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SOME CHARACTERIZATIONS OF CONICS AND HYPERSURFACES WITH CENTRALLY SYMMETRIC HYPERPLANE SECTIONS

Shin-Ok Bang, Dong Seo Kim, Dong-Soo Kim, and Wonyong Kim

Abstract. Parallel conics have interesting area and chord properties. In this paper, we study such properties of conics and conic hypersurfaces. First of all, we characterize conics in the plane with respect to the above mentioned properties. Finally, we establish some characterizations of hypersurfaces with centrally symmetric hyperplane sections.

1. Introduction

The following properties of parallel conics are well-known.

Proposition 1. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are the parallel parabolas $y = a^2x^2 + k_1$ and $y = a^2x^2 + k_2$, $k_2 > k_1 > 0$, then for a point $P \in \mathcal{P}_2$ the tangent line to $\mathcal{P}_2$ at P meets $\mathcal{P}_1$ at A and B such that the area enclosed between $\mathcal{P}_1$ and the chord AB does not depend on P and the point P of tangency is the midpoint of the chord AB. The same holds for parallel ellipses $x^2/a^2 + y^2/b^2 = k_1^2$ and $x^2/a^2 + y^2/b^2 = k_2^2$ and for parallel hyperbolas $x^2/a^2 - y^2/b^2 = k_1$ and $x^2/a^2 - y^2/b^2 = k_2$.*

Conversely, in [3] using Clairaut's first order differential equation, it was shown that the above property characterizes parabolas as follows. For a proof, see Theorem 1 of [3]. The same characterization holds for parallel ellipses and for parallel hyperbolas ([3]).

Proposition 2. *Let $f \in C^{(2)}(\mathbb{R})$ be such that $f(x) > ax^2$ for all $x \in \mathbb{R}$. If the tangent line to the curve $y = f(x)$ at each one of its points cuts off from the parabola $\mathcal{P}(y = ax^2)$ a segment of constant area, then $f(x) = ax^2 + k$ for a positive constant $k \in \mathbb{R}$.*

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Dualizing Proposition 2, first of all in this paper we prove the following characterization theorems for conics. See Section 3.

Theorem 3. *For a positive constant $a \in \mathbb{R}$ we consider the parabola $\mathcal{P} : y = ax^2$ and the convex differentiable curve $\mathcal{X} : y = f(x)$ such that $f(x) < ax^2$ for all $x \in \mathbb{R}$. Suppose that every tangent line to the parabola $\mathcal{P}$ at a point $P \in \mathcal{P}$ meets the curve $\mathcal{X}$ at two points A and B. Then the following are equivalent.*

(1) *The tangent line to the parabola $\mathcal{P}$ at each $P \in \mathcal{P}$ cuts off the curve $\mathcal{X}$ a segment of constant area, which is independent of the point P.*
(2) *The point P is always the midpoint of the chord AB of the curve $\mathcal{X}$.*
(3) *The curve $\mathcal{X}$ is given by $f(x) = ax^2 - k$ with $k > 0$.*

For ellipses, we prove:

Theorem 4. *For two positive constants $a, b \in \mathbb{R}$, we consider the ellipse $\mathcal{E} : x^2/a^2 + y^2/b^2 = 1$ and a convex differentiable curve $\mathcal{X}$ outside of $\mathcal{E}$. Suppose that every tangent line to the ellipse $\mathcal{E}$ at a point $P \in \mathcal{E}$ meets the curve $\mathcal{X}$ at two points A and B. Then the following are equivalent.*

(1) *The tangent line to the ellipse $\mathcal{E}$ at each $P \in \mathcal{E}$ cuts off the curve $\mathcal{X}$ a segment of constant area, which is independent of the point P.*
(2) *The point P is always the midpoint of the chord AB of the curve $\mathcal{X}$.*
(3) *The curve $\mathcal{X}$ is the ellipse given by $\mathcal{X} : x^2/a^2 + y^2/b^2 = k$ with $k > 1$.*

In order to establish the same characterization for hyperbolas, we need some additional conditions as follows.

Theorem 5. *For two positive constants $a, b \in \mathbb{R}$, we consider a branch of the hyperbola $\mathcal{H} : x^2/a^2 - y^2/b^2 = 1$, $x > 0$ and a convex differentiable curve $\mathcal{X}$ in the concave side of $\mathcal{H}$. Suppose that every tangent line to the hyperbola $\mathcal{H}$ at a point $P \in \mathcal{H}$ meets the curve $\mathcal{X}$ at two points A and B. In addition, we suppose that for each point of $\mathcal{X}$, two tangent lines to $\mathcal{H}$ pass through the point. Then the following are equivalent.*

(1) *The tangent line to the hyperbola $\mathcal{H}$ at each $P \in \mathcal{H}$ cuts off the curve $\mathcal{X}$ a segment of constant area, which is independent of the point P.*
(2) *The point P is always the midpoint of the chord AB of the curve $\mathcal{X}$.*
(3) *The curve $\mathcal{X}$ is the hyperbola given by $\mathcal{X} : x^2/a^2 - y^2/b^2 = k$ with $x > 0$ and $0 < k < 1$.*

Finally, in Section 4 we use Theorems 3, 4 and 5 in order to characterize conic hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ as follows. See Section 4.

For elliptic paraboloids, we prove:

Theorem 6. *For an elliptic paraboloid $\mathcal{P} : y = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$ with $a_i > 0$, $i = 1, 2, \ldots, n$ and a convex continuous function $f(x), x = (x_1, \ldots, x_n)$ on $\mathbb{R}^n$ with $f(x) < a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$, we suppose that every tangent hyperplane to the paraboloid $\mathcal{P}$ at a point $P \in \mathcal{P}$ intersects the hypersurface $\mathcal{X} : y = f(x)$ in the boundary of a bounded region $D(P)$ in the hyperplane. Then the following are equivalent.*

(1) *The region $D(P)$ is centrally symmetric with respect to the point P.*
(2) $f(x) = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 - k$ *with* $k > 0$.

For ellipsoids, we prove:

Theorem 7. *For an ellipsoid $\mathcal{E} : x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2 + y^2/b^2 = 1$ with a_i, $b > 0$, $i = 1, 2, \ldots, n$ and a convex hypersurface $\mathcal{X}$ outside of $\mathcal{E}$, we suppose that every tangent hyperplane to the ellipsoid $\mathcal{E}$ at a point $P \in \mathcal{E}$ intersects the hypersurface $\mathcal{X}$ in the boundary of a bounded region $D(P)$ in the hyperplane. Then the following are equivalent.*

(1) *The region $D(P)$ is centrally symmetric with respect to the point P.*
(2) $\mathcal{X} : x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2 + y^2/b^2 = k$ *with* $k > 1$.

In order to establish the similar characterization for elliptic hyperboloids, we need some additional conditions as follows.

Theorem 8. *For an elliptic hyperboloid $\mathcal{H} : x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2 - y^2/b^2 = -1$ with $y > 0$ and a_i, $b > 0$, $i = 1, 2, \ldots, n$ and a convex hypersurface $\mathcal{X}$ in the concave side of $\mathcal{E}$, we suppose that every tangent hyperplane to the hyperboloid $\mathcal{H}$ at a point $P \in \mathcal{H}$ intersects the hypersurface $\mathcal{X}$ in the boundary of a bounded region $D(P)$ in the hyperplane. In addition, we suppose that the convex hypersurface $\mathcal{X}$ lies in the convex open cone $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2 - y^2/b^2 < 0$ with $y > 0$. Then the following are equivalent.*

(1) *The region $D(P)$ is centrally symmetric with respect to the point P.*
(2) $\mathcal{X} : x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_n^2/a_n^2 - y^2/b^2 = -k$ *with* $y > 0$ *and* $0 < k < 1$.

Various properties of conics (especially, parabolas) have been proved to be characteristic ones ([1–3, 5, 6, 10, 12–14, 16, 17]). Some characterization theorems for hyperplanes, circular hypercylinders, hyperspheres, ellipsoids, elliptic paraboloids and elliptic hyperboloids in the Euclidean space $\mathbb{E}^{n+1}$ were established in [3, 4, 7–9, 11, 15].

2. Some lemmas

In this section, we prove two lemmas which are crucial in the proof of our theorems.

Lemma 9. *Suppose that a positive continuous function $g : \mathbb{R} \to \mathbb{R}^+$ satisfies*

$$g(x + g(x)) = g(x) \tag{2.1}$$

for all $x \in \mathbb{R}$, where $\mathbb{R}^+$ denotes the set of positive real numbers. Then g is a constant function.

Proof. We give a proof using several steps as follows. Each step can be shown easily. Otherwise, we give a brief proof.

Step 1. The function h defined by $h(x) = x + g(x)$ is an injective function satisfying

$$g(h(x)) = g(x). \tag{2.2}$$

Step 2. We put $g(0) = a > 0$. Then we have

$$g(na) = a, \quad \text{and} \quad h(na) = (n+1)a, \quad n = 1, 2, \ldots. \tag{2.3}$$

Together with the continuity of h, Steps 1 and 2 show that the function h is strictly increasing. Hence we have

Step 3. For every $x_1 \in I_0 := [0, a]$, we put $x_2 = h(x_1), \ldots, x_{n+1} = h(x_n)$. Then we have for all $n \in N$, $x_n \in I_{n-1} := [(n-1)a, na]$ and $g(x_n) = g(x_1)$. Furthermore, for all $n \in N$ we have $h(I_{n-1}) = I_n$.

Step 4. For every $x_1 \in [0, a]$, we have $g(x_1) \leq a$.

Proof of Step 4. Suppose that $\epsilon := g(x_1) - a > 0$ for some $x_1 \in (0, a)$. Then we have

$$\begin{aligned} x_2 &= h(x_1) = x_1 + g(x_1) = x_1 + a + \epsilon, \\ x_3 &= h(x_2) = x_2 + g(x_2) = x_1 + 2(a + \epsilon), \ldots, \\ x_{n+1} &= h(x_n) = x_1 + n(a + \epsilon), \quad n = 1, 2, \ldots. \end{aligned} \tag{2.4}$$

This shows that for a sufficiently large n, $x_{n+1} - (n+1)a = x_1 + n\epsilon - a > 0$, which contradicts to Step 3. This completes the proof of Step 4.

In a similar manner as in the proof of Step 4, we may prove

Step 5. For every $x_1 \in [0, a]$, we have $g(x_1) \geq a$.

Hence we have $g(x) = a$ for all $x \in [0, a]$. Together with Step 3, this shows that $g(x) = a$ for all $x \in [0, \infty)$.

Step 6. For every $x \in \mathbb{R}$, we have $g(x) = a$.

Proof of Step 6. Suppose that $x_0 := \text{glb}\{b \in \mathbb{R} \mid g(x) = a \text{ for all } x \in [b, \infty)\} > -\infty$. Then we have $x_0 \leq 0$. On $[x_0, \infty)$, we have $g(x) = a$, and hence $h(x) = x + a$. Since $h(x_0) = x_0 + a > x_0$, the continuity of h shows that there exists a positive ϵ such that $h((x_0 - \epsilon, x_0]) \subset [x_0, \infty)$. Hence it follows from (2.2) that for all $x \in (x_0 - \epsilon, x_0]$, $g(x) = g(h(x)) = a$, which is a contradiction. This completes the proof of Step 6. □

Lemma 10. *Suppose that a positive continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies*

$$\frac{g(x + g(x))}{x + g(x)} = \frac{g(x)}{x} \tag{2.5}$$

for all $x \in \mathbb{R}^+$. Then $k(x) = g(x)/x$ is a constant function.

Proof. As in the proof of Lemma 9, we can prove Lemma 10 using following steps. We omit the proof of each step.

We put $h(x) = x + g(x)$ and $k(x) = g(x)/x$. Then the functions h and k satisfy

$$h(x) = x + g(x) = x(1 + k(x)), \tag{2.6}$$

$$k(h(x)) = k(x). \tag{2.7}$$

Step 1. The function h is an injective function.

Step 2. We put $g(1) = a > 0$. Then we have $k(1) = a, h(1) = b(:= 1 + a) > 1$, and $k(b) = a, h(b) = b^2$. Hence $h(x)$ is a strictly increasing function on $\mathbb{R}^+$. The functions satisfy

$$k(b^n) = a, \quad h(b^n) = b^{n+1}, \quad g(b^n) = ab^n, \quad n = 1, 2, \ldots. \tag{2.8}$$

Step 3. Since $h(x)$ is strictly increasing, we see that $h([b^{n-1}, b^n]) = [b^n, b^{n+1}]$ for all $n \in N$. For every $x_1 \in [1, b]$, we put $x_2 = h(x_1), \ldots, x_{n+1} = h(x_n)$. Then we have $x_{n+1} \in [b^n, b^{n+1}]$, and $k(x_n) = k(x_1)$ for all $n \in N$.
Step 4. For every $x_1 \in [1, b]$, we have $k(x_1) \leq a$.
Step 5. For every $x_1 \in [1, b]$, we have $k(x_1) \geq a$.

It follows from Steps 4 and 5 that $k(x) = a$ for all $x \in [1, b]$. Hence Step 3 implies $k(x) = a$ for all $x \in [1, \infty)$.
Step 6. The assumption that $x_0 := \text{glb}\{c \in \mathbb{R}^+ \mid k(x) = a \text{ for all } x \in [c, \infty)\} > 0$ leads a contradiction. Thus we see that $x_0 = 0$, that is, $k(x) = a$ for all $x \in \mathbb{R}^+$. □

3. Proofs of Theorems 3, 4 and 5

In this section, we prove Theorems 3, 4 and 5 as follows.

Proof of Theorem 3. Using the linear transformation $L = \text{diag}(\sqrt{a}, 1)$, it suffices to prove when the parabola is given by $\mathcal{P} : y = x^2$.

(1) ⇒ (2). The tangent line at $P(u, u^2)$ meets the convex differentiable curve $\mathcal{X} : y = f(x)$ at $A(u - g(u), f(u - g(u)))$ and $B(u + h(u), f(u + h(u)))$, where $g(u)$ and $h(u)$ are some positive functions. Hence the tangent line ℓ is given by $y = 2ux - u^2$. Since ℓ passes through A and B, we have

$$f(u - g(u)) = u(u - 2g(u)), \quad f(u + h(u)) = u(u + 2h(u)). \tag{3.1}$$

The area $S(u)$ of the segment is given by

$$S(u) = \int_{u-g(u)}^{u+h(u)} \phi(x)dx, \quad \phi(x) = 2ux - u^2 - f(x). \tag{3.2}$$

Together with (3.1), differentiating $S(u)$ with respect to u gives

$$S'(u) = (h(u) + g(u))(h(u) - g(u)). \tag{3.3}$$

Since $h(u) + g(u) > 0$, the assumption implies $h(u) - g(u) = 0$. This completes the proof.

(2) ⇒ (3). For an arbitrary point $A(u, f(u))$ of the convex curve $\mathcal{X} : y = f(x)$, there exists a tangent line ℓ to the parabola $\mathcal{P}$ at $P(u + g(u), (u + g(u))^2)$ which passes through the point $A(u, f(u))$, where $g(u)$ is a positive function. The assumption implies that the tangent line ℓ meets the curve $\mathcal{X}$ at $B(u + 2g(u), f(u + 2g(u)))$. Hence the tangent line ℓ is given by $y = 2(u + g(u))x - (u + g(u))^2$. Since ℓ passes through A and B, we have

$$f(u) = (u - g(u))(u + g(u)), \tag{3.4}$$

$$f(u + 2g(u)) = (u + g(u))(u + 3g(u)). \tag{3.5}$$

We put $w = u + 2g(u)$, then (3.5) becomes

$$f(w) = (w - g(u))(w + g(u)). \tag{3.6}$$

Let us replace u in (3.4) with w. Then we get

$$f(w) = (w - g(w))(w + g(w)). \tag{3.7}$$

It follows from (3.6) and (3.7) that $g(u)^2 = g(w)^2$, which shows $g(u) = g(w) = g(u + 2g(u))$ for all $u \in \mathbb{R}$.

Finally, Lemma 9 implies $2g(u)$ is a constant function, and hence $g(u) = a$ for a constant a. Thus (3.4) shows that $f(u) = u^2 - k$ with $k = a^2 > 0$. This completes the proof.

(3) $\Rightarrow$ (1). Proposition 1 completes the proof. □

Proof of Theorem 4. Using the linear transformation $L = \text{diag}(1/a, 1/b)$, it suffices to prove when the ellipse is given by $\mathcal{E} : x^2 + y^2 = 1$.

(1) $\Rightarrow$ (2). The tangent line ℓ at $P \in \mathcal{E}$ meets the convex differentiable curve $\mathcal{X}$ at A and B. By a suitable rotation around the origin if necessary, we may assume that a neighborhood of the arc $\overset{\frown}{AB}$ of the curve $\mathcal{X}$ is given by the graph of a convex differentiable function f satisfying

$$\begin{aligned} P &= (u, -\sqrt{1 - u^2}), \\ A &= (u - g(u), f(u - g(u))), \\ B &= (u + h(u), f(u + h(u))), \end{aligned} \tag{3.8}$$

where $g(u)$ and $h(u)$ are two positive functions. Hence the tangent line ℓ is given by

$$y = j(u)(ux - 1), \ j(u) = (1 - u^2)^{-1/2}. \tag{3.9}$$

Since ℓ passes through A and B, we have

$$f(u - g(u)) = j(u)[-ug(u) + u^2 - 1], \tag{3.10}$$

$$f(u + h(u)) = j(u)[uh(u) + u^2 - 1]. \tag{3.11}$$

The area $S(u)$ of the segment is given by

$$S(u) = \int_{u-g(u)}^{u+h(u)} \phi(x)dx, \quad \phi(x) = j(u)(ux - 1) - f(x). \tag{3.12}$$

Now, we differentiate $S(u)$ with respect to u. Then (3.10) and (3.11) show

$$S'(u) = \frac{1}{2(1 - u^2)^{3/2}}(h(u) + g(u))(h(u) - g(u)). \tag{3.13}$$

Since $h(u) + g(u) > 0$, the assumption implies $h(u) - g(u) = 0$. This completes the proof.

(2) $\Rightarrow$ (3). For an arbitrary point $P(\cos u, \sin u)$ of the ellipse $\mathcal{E}$, the tangent line ℓ to $\mathcal{E}$ at P meets the convex curve $\mathcal{X}$ at

$$\begin{aligned} A &= r(u)(\cos(u-g(u)), \sin(u-g(u))), \quad \text{and} \\ B &= r(u)(\cos(u+g(u)), \sin(u+g(u))), \end{aligned} \tag{3.14}$$

where $g(u) \in (0, \pi/2)$, $s(u) = \overline{PA} = \overline{PB}$ and $r(u) = \overline{OA} = \overline{OB} = \sqrt{1+s(u)^2}$. Obviously, we have $s(u) = \tan g(u)$ and $r(u) = \sec g(u)$.

Another tangent line ℓ' to $\mathcal{E}$ at Q passes through the point B. Since $\angle POB = \angle QOB$, the point Q of tangency is given by $Q(\cos(u+2g(u)), \sin(u+2g(u)))$ and $s(u+2g(u)) = \overline{QB}$. Since $\overline{QB} = \overline{PB}$, we see that $s(u+2g(u)) = s(u)$, and hence $g(u+2g(u)) = g(u)$ for all $u \in \mathbb{R}$.

Finally, Lemma 9 implies $2g(u)$ is a constant function, and hence $r(u)$ is a constant a. Thus the curve is given by $\mathcal{X} : x^2 + y^2 = a^2$. This completes the proof.

(3) $\Rightarrow$ (1). Proposition 1 completes the proof. $\square$

Proof of Theorem 5. Using the linear transformation L defined by

$$L = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}, \tag{3.15}$$

it suffices to prove when the hyperbola is given by $\mathcal{H} : y = 1/x$, $x > 0$.

(1) $\Rightarrow$ (2). The tangent line ℓ at $P(u, 1/u) \in \mathcal{H}$ meets the convex differentiable curve $\mathcal{X} : y = f(x)$ at $A(u-g(u), f(u-g(u)))$ and $B(u+h(u), f(u+h(u)))$, where $g(u)$ and $h(u)$ are positive functions. The tangent line ℓ is given by

$$y = -\frac{1}{u^2}(x - 2u). \tag{3.16}$$

Since ℓ passes through A and B, we have

$$f(u-g(u)) = \frac{u+g(u)}{u^2} \quad \text{and} \quad f(u+h(u)) = \frac{u-h(u)}{u^2}. \tag{3.17}$$

The area $S(u)$ of the segment is given by

$$S(u) = \int_{u-g(u)}^{u+h(u)} \phi(x)dx, \quad \phi(x) = -\frac{1}{u^2}(x-2u) - f(x). \tag{3.18}$$

Now, we differentiate $S(u)$ with respect to u. Then using (3.17), a lengthy calculation implies

$$S'(u) = \frac{1}{u^3}(h(u)+g(u))(h(u)-g(u)). \tag{3.19}$$

Since $h(u) + g(u) > 0$, the assumption implies $h(u) - g(u) = 0$. This completes the proof.

(2) $\Rightarrow$ (3). The additional assumption shows that the curve $\mathcal{X} : y = f(x)$ lies in the first quadrant, that is, $f(x)$ is defined for $x > 0$ and $0 < f(x) < 1/x$. For an arbitrary point $A(u, f(u))$ with $u > 0$ of the convex curve $\mathcal{X} : y = f(x)$, there exists a tangent line ℓ to the hyperbola $\mathcal{H}$ at $P(u+g(u), (u+g(u))^{-1})$

which passes through the point $A(u, f(u))$, where $g(u)$ is a positive function defined on $(0, \infty)$. The assumption implies that the tangent line ℓ meets the curve $\mathcal{X}$ at $B(u + 2g(u), f(u + 2g(u)))$. Hence the tangent line ℓ is given by

$$y = a(u)x + b(u),\ a(u) = -(u + g(u))^{-2},\ b(u) = 2(u + g(u))^{-1}. \tag{3.20}$$

Since ℓ passes through A and B, we have

$$f(u) = a(u)u + b(u) = w(u)(u + g(u))^{-2} \tag{3.21}$$

and

$$f(w(u)) = a(u)w(u) + b(u) = u(u + g(u))^{-2}, \tag{3.22}$$

where we put $w(u) = u + 2g(u)$.

Now, let us replace u in (3.21) with $w(u)$. Then we get

$$f(w(u)) = [w(u) + 2g(w(u))][w(u) + g(w(u))]^{-2}. \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$u[w(u) + g(w(u))]^2 = [w(u) + 2g(w(u))](u + g(u))^2, \tag{3.24}$$

which can be rewritten as

$$\{ug(w(u)) - g(u)w(u)\}\{g(w(u)) + g(u)\} = 0. \tag{3.25}$$

Since $g(u) > 0$ for all $u \in \mathbb{R}^+$, we see that $ug(w(u)) = g(u)w(u)$, that is,

$$\frac{g(u + 2g(u))}{u + 2g(u)} = \frac{g(u)}{u} \tag{3.26}$$

for all $u \in \mathbb{R}^+$.

Finally, Lemma 10 implies $g(u) = bu$ for a positive constant b. Hence it follows from (3.21) that $f(x) = a/x$, where $a = (1 + 2b)/(1 + b)^2 \in (0, 1)$. This completes the proof.

(3) $\Rightarrow$ (1). Proposition 1 completes the proof. □

Remark. In the proof of (2) $\Rightarrow$ (3) of each proof of Theorems 3, 4 and 5, we do not assume the differentiability of the curve $\mathcal{X}$.

4. Hypersurfaces with centrally symmetric hyperplane sections

In this section, we prove Theorems 6, 7 and 8 as follows. Note that in the proof of (2) $\Rightarrow$ (3) of Theorems 3, 4 and 5, we do not assume the differentiability of the curve $\mathcal{X}$.

Proof of Theorem 6. Using the linear transformation $L = \mathrm{diag}(\sqrt{a_1}, \ldots, \sqrt{a_n}, 1)$, it suffices to prove when the elliptic paraboloid is given by $\mathcal{P} : y = x_1^2 + x_2^2 + \cdots + x_n^2$.

(1) $\Rightarrow$ (2). For every tangent unit vector $v = (v_1, \ldots, v_n, 0)$ to $\mathcal{P}$ at the origin, we consider the intersection $\mathcal{P}_v = \mathcal{P} \cap vy$-plane. Then $\mathcal{P}_v : y = t^2$ is a parabola. We put $g(t) = f(tv)$. Then the parabola $\mathcal{P}_v$ and the convex curve $\mathcal{X} : y = g(t)$ satisfy the assumptions in Theorem 3. Hence we see that

$f(tv) = g(t) = t^2 + k$, $k < 0$. Since $k = f(0)$, the constant k is independent of the unit tangent vector v. For any $x = (x_1, \dots, x_n, 0) \neq 0$, we put $t = |x|$ and $v = x/t$, then we have

$$f(x) = f(tv) = t^2 + k = x_1^2 + x_2^2 + \cdots + x_n^2 + k. \tag{4.1}$$

This completes the proof.

(2) $\Rightarrow$ (1). It is straightforward to complete the proof. □

Proof of Theorem 7. Using the linear transformation $L = \operatorname{diag}(1/a_1, \dots, 1/a_n, 1/b)$, it suffices to prove when the ellipsoid is given by $\mathcal{E} : x_1^2 + x_2^2 + \cdots + x_n^2 + y^2 = 1$.

(1) $\Rightarrow$ (2). For every tangent unit vector $v = (v_1, \dots, v_n, 0)$ to $\mathbb{R}^n$ at the origin, we consider the intersection $\mathcal{E}_v = \mathcal{E} \cap vy$-plane. Then $\mathcal{E}_v$ is a great circle. We put $\mathcal{X}_v = \mathcal{X} \cap vy$-plane. Then the circle $\mathcal{E}_v$ and the convex curve $\mathcal{X}_v$ satisfy the assumptions in Theorem 4. Hence we see that $\mathcal{X}_v$ is a circle of radius $r > 1$. Note that $(0, \dots, 0, r)$ is the intersection point $\mathcal{X} \cap y$-axis. Hence the radius r is independent of the unit tangent vector v. Thus, the hypersurface $\mathcal{X}$ is an n-dimensional sphere. This completes the proof.

(2) $\Rightarrow$ (1). It is straightforward to complete the proof. □

Proof of Theorem 8. Using the linear transformation $L = \operatorname{diag}(1/a_1, \dots, 1/a_n, 1/b)$, it suffices to prove when the elliptic hyperboloid is given by $\mathcal{H} : x_1^2 + x_2^2 + \cdots + x_n^2 - y^2 = -1$ with $y > 0$.

(1) $\Rightarrow$ (2). For every tangent unit vector $v = (v_1, \dots, v_n, 0)$ to $\mathbb{R}^n$ at the origin, we consider the intersection $\mathcal{H}_v = \mathcal{H} \cap vy$-plane. Then $\mathcal{H}_v = \{sv + ye_{n+1} \,|\, s^2 - y^2 = -1\}$ is a hyperbola. We put $\mathcal{X}_v = \mathcal{X} \cap vy$-plane. Then the hyperbola $\mathcal{H}_v$ and the convex curve $\mathcal{X}_v$ satisfy the assumptions in Theorem 5. Furthermore, $\mathcal{X}_v$ satisfies the additional assumption in Theorem 5. Hence we see that

$$\mathcal{X}_v = \{sv + ye_{n+1} \,|\, s^2 - y^2 = -k,\ y > 0\},\ 0 < k < 1. \tag{4.2}$$

Since $(0, \dots, 0, \sqrt{k}) = \mathcal{X} \cap y$-axis, the constant k is independent of the unit tangent vector v. For any $x = (x_1, \dots, x_n, 0) \neq 0$, we put $s = |x|$ and $v = x/s$, then we have

$$y^2 = s^2 + k = x_1^2 + x_2^2 + \cdots + x_n^2 + k,\ y > 0. \tag{4.3}$$

This completes the proof.

(2) $\Rightarrow$ (1). It is straightforward to complete the proof. □

References

[1] Á. Bényi, P. Szeptycki, and F. Van Vleck, *Archimedean properties of parabolas*, Amer. Math. Monthly **107** (2000), no. 10, 945–949. `https://doi.org/10.2307/2695591`

[2] Á. Bényi, P. Szeptycki, and F. Van Vleck, *A generalized Archimedean property*, Real Anal. Exchange **29** (2003/04), no. 2, 881–889.

[3] Ó. Ciaurri, E. Fernández, and L. Roncal, *Revisiting floating bodies*, Expo. Math. **34** (2016), no. 4, 396–422. `https://doi.org/10.1016/j.exmath.2016.06.001`

[4] D.-S. Kim, *Ellipsoids and elliptic hyperboloids in the Euclidean space* $\mathbb{E}^{n+1}$, Linear Algebra Appl. **471** (2015), 28–45. https://doi.org/10.1016/j.laa.2014.12.014

[5] D.-S. Kim and S. H. Kang, *A characterization of conic sections*, Honam Math. J. **33** (2011), no. 3, 335–340. https://doi.org/10.5831/HMJ.2011.33.3.335

[6] D.-S. Kim and Y. H. Kim, *A characterization of ellipses*, Amer. Math. Monthly **114** (2007), no. 1, 66–70. https://doi.org/10.1016/j.laa.2012.02.013

[7] D.-S. Kim and Y. H. Kim, *New characterizations of spheres, cylinders and W-curves*, Linear Algebra Appl. **432** (2010), no. 11, 3002–3006. https://doi.org/10.1016/j.laa.2010.01.006

[8] D.-S. Kim and Y. H. Kim, *Some characterizations of spheres and elliptic paraboloids*, Linear Algebra Appl. **437** (2012), no. 1, 113–120. https://doi.org/10.1016/j.laa.2012.02.013

[9] D.-S. Kim and Y. H. Kim, *Some characterizations of spheres and elliptic paraboloids II*, Linear Algebra Appl. **438** (2013), no. 3, 1356–1364. https://doi.org/10.1016/j.laa.2012.08.024

[10] D.-S. Kim and Y. H. Kim, *On the Archimedean characterization of parabolas*, Bull. Korean Math. Soc. **50** (2013), no. 6, 2103–2114. https://doi.org/10.4134/BKMS.2013.50.6.2103

[11] D.-S. Kim and Y. H. Kim, *A characterization of concentric hyperspheres in* $\mathbb{R}^n$, Bull. Korean Math. Soc. **51** (2014), no. 2, 531–538. https://doi.org/10.4134/BKMS.2014.51.2.531

[12] D.-S. Kim, J. H. Park, and Y. H. Kim *Some characterizations of parabolas*, Kyungpook Math. J. **53** (2013), no. 1, 99–104. https://doi.org/10.5666/KMJ.2013.53.1.99

[13] D.-S. Kim, S. Park, and Y. H. Kim, *Center of gravity and a characterization of parabolas*, Kyungpook Math. J. **55** (2015), no. 2, 473–484. https://doi.org/10.5666/KMJ.2015.55.2.473

[14] D.-S. Kim and K.-C. Shim, *Area of triangles associated with a curve*, Bull. Korean Math. Soc. **51** (2014), no. 3, 901–909. https://doi.org/10.4134/BKMS.2014.51.3.901

[15] D.-S. Kim and B. Song, *A characterization of elliptic hyperboloids*, Honam Math. J. **35** (2013), no. 1, 37–49. https://doi.org/10.5831/HMJ.2013.35.1.37

[16] J. Krawczyk, *On areas associated with a curve*, Zesz. Nauk. Uniw. Opol. Mat. **29** (1995), 97–101.

[17] B. Richmond and T. A. Richmond, *How to recognize a parabola*, Amer. Math. Monthly **116** (2009), no. 10, 910–922. https://doi.org/10.4169/000298909X477023

Shin-Ok Bang
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: bso712@hanmail.net

Dong Seo Kim
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: dongseo@chonnam.ac.kr

Dong-Soo Kim
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: dosokim@chonnam.ac.kr

Wonyong Kim
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
Email address: yong4625@naver.com