{W}_1\right)d\rho\left(\mathcal{U}_1\right)-g\left(\mathcal{U}_1,\mathcal{W}_1\right)d\rho\left(\mathcal{V}_1\right)\right\}\Big].\end{aligned} \tag{1.7}$$

The m-projective curvature tensor $\mathcal{M}$ in a semi-Riemannian manifold (M^n,g) $(n\geq 2)$ is defined as [20]

$$\mathcal{M}\left(\mathcal{U}_1,\mathcal{V}_1\right)\mathcal{W}_1=\mathcal{R}\left(\mathcal{U}_1,\mathcal{V}_1\right)\mathcal{W}_1-\frac{1}{2(n-1)}\left[\mathcal{S}\left(\mathcal{V}_1,\mathcal{W}_1\right)\mathcal{U}_1-\mathcal{S}\left(\mathcal{U}_1,\mathcal{W}_1\right)\mathcal{V}_1\right.$$

$$+g(\mathcal{V}_1,\mathcal{W}_1)\,\mathcal{Q}\mathcal{U}_1 - g(\mathcal{U}_1,\mathcal{W}_1)\,\mathcal{Q}\mathcal{V}_1\right], \tag{1.8}$$

where $\mathcal{R}$ stands for the curvature tensor. In [20], the authors obtained the relativistic significance of m-projective curvature tensor. The following is the outline of the paper: After preliminaries in Section 3, we investigate a perfect fluid spacetime admitting $\mathrm{div}\mathcal{M} = 0$. The analysis of Ricci recurrent spacetimes with $\mathrm{div}\mathcal{M} = 0$ is presented in the last section.

2. Preliminaries

At any point on the manifold, considering an orthonormal frame field and contracting $\mathcal{U}_1$ and $\mathcal{V}_1$ in (1.1) yields

$$\rho = n\gamma - \delta. \tag{2.1}$$

Differentiating (1.8) covariantly we obtain that

$$\begin{aligned}(\nabla_{\mathcal{Z}}\mathcal{M})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1 = {}& (\nabla_{\mathcal{Z}}\mathcal{R})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1 - \frac{1}{2(n-1)}\left[(\nabla_{\mathcal{Z}}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)\mathcal{U}_1\right.\\ & - (\nabla_{\mathcal{Z}}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)\mathcal{V}_1 + g(\mathcal{V}_1,\mathcal{W}_1)(\nabla_{\mathcal{Z}}\mathcal{Q})\mathcal{U}_1\\ & \left.-g(\mathcal{U}_1,\mathcal{W}_1)(\nabla_{\mathcal{Z}}\mathcal{Q})\mathcal{V}_1\right].\end{aligned} \tag{2.2}$$

It follows that

$$\begin{aligned}(\mathrm{div}\mathcal{M})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1 = {}& (\mathrm{div}\,\mathcal{R})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1 - \frac{1}{2(n-1)}\left[(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)\right.\\ & \left.- (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)\right]\\ & - \frac{1}{4(n-1)}\left[g(\mathcal{V}_1,\mathcal{W}_1)\,d\rho(\mathcal{U}_1) - g(\mathcal{U}_1,\mathcal{W}_1)\,d\rho(\mathcal{V}_1)\right].\end{aligned}$$

Making use of $(\mathrm{div}\,\mathcal{R})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1 = (\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) - (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)$, the above equation implies

$$\begin{aligned}(\mathrm{div}\mathcal{M})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1 = {}& \frac{2n-3}{2(n-1)}\left[(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) - (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)\right]\\ & - \frac{1}{4(n-1)}\left[g(\mathcal{V}_1,\mathcal{W}_1)\,d\rho(\mathcal{U}_1) - g(\mathcal{U}_1,\mathcal{W}_1)\,d\rho(\mathcal{V}_1)\right].\end{aligned} \tag{2.3}$$

Proposition 2.1. *On a spacetime, the divergence of the m-projective curvature tensor vanishes if and only if the Ricci tensor is of Codazzi type.*

Proof. Assume that the spacetime obeys $\mathrm{div}\mathcal{M} = 0$. Then the equation (2.3) takes the form:

$$\begin{aligned}& \frac{2n-3}{2(n-1)}\left[(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) - (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)\right]\\ & - \frac{1}{4(n-1)}\left[g(\mathcal{V}_1,\mathcal{W}_1)\,d\rho(\mathcal{U}_1) - g(\mathcal{U}_1,\mathcal{W}_1)\,d\rho(\mathcal{V}_1)\right] = 0.\end{aligned} \tag{2.4}$$

Contracting $\mathcal{V}_1$ and $\mathcal{W}_1$ in (2.4) yields

$$d\rho(\mathcal{U}_1) = 0. \tag{2.5}$$

Utilizing the equations (2.4) and (2.5), we find that

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)-(\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)=0. \tag{2.6}$$

Thus the Ricci tensor is of Codazzi type.

Conversely, suppose that the Ricci tensor $\mathcal{S}$ is of Codazzi type. Then the scalar curvature ρ is constant. Hence from (2.3), we get $\mathrm{div}\mathcal{M}=0$. □

Remark 2.2. In a semi-Riemannian manifold, the Ricci tensor is of Codazzi type, that is, $(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)=(\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1)$ implies constant scalar curvature. These manifolds generalize the locally symmetric and Einstein manifolds and on the other hand, the critical points of the functional $\hat{E}(g)=\frac{1}{2}\int_M\|\mathcal{R}^{\nabla}\|^2dV_g$ are their metrics, where the curvature tensor of the Levi-Civita connection of g is $\mathcal{R}^{\nabla}$ and whenever the total volume $\int_M dV_g$ is normalized. The Levi-Civita connection ∇ of such metrics is a Yang-Mills connection if the metric on M is unchanged and $(\mathrm{div}\,\mathcal{R})(\mathcal{U}_1,\mathcal{V}_1)\mathcal{W}_1=0$. Derdziński [8] established conformally flat metrics on $S^1\times N$ for every N containing the Einstein metric with positive scalar curvature, with the Ricci tensor is not parallel. According to Bourguignon [4], any metric with harmonic curvature on a compact orientable 4-dimensional manifold with non-vanishing signature is Einstein.

Remark 2.3. If $\mathrm{div}\mathcal{M}$ vanishes, then from Proposition 2.1, it follows that the Ricci tensor is of Codazzi type and consequently the scalar curvature is constant. Hence from (1.7) we deduce that $\mathrm{div}\,\mathcal{C}=0$. Thus, we conclude that $\mathrm{div}\mathcal{M}=0$ implies $\mathrm{div}\,\mathcal{C}=0$, but in general the converse does not hold.

3. Divergence free m-projective curvature tensor on perfect fluid spacetimes

Here we characterize perfect fluid spacetimes with $\mathrm{div}\mathcal{M}=0$. If $\mathrm{div}\mathcal{M}=0$, then from Proposition 2.1, we find

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1)=(\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1). \tag{3.1}$$

By virtue of (1.1) and (3.1) we acquire

$$\begin{aligned}&d\gamma(\mathcal{U}_1)\,g(\mathcal{V}_1,\mathcal{W}_1)+d\delta(\mathcal{U}_1)\,\Pi(\mathcal{V}_1)\,\Pi(\mathcal{W}_1)\\&+\delta\left[(\nabla_{\mathcal{U}_1}\Pi)(\mathcal{V}_1)\,\Pi(\mathcal{W}_1)+\Pi(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}\Pi)(\mathcal{W}_1)\right]\\&-d\gamma(\mathcal{V}_1)\,g(\mathcal{U}_1,\mathcal{W}_1)-d\delta(\mathcal{V}_1)\,\Pi(\mathcal{U}_1)\,\Pi(\mathcal{W}_1)\\&-\delta\left[(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{U}_1)\,\Pi(\mathcal{W}_1)+\Pi(\mathcal{U}_1)(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{W}_1)\right]=0.\end{aligned} \tag{3.2}$$

Now, contraction of (3.2) gives

$$(1-n)\,d\gamma(\mathcal{V}_1)+d\delta(\mu)\,\Pi(\mathcal{V}_1)+d\delta(\mathcal{V}_1)+\delta\left[(\nabla_\mu\Pi)(\mathcal{V}_1)+\Pi(\mathcal{V}_1)\,\Omega\Pi\right]=0, \tag{3.3}$$

where $\Omega\Pi=\sum_{i=1}^n\varepsilon_i(\nabla_{e_i}\Pi)(e_i)$.

Setting $\mathcal{U}_1=\mathcal{W}_1=\mu$ in (3.2), we arrive at

$$\delta(\nabla_\mu\Pi)(\mathcal{V}_1)=d\gamma(\mu)\,\Pi(\mathcal{V}_1)-d\delta(\mu)\,\Pi(\mathcal{V}_1)+d\gamma(\mathcal{V}_1)-d\delta(\mathcal{V}_1). \tag{3.4}$$

Now the scalar curvature ρ is constant as the Ricci tensor is of Codazzi type. Hence from (2.1), we have

$$nd\gamma\left(\mathcal{U}_1\right) = d\delta\left(\mathcal{U}_1\right). \tag{3.5}$$

Using (3.5) in (3.4) we reach

$$\delta\left(\nabla_\mu \Pi\right)\left(\mathcal{V}_1\right) = (1-n)\left[d\gamma\left(\mu\right)\Pi\left(\mathcal{V}_1\right) + d\gamma\left(\mathcal{V}_1\right)\right]. \tag{3.6}$$

In light of (3.3), (3.5) and (3.6) we obtain

$$(1-n)\, d\gamma\left(\mathcal{V}_1\right) + d\gamma\left(\mu\right)\Pi\left(\mathcal{V}_1\right) + d\gamma\left(\mathcal{V}_1\right) + \delta\Pi\left(\mathcal{V}_1\right)\Omega\Pi = 0. \tag{3.7}$$

Replacing $\mathcal{V}_1$ by μ in (3.7) reveals that

$$\delta\Omega\Pi = (1-n)\, d\gamma\left(\mu\right). \tag{3.8}$$

Equations (3.7) and (3.8) implies

$$d\gamma\left(\mathcal{V}_1\right) = -d\gamma\left(\mu\right)\Pi\left(\mathcal{V}_1\right). \tag{3.9}$$

Using (3.9) in (3.5) infers

$$d\delta\left(\mathcal{V}_1\right) = -nd\gamma\left(\mu\right)\Pi\left(\mathcal{V}_1\right). \tag{3.10}$$

Substituting $\mathcal{W}_1 = \mu$ in (3.2), we reveal

$$\begin{aligned} &d\gamma\left(\mathcal{U}_1\right)\Pi\left(\mathcal{V}_1\right) - d\delta\left(\mathcal{U}_1\right)\Pi\left(\mathcal{V}_1\right) - \delta\left(\nabla_{\mathcal{U}_1}\Pi\right)\left(\mathcal{V}_1\right) \\ &- d\gamma\left(\mathcal{V}_1\right)\Pi\left(\mathcal{U}_1\right) + d\delta\left(\mathcal{V}_1\right)\Pi\left(\mathcal{U}_1\right) + \delta\left(\nabla_{\mathcal{V}_1}\Pi\right)\left(\mathcal{U}_1\right) = 0. \end{aligned} \tag{3.11}$$

In view of (3.9), (3.10) and (3.11) we arrive

$$\delta\left[\left(\nabla_{\mathcal{U}_1}\Pi\right)\left(\mathcal{V}_1\right) - \left(\nabla_{\mathcal{V}_1}\Pi\right)\left(\mathcal{U}_1\right)\right] = 0. \tag{3.12}$$

Since in a perfect fluid spacetime $\delta \neq 0$, therefore from (3.12) we obtain the one-form Π is closed and Π is closed entails that the velocity vector field μ is irrotational. Thus the perfect fluid spacetime has zero vorticity.

Setting $\mathcal{V}_1 = \mu$ in (3.12) we reach

$$\left(\nabla_\mu \Pi\right)\left(\mathcal{U}_1\right) = 0, \quad \text{that is,} \quad g\left(\mathcal{U}_1, \nabla_\mu \mu\right) = 0$$

for all $\mathcal{U}_1$. Hence we write:

Theorem 3.1. *A perfect fluid spacetime admitting divergence free m-projective curvature tensor is vorticity free and the integral curves of the velocity vector field are geodesics.*

Mantica and Molinari [15] proved the following theorem:

Theorem A. *A Lorentzian manifold of dimension $n \geq 3$ is a generalized Robertson-Walker spacetime if and only if it admits a unit timelike torse-forming vector field μ: $\left(\nabla_{\mathcal{U}_1}\Pi\right)\left(\mathcal{V}_1\right) = \psi\left[g\left(\mathcal{U}_1, \mathcal{V}_1\right) + \Pi\left(\mathcal{U}_1\right)\Pi\left(\mathcal{V}_1\right)\right]$, that is also an eigenvector of the Ricci tensor.*

The equations (3.9), (3.10) and (3.2) together yield

$$\begin{aligned}
&- d\gamma(\mu)\,\Pi(\mathcal{U}_1)\, g(\mathcal{V}_1,\mathcal{W}_1)\\
&+ \delta\left[(\nabla_{\mathcal{U}_1}\Pi)(\mathcal{V}_1)\,\Pi(\mathcal{W}_1) + \Pi(\mathcal{V}_1)(\nabla_{\mathcal{U}_1}\Pi)(\mathcal{W}_1)\right]\\
&+ d\gamma(\mu)\,\Pi(\mathcal{V}_1)\, g(\mathcal{U}_1,\mathcal{W}_1)\\
&- \delta\left[(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{U}_1)\,\Pi(\mathcal{W}_1) + \Pi(\mathcal{U}_1)(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{W}_1)\right] = 0. \qquad (3.13)
\end{aligned}$$

Replacing $\mathcal{U}_1$ by μ in (3.13) entails that

$$\begin{aligned}
&d\gamma(\mu)\left[g(\mathcal{V}_1,\mathcal{W}_1) + \Pi(\mathcal{V}_1)\,\Pi(\mathcal{W}_1)\right] + \delta\,(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{W}_1)\\
&+ \delta\left[(\nabla_{\mu}\Pi)(\mathcal{V}_1)\,\Pi(\mathcal{W}_1) + \Pi(\mathcal{V}_1)(\nabla_{\mu}\Pi)(\mathcal{W}_1)\right] = 0. \qquad (3.14)
\end{aligned}$$

Utilizing the equations (3.6) and (3.14), we find that

$$\begin{aligned}
&(1-n)\left[2d\gamma(\mu)\,\Pi(\mathcal{V}_1)\,\Pi(\mathcal{W}_1) + d\gamma(\mathcal{V}_1)\,\Pi(\mathcal{W}_1) + d\gamma(\mathcal{W}_1)\,\Pi(\mathcal{V}_1)\right]\\
&+ d\gamma(\mu)\left[g(\mathcal{V}_1,\mathcal{W}_1) + \Pi(\mathcal{V}_1)\,\Pi(\mathcal{W}_1)\right] + \delta\,(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{W}_1) = 0. \qquad (3.15)
\end{aligned}$$

From (3.9) and (3.15), we have

$$(\nabla_{\mathcal{V}_1}\Pi)(\mathcal{W}_1) = -\frac{d\gamma(\mu)}{\delta}\left[g(\mathcal{V}_1,\mathcal{W}_1) + \Pi(\mathcal{V}_1)\,\Pi(\mathcal{W}_1)\right], \qquad (3.16)$$

which shows that the unit timelike vector field μ is a torse-forming vector field. Putting $\mathcal{V}_1 = \mu$ in (1.1), we acquire

$$\mathcal{S}(\mathcal{U}_1,\mu) = (\gamma-\delta)\,\Pi(\mathcal{U}_1). \qquad (3.17)$$

After adopting (2.1) and (3.17) we get the form

$$\mathcal{S}(\mathcal{U}_1,\mu) = [\rho + (1-n)\gamma]\, g(\mathcal{U}_1,\mu). \qquad (3.18)$$

Since $\gamma \neq \delta$ in general, $\rho + (1-n)\gamma \neq 0$. This shows that the unit timelike torse-forming vector field μ is an eigenvector of the Ricci tensor $\mathcal{S}$ corresponding to the eigenvalue $\rho + (1-n)\gamma$.

In view of this observation and Theorem A, we conclude the following:

Theorem 3.2. *A perfect fluid spacetime with* $\operatorname{div}\mathcal{M} = 0$ *is a generalized Robertson-Walker spacetime.*

Remark 3.3. Mantica et al. [16] proved that a perfect fluid spacetime with $\operatorname{div}\mathcal{C} = 0$ is a generalized Robertson-Walker spacetime, provided the velocity vector field is irrotational. In the above theorem, we establish that a perfect fluid spacetime with $\operatorname{div}\mathcal{M} = 0$ is a generalized Robertson-Walker spacetime without assuming the velocity vector field is irrotational.

A 4-dimensional Lorentzian manifold is named a Yang pure space [10] whose metric satisfies the Yang's equation:

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) = (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1,\mathcal{W}_1).$$

Guilfoyle and Nolan [10] established the following result:

Proposition 3.4. *A four-dimensional perfect fluid spacetime with* $p + \sigma \neq 0$ *is a Yang pure space if and only if it is a Robertson-Walker spacetime.*

Since in a perfect fluid spacetime $\delta \neq 0$, therefore from (1.4) we get $p+\sigma \neq 0$. Thus we have the following result from Proposition 3.4.

Corollary 3.5. *A four-dimensional perfect fluid spacetime satisfying* $\operatorname{div}\mathcal{M} = 0$ *is a Robertson-Walker spacetime.*

Also the equations (1.3) and (3.1) reflect that the energy momentum tensor is of Codazzi type, that is,

$$(\nabla_{\mathcal{U}_1}\mathcal{T})(\mathcal{V}_1,\mathcal{W}_1) = (\nabla_{\mathcal{V}_1}\mathcal{T})(\mathcal{U}_1,\mathcal{W}_1) \tag{3.19}$$

for all $\mathcal{U}_1,\mathcal{V}_1,\mathcal{W}_1$.

However, it has been established [9] *that if the energy momentum tensor is of Codazzi type in a perfect fluid spacetime, then the fluid is of vanishing shear and vorticity, and its velocity vector field becomes hypersurface orthogonal.*

Barnes [1] *observed that the probable local cosmological structures of the perfect fluid spacetimes are of Petrov types I, D, or O if the perfect fluid spacetime is of vanishing shear and vorticity, the velocity vector field μ is hypersurface orthogonal and the constant energy density over a hypersurface orthogonal to μ.*

As a result of the foregoing facts, we arrive:

Theorem 3.6. *For a perfect fluid spacetime with* $\operatorname{div}\mathcal{M} = 0$*, the probable local cosmological structures of the spacetime are of Petrov types I, D, or O.*

By virtue of (1.2) and (1.3) we acquire

$$\mathcal{S}(\mathcal{U}_1,\mathcal{V}_1) - \frac{\rho}{2} g(\mathcal{U}_1,\mathcal{V}_1) = \kappa p g(\mathcal{U}_1,\mathcal{V}_1) + \kappa(p+\sigma)\Pi(\mathcal{U}_1)\Pi(\mathcal{V}_1). \tag{3.20}$$

Contracting the foregoing equation, we can derive

$$\rho = \kappa(\sigma - 3p). \tag{3.21}$$

The scalar curvature ρ is constant as the Ricci tensor is of Codazzi type. Hence from (3.21), we have

$$\sigma = 3p + \text{constant}. \tag{3.22}$$

Theorem 3.7. *For a* 4*-dimensional perfect fluid spacetime obeys* $\operatorname{div}\mathcal{M} = 0$ *the equation of state is* $\sigma = 3p + \text{constant}$.

Remark 3.8. If the scalar curvature vanishes, then a 4-dimensional perfect fluid spacetime satisfying $\operatorname{div}\mathcal{M} = 0$ represents a radiation era [5].

4. Ricci recurrent spacetimes admitting divergence free m-projective curvature tensor

A semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a Ricci recurrent manifold [19] if the Ricci tensor $\mathcal{S}$ satisfies

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1,\mathcal{W}_1) = \Pi(\mathcal{U}_1)\mathcal{S}(\mathcal{V}_1,\mathcal{W}_1), \tag{4.1}$$

where Π is a non-vanishing one-form associated with the vector field μ, as $\Pi(\mathcal{U}_1) = g(\mathcal{U}_1, \mu)$. If the Ricci tensor fulfills the condition (4.1), a Lorentzian manifold is referred to as Ricci recurrent spacetime. In this case, suppose the vector field μ related to the one-form Π is treated as a unit timelike vector field, that is, $\Pi(\mu) = -1$.

If $\mathrm{div}\mathcal{M} = 0$, then from Proposition 2.1, we find

$$(\nabla_{\mathcal{U}_1}\mathcal{S})(\mathcal{V}_1, \mathcal{W}_1) = (\nabla_{\mathcal{V}_1}\mathcal{S})(\mathcal{U}_1, \mathcal{W}_1). \tag{4.2}$$

Using (4.1) in (4.2) reveals that

$$\Pi(\mathcal{U}_1)\,\mathcal{S}(\mathcal{V}_1, \mathcal{W}_1) = \Pi(\mathcal{V}_1)\,\mathcal{S}(\mathcal{U}_1, \mathcal{W}_1). \tag{4.3}$$

Setting $\mathcal{U}_1 = \mu$ in (4.3), we infer that

$$\mathcal{S}(\mathcal{V}_1, \mathcal{W}_1) = -\Pi(\mathcal{V}_1)\,\mathcal{S}(\mathcal{W}_1, \mu). \tag{4.4}$$

Contracting (4.3), we can derive

$$\rho\Pi(\mathcal{U}_1) = \mathcal{S}(\mathcal{U}_1, \mu). \tag{4.5}$$

Using (4.5) in (4.4) we deduce that

$$\mathcal{S}(\mathcal{V}_1, \mathcal{W}_1) = -\rho\Pi(\mathcal{V}_1)\,\Pi(\mathcal{W}_1), \tag{4.6}$$

which implies the spacetime is Ricci simple [17].

Proposition 4.1. *A Ricci recurrent spacetime admitting* $\mathrm{div}\mathcal{M} = 0$ *is Ricci simple.*

Remark 4.2. The physical interpretation of a Ricci simple spacetime is explored in the paper [17]. The authors proved that a Ricci simple spacetime becomes a stiff matter fluid [22]. Thus we conclude that a Ricci recurrent spacetime admitting divergence free m-projective curvature tensor becomes a stiff matter fluid.

Mantica, Suh and De [17] proved the following theorem:

Theorem B. *If an n-dimensional $(n > 3)$ Lorentzian manifold (M^n, g) with the Ricci tensor of the form $\mathcal{S}(\mathcal{U}_1, \mathcal{V}_1) = -\rho\Pi(\mathcal{U}_1)\,\Pi(\mathcal{V}_1)$ satisfies the curvature condition* $\mathrm{div}\,\mathcal{C} = 0$, *then (M^n, g) is a generalized Robertson-Walker spacetime.*

In virtue of Remark 2.3, Proposition 4.1 and Theorem B, we can say:

Theorem 4.3. *A Ricci recurrent spacetime admitting* $\mathrm{div}\mathcal{M} = 0$ *is a generalized Robertson-Walker spacetime.*

5. Conclusion

The physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field can be effectively modeled by some Lorentzian metric defined on a suitable four-dimensional manifold, since the matter content of the universe is assumed to behave like a perfect fluid in standard cosmological models. The Einstein's field equations are crucial in the development of cosmological models because they imply that matter influences the geometry of spacetime and that matter's velocity is determined by the non-flat metric tensor of space. Relativistic fluid models are of great interest in astrophysics, plasma physics, and nuclear physics, among other fields.

In general relativity and cosmology, the physical motivation for studying different forms of spacetime models is to learn about distinct phases in the evolution of the universe, that can be split into three phases, namely,

I. Viscous fluid phase admitting heat flux,

II. Non-viscous fluid phase admitting heat flux and

III. Perfect fluid phase with thermal equilibrium.

In this work, the last phase is chosen and it has been revealed that under the condition $\mathrm{div}\mathcal{M} = 0$, perfect fluid spacetime becomes a generalized Robertson-Walker spacetime. Utilizing $\mathrm{div}\mathcal{M} = 0$, it is shown that the Ricci recurrent spacetime is a generalized Robertson-Walker spacetime. Furthermore, we discuss their physical relevance.

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