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TERMINAL SPACES OF MONOIDS

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Abstract. The purpose of this note is a wide generalization of the topological results of various classes of ideals of rings, semirings, and modules, endowed with Zariski topologies, to r-strongly irreducible r-ideals (endowed with Zariski topologies) of monoids, called terminal spaces. We show that terminal spaces are T_0, quasi-compact, and every nonempty irreducible closed subset has a unique generic point. We characterize r-arithmetic monoids in terms of terminal spaces. Finally, we provide necessary and sufficient conditions for the subspaces of r-maximal r-ideals and r-prime r-ideals to be dense in the corresponding terminal spaces.

1. Introduction and preliminaries

Under the name primitive ideals, in [7], the notion of strongly irreducible ideals was introduced for commutative rings. In [6, p. 301, Exercise 34], the ideals of the same spectrum are called quasi-prime ideals. The term "strongly irreducible" was first used for noncommutative rings in [5]. Since then, several algebraic and topological studies have been done on these types of ideals of rings (see [3, 13, 18]). The notion of strongly irreducible ideals has been generalized to semirings (see [2, 14]) and modules (see [15, 17]).

The aim of this note is to study the topological properties of the space of r-strongly irreducible r-ideals of a monoid endowed with a Zariski topology. This is a wide generalization of Zariski spaces. Moreover, r-strongly irreducible r-ideals are the "largest" class of r-ideals on which one can impose a Zariski topology. Therefore, we not only generalize some of the topological results from the above-mentioned works on strongly irreducible ideals of rings, semirings, and semimodules to monoids, but also generalize topological results on maximal, prime, minimal prime, and primary ideals of those structures to r-strongly irreducible r-ideals of monoids. We highlight the results that have been generalized here. Although our setup is on monoids, many of the results still hold for (commutative) semigroups.

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Let us recall some elementary definitions from [9] (see also [10]). A monoid $M = (M, \cdot)$ consists of a set $M \neq \emptyset$, together with an associative and commutative binary operation $*\colon M \times M \to M$ such that M possesses an identity element $1 \in M$ satisfying $1 * m = m$ for all $m \in M$, and a zero element $0 \in M$ satisfying $m * 0 = 0$ for all $m \in M$. The identity element and the zero element are uniquely determined, and we shall always assume that $0 \neq 1$. We shall write xy for $x * y$ and we shall assume all our monoids are commutative. For a set X, we denote by $\mathcal{P}(X)$ the set of all subsets of X. If M is a monoid, $S, T \in \mathcal{P}(M)$, and $m \in M$, we set

$$S * T = ST = \{st \mid s \in S,\ t \in T\}, \qquad mT = \{m\}T = \{mt \mid t \in T\}.$$

An *ideal system* on a monoid M is a map $r\colon \mathcal{P}(M) \to \mathcal{P}(M)$ defined by $X \mapsto X_r$ such that the following conditions are satisfied for all subsets $X, Y \subseteq M$ and all elements $m \in M$:

- $X \cup \{0\} \subseteq X_r$,
- $X \subseteq Y_r$ implies that $X_r \subseteq Y_r$,
- $mM \subseteq \{m\}_r$, and
- $mX_r = (mX)_r$.

Let r be an ideal system. A subset $I \subseteq M$ is called an *r-ideal* if $I = I_r$. By $\mathcal{I}_r(M)$, we shall denote the set of nonempty r-ideals of M. An r-ideal I is called *proper* if $I \neq M$. Let r be an ideal system. A monoid M is called *r-Noetherian* if $(\mathcal{I}_r(M), \subseteq)$ satisfies the ascending chain condition. If X is a nonempty subset of a monoid M, then the following equality

$$X_r = \bigcap_{J \in \mathcal{I}_r(M),\, J \supseteq X} J$$

holds, and thus X_r is the smallest r-ideal containing X. For r-ideals $I, J \in \mathcal{I}_r(M)$, we call $I *_r J = (IJ)_r \in \mathcal{I}_r(M)$, the *r-product* of I and J. For all $I, J \in \mathcal{I}_r(M)$, it is easy to show that $I *_r J \subseteq I \cap J$ (see [9, §2.3, Proposition (iii)]). An r-ideal P is called *r-prime* if $P \neq M$, and $i, i' \in M$, $ii' \in P$ implies $i \in P$ or $i' \in P$. If I is an r-ideal of M, the *r-radical* of I is defined by

$$\sqrt{I} = \{m \in M \mid m^k \in I \text{ for some } k \in \mathbb{Z}^+\}.$$

An r-ideal L of a monoid M is called *r-irreducible* if $L \neq M$, and for all r-ideals $I, J \in \mathcal{I}_r(M)$, $L = I \cap J$ implies that $L = I$ or $L = J$. An r-ideal K of a monoid M is called *r-strongly irreducible* if $K \neq M$ and, for all r-ideals $I, J \in \mathcal{I}_r(M)$, $I \cap J \subseteq K$ implies that $I \subseteq K$ or $J \subseteq K$. An r-ideal $I \in \mathcal{I}_r(M)$ is called *r-maximal* if $M \neq I$ and there is no r-ideal $J \in \mathcal{I}_r(M)$ such that $I \subsetneq J \subsetneq M$.

2. Terminal spaces

Let M be a monoid and let $\mathcal{S}_r(M)$ be the set of all r-strongly irreducible r-ideals of M. We impose a Zariski topology (in the sense of [8, §1.1.1]) on

$\mathcal{S}_r(M)$ by defining closed sets by

$$\mathcal{HK}(X) = \begin{cases} \{J \in \mathcal{S}_r(M) \mid J \supseteq \mathcal{K}(X)\}, & X \neq \emptyset; \\ \emptyset, & X = \emptyset, \end{cases} \tag{2.1}$$

where $X \subseteq \mathcal{S}_r(M)$ and $\mathcal{K}(X) = \bigcap_{I \in X} I$. The following theorem shows that $\mathcal{HK}$ is a Kuratowski closure operator on $\mathcal{S}_r(M)$, and hence indeed induces a closed-set topology on $\mathcal{S}_r(M)$.

Theorem 2.1. *Let M be a monoid and let $\mathcal{HK}$ be defined as in* (2.1)*.*

(1) $\mathcal{HK}(\emptyset) = \emptyset$.
(2) *For all* $X \subseteq \mathcal{S}_r(M)$, $X \subseteq \mathcal{HK}(X)$.
(3) *For all* $X \subseteq \mathcal{S}_r(M)$, $\mathcal{HK}(\mathcal{HK}(X)) = \mathcal{HK}(X)$.
(4) *For all* $X, X' \subseteq \mathcal{S}_r(M)$, $\mathcal{HK}(X \cup X') = \mathcal{HK}(X) \cup \mathcal{HK}(X')$.

Proof. (1)–(2) Follows from (2.1).

(3) By (2), $X \subseteq \mathcal{HK}(X)$ and hence $\mathcal{HK}(\mathcal{HK}(X)) \supseteq \mathcal{HK}(X)$ by increasing property of $\mathcal{HK}$. The other inclusion follows from (2.1).

(4) By (2) and by the increasing property of $\mathcal{HK}$, we have $\mathcal{HK}(X \cup X') \supseteq \mathcal{HK}(X) \cup \mathcal{HK}(X')$. Suppose $J \in \mathcal{HK}(X \cup X')$. Then $\mathcal{K}(X) \cap \mathcal{K}(X') \subseteq J$. Since J is strongly irreducible, $\mathcal{K}(X) \subseteq J$ or $\mathcal{K}(X') \subseteq J$, and hence $J \in \mathcal{HK}(X) \cup \mathcal{HK}(X')$. □

From Theorem 2.1(4), it is clear that the class of r-strongly irreducible r-ideals is the "largest" class of r-ideals of a monoid on which we can endow a hull-kernel topology (= Zariski topology). The set $\mathcal{S}_r(M)$ endowed with the above-mentioned hull-kernel topology will be called a *terminal space*. The following proposition characterizes strongly irreducible ideals as terminal spaces, and it generalizes the ring-theoretic result [16, §2.2, p. 11].

Proposition 2.2. *The operation defined in* (2.1) *is a Kuratowski closure operator on a class $\mathcal{F}$ of r-ideals of M if and only if*

$$J \cap K \subseteq I \quad \text{implies} \quad J \subseteq I \ \text{ or } \ K \subseteq I$$

for all $J, K \in \mathcal{I}_r(M)$ and for all $I \in \mathcal{F}$.

Before we discuss topological properties of terminal spaces, let us note down a few more elementary results about the closure operator $\mathcal{HK}$, which will be used in the sequel.

Lemma 2.3. *Let M be a monoid and let X, X', $\{X_\lambda\}_{\lambda \in \Lambda}$ be nonempty subsets of $\mathcal{S}_r(M)$. Then the following hold.*

(1) $\mathcal{HK}(M) = \emptyset$.
(2) $\mathcal{HK}(X) = \overline{X}$.
(3) $\mathcal{HK}(X) \cup \mathcal{HK}(X') = \mathcal{HK}(X \cap X')$.
(4) $\bigcap_{\lambda \in \Lambda} \mathcal{HK}(X_\lambda) = \mathcal{HK}\left(\bigcap_{\lambda \in \Lambda} X_\lambda\right)$.
(5) $\mathcal{HK}(X) \subseteq \mathcal{HK}(\langle X \rangle) \subseteq \mathcal{HK}(\sqrt{\langle X \rangle})$.

Proof. (1) Follows from the definition of a r-strongly irreducible r-ideal of M.

(2) From Theorem 2.1(2), we have $\overline{X} \subseteq \overline{\mathcal{HK}(X)} = \mathcal{HK}(X)$. Let $\mathcal{HK}(Y)$ be an arbitrary closed subset of $\mathcal{S}_r(M)$ containing X. Then

$$\mathcal{HK}(Y) = \mathcal{HK}(\mathcal{HK}(Y)) \supseteq \mathcal{HK}(X).$$

Since $\mathcal{HK}(X)$ is the smallest closed set containing X by Theorem 2.1(2), we have the claim.

(3)–(5) Straightforward. □

The next result generalizes Theorem 4.1 and Theorem 3.1 in [12], Theorem 9 in [14], Theorem 4.1(v)–(vi) in [3], and Proposition 2.4 in [20].

Theorem 2.4. *Suppose that r is finitary. Then every terminal space $\mathcal{S}_r(M)$ is quasi-compact and a T_0-space.*

Proof. Let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of closed sets of $\mathcal{S}_r(M)$ and let $\bigcap_{\lambda\in\Lambda} C_\lambda = \emptyset$. Then $C_\lambda = \mathcal{HK}(X_\lambda)$ for some subsets X_λ of $\mathcal{S}_r(M)$, and by Lemma 2.3(4), we have

$$\bigcap_{\lambda\in\Lambda} \mathcal{HK}(X_\lambda) = \mathcal{HK}\left(\bigcap_{\lambda\in\Lambda} X_\lambda\right) = \emptyset.$$

Let K be the r-closure of $\left\langle \bigcup_{\lambda\in\Lambda} \mathcal{K}(X_\lambda)\right\rangle$. We claim that $K = M$. If not, then by [9, §6.4, Theorem (ii)], there exists a r-maximal r-ideal J of M such that

$$\bigcap_{I\in X_\lambda} I \subseteq K \subseteq J$$

for all $\lambda \in \Lambda$. Therefore, $J \in \mathcal{H}(C_\lambda) = C_\lambda$ for all $\lambda \in \Lambda$, a contradiction. Since $1 \in K$, we have $1 \in \bigcup_{i=1}^{n} \mathcal{K}(X_{\lambda_i})$ for a finite subset $\{\lambda_1, \dots, \lambda_n\}$ of Λ. Hence, $\bigcap_{i=1}^{n} C_{\lambda_i} = \emptyset$, and by the finite intersection property, we have the quasi-compactness of $\mathcal{S}_r(M)$.

To show the T_0 separation property, let $I, I' \in \mathcal{S}_r(M)$ such that $\mathcal{HK}(\{I\}) = \mathcal{HK}(\{I'\})$. It suffices to show $I = I'$. Since $I' \in \mathcal{HK}(\{I\})$, we have $I \subseteq I'$. Similarly, we obtain $I' \subseteq I$. Hence $I = I'$. □

The following result characterizes T_1 terminal spaces, and generalizes Theorem 3.2 in [12], Theorem 3.7 in [11], and Theorem 3 in [19].

Theorem 2.5. *Let M be a monoid. A terminal space $\mathcal{S}_r(M)$ is a T_1-space if and only if every r-strongly irreducible r-ideal of M does not contain other r-strongly irreducible r-ideals of M.*

Proof. If $\mathcal{S}_r(M)$ is a T_1-space, then for every $I \in \mathcal{S}_r(M)$ we have $\bar{I} = \{I\}$. By Lemma 2.3(2), $\bar{I} = \mathcal{HK}(\{I\}) = \mathcal{H}(I)$, and so, $\{I\} = \mathcal{H}(I)$, implying that the only r-strongly irreducible r-ideal of M containing I is I itself. For the converse, let I be the unique r-strongly irreducible r-ideal of M that contains I. Then by Lemma 2.3(2),

$$\overline{\{I\}} = \mathcal{HK}(\{I\}) = \mathcal{H}(I) = \{I\}.$$

Thus $\{I\}$ is a closed set, proving that $\mathcal{S}_r(M)$ is a T_1-space. $\square$

Our next goal is to study generic points of irreducible closed sets of terminal spaces. Recall that a subset Y of a topological space X is called *irreducible* if for any closed subsets Y_1 and Y_2 in X, $Y \subseteq Y_1 \cup Y_2$ implies that $Y \subseteq Y_1$ or $Y \subseteq Y_2$. A maximal irreducible subset Y of X is called an *irreducible component.* An element y of a closed subset Y of X is called a *generic point of* Y if $Y = \overline{\{y\}}$.

The following result characterizes irreducible closed subsets of a terminal space. Moreover, this result generalizes Theorem 3.3 in [12], Proposition 3 in [19], Theorem 2.6(1) in [20], and Corollary 3.1 in [12].

Theorem 2.6. *Every terminal space $\mathcal{S}_r(M)$ is sober.*

Proof. We prove more, namely, a nonempty closed subset X of a terminal space $\mathcal{S}_r(M)$ is irreducible if and only if $\mathcal{K}(X)$ is a r-strongly irreducible r-ideal of M. It is clear that $\mathcal{K}(X)$ is a proper ideal of M. Let $I \cap J \subseteq \mathcal{K}(X)$ for some $I, J \in \mathcal{I}_r(M)$. Then for any $L \in X$, we have $I \subseteq L$ or $J \subseteq L$ since $L \in \mathcal{S}_r(M)$. Hence $X \subseteq \mathcal{H}(I) \cup \mathcal{H}(J)$. Since X is irreducible, $X \subseteq \mathcal{H}(I)$ or $X \subseteq \mathcal{H}(J)$, which implies that $I \subseteq \mathcal{K}(X)$ or $J \subseteq \mathcal{K}(X)$. Therefore, $\mathcal{K}(X)$ is r-strongly irreducible.

For the converse, let $\mathcal{K}(X)$ be a r-strongly irreducible r-ideal of M. Since $\mathcal{K}(X) \neq M$, $\mathcal{K}(X)$ is nonempty. Let $X = X_1 \cup X_2$ for some nonempty closed subsets of the terminal space $\mathcal{S}_r(M)$. Then $\mathcal{K}(X) \supseteq \mathcal{K}(X_1) \cap \mathcal{K}(X_2)$. Since $\mathcal{K}(X)$ is r-strongly irreducible, $\mathcal{K}(X) \in \mathcal{H}(\mathcal{K}(X_1) \cap \mathcal{K}(X_2))$. By Lemma 2.3(3), this implies $\mathcal{K}(X) \in \mathcal{HK}(X_1) \cup \mathcal{HK}(X_2)$. If $\mathcal{K}(X) \in \mathcal{HK}(X_1)$, then

$$X \subseteq \overline{X} = \mathcal{HK}(X) \subseteq \mathcal{HK}(X_1) = \overline{X_1} = X_1,$$

where the first and the second equalities follow from Lemma 2.3(2). Similarly, if $\mathcal{K}(X) \in \mathcal{HK}(X_2)$, then $X \subseteq X_2$. This proves that X is irreducible.

Let $\mathcal{H}(I)$ be a nonempty irreducible subset of $\mathcal{S}_r(M)$. Then by the above, I is r-strongly irreducible. Hence $\overline{\{I\}} = \mathcal{HK}(I) = \mathcal{H}(I)$, where the first equality follows from Lemma 2.3(2). Thus I is a generic point of $\mathcal{H}(I)$. The uniqueness of this point follows from the fact that $\mathcal{S}_r(M)$ is a T_0-space (see Theorem 2.4). $\square$

The following one-to-one correspondence generalizes Theorem 3.4 in [1].

Theorem 2.7. *Let M be a monoid. Then there is a bijection between the set of irreducible components of the terminal space $\mathcal{S}_r(M)$ and the set of minimal r-strongly irreducible r-ideals of M.*

Proof. If X is an irreducible component of the terminal space $\mathcal{S}_r(M)$, then by Theorem 2.6, $X = \mathcal{H}(I)$ for some $I \in \mathcal{S}_r(M)$. If $J \in \mathcal{S}_r(M)$ such that $I \supseteq J$, then $\mathcal{H}(I) \subseteq \mathcal{H}(J)$ so that $I = J$. Conversely, let I be a minimal r-strongly irreducible r-ideal of M and let $\mathcal{H}(I) \subseteq \mathcal{H}(J)$ for some $J \in \mathcal{S}_r(M)$. Then

$$\overline{\{I\}} = \mathcal{H}(I) \subseteq \mathcal{H}(J) = \overline{\{J\}},$$

implying that $I = J$. Hence, $\mathcal{H}(I)$ is an irreducible component of $\mathcal{S}_r(M)$. $\square$

It is well-known that the prime spectrum of a Noetherian (commutative) ring endowed with Zariski topology is a Noetherian space. The following proposition generalizes this to r-strongly irreducible r-ideals of monoids, and it also generalizes Proposition 4.2(i) in [3]. The proof is easy, and so will be omitted.

Proposition 2.8. *If M is a Noetherian monoid, then $\mathcal{S}_r(M)$ is a Noetherian terminal space.*

A monoid M is called *r-arithmetic* if $\mathcal{I}_r(M)$ is a distributive lattice. The following theorem characterizes r-arithmetic monoids in terms of r-strongly irreducible r-ideals. This result is a generalization of Theorem 10 in [14]. The half of the implications uses the Zariski topology on $\mathcal{S}_r(M)$.

Theorem 2.9. *A monoid M is r-arithmetic if and only if each r-ideal is the intersection of all r-strongly irreducible r-ideals containing it.*

Proof. Let $I \in \mathcal{I}_r(M)$ and let $I = \bigcap_{I \subseteq J}\{J \mid J \in \mathcal{S}_r(M)\}$. To show $\mathcal{I}_r(M)$ is distributive, it suffices to show that the lattice $\mathcal{I}_r(M)$ is isomorphic to the lattice of some closed sets of the terminal space $\mathcal{S}_r(M)$, because following [14, Theorem 10], we can show that $\mathcal{I}_r(M)$ is distributive if and only if each ideal is the intersection of all strongly irreducible ideals containing it. Note that the map $I \mapsto \{J \in \mathcal{S}_r(M) \mid J \supseteq I\} = \mathcal{H}(I)$ is a bijection and since $\mathcal{H}(I)$ is a closed set, this map is also an lattice isomorphism.

For the converse, we first observe that by [4], in a distributive lattice, r-irreducible ideals and r-strongly irreducible r-ideals coincide. The rest of the proof now follows from Theorem 6 and Theorem 7 in [14]. $\square$

Finally, we wish to see relations between a terminal space and its subspaces of r-maximal r-ideals $\mathrm{Max}_r(M)$ and r-prime r-ideals $\mathrm{Spec}_r(M)$. To do so, we first talk about radicals induced by r-maximal, r-prime, and r-strongly irreducible r-ideals of a monoid M. The *m_r-radical* $\sqrt[m]{M}$ (respectively, *p_r-radical* $\sqrt[p]{M}$ and *s_r-radical* $\sqrt[s]{M}$) of M is the intersection of all r-maximal r-ideals (respectively, r-prime r-ideals and r-strongly irreducible r-ideals) of M.

Proposition 2.10. *Let M be a monoid.*

(1) *The subspace $\mathrm{Max}_r(M)$ is dense in the terminal space $\mathcal{S}_r(M)$ if and only if $\sqrt[p]{M} = \sqrt[s]{M}$.*
(2) *The subspace $\mathrm{Spec}_r(M)$ is dense in the terminal space $\mathcal{S}_r(M)$ if and only if $\sqrt[m]{M} = \sqrt[s]{M}$.*

Proof. (1) Although the claim essentially follows from the fact that if $X \subseteq S_r(M)$, then

$$\overline{X} = \left\{ J \in S_r(M) \mid J \supseteq \bigcap_{I \in X} I \right\},$$

however, we provide some details. Let $\overline{\mathrm{Spec}_r(M)} = \mathcal{S}_r(M)$. Then $\{J \in \mathcal{S}_r(M) \mid \bigcap_{P \in \mathrm{Spec}_r(M)} P \subseteq J\} = \mathcal{S}_r(M)$. This implies that $\sqrt[p]{M} \subseteq \sqrt[s]{M}$. Furthermore, $\mathrm{Max}_r(M) \subseteq \mathcal{S}_r(M)$ implies $\sqrt[s]{M} \subseteq \sqrt[p]{M}$. Hence, we have the desired equality. To obtain the converse, let $\mathcal{S}_r(M) \setminus \overline{\mathrm{Spec}_r(M)} \neq \emptyset$. This implies $J \notin \overline{\mathrm{Spec}_r(M)}$, but $J \in \mathcal{S}_r(M)$. Therefore, there exists a neighbourhood N_J of J such that $N_J \cap \mathrm{Spec}(M) = \emptyset$, and $\sqrt[s]{M} \subsetneq \sqrt[p]{M}$. In other words, we have $\sqrt[s]{M} \neq \sqrt[p]{M}$.

(2) Follows from (1). □

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References

[1] H. Ansari-Toroghy and D. Hassanzadeh-Lelekaami, *On the prime spectrum of top modules*, Algebra Discrete Math. **11** (2011), no. 1, 1–16.

[2] R. E. Atani and S. E. Atani, *Ideal theory in commutative semirings*, Bul. Acad. Ştiinţe Repub. Mold. Mat. **2008** (2008), no. 2, 14–23.

[3] A. Azizi, *Strongly irreducible ideals*, J. Aust. Math. Soc. **84** (2008), no. 2, 145–154. https://doi.org/10.1017/S1446788708000062

[4] G. Birkhoff and O. Frink Jr., *Representations of lattices by sets*, Trans. Amer. Math. Soc. **64** (1948), 299–316. https://doi.org/10.2307/1990504

[5] R. L. Blair, *Ideal lattices and the structure of rings*, Trans. Amer. Math. Soc. **75** (1953), 136–153. https://doi.org/10.2307/1990782

[6] N. Bourbaki, *Elements of mathematics*: *Commutative algebra*, Addison-Wesley, Reading, MA, 1972.

[7] L. Fuchs, *Über die Ideale arithmetischer Ringe*, Comment. Math. Helv. **23** (1949), 334–341. https://doi.org/10.1007/BF02565607

[8] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. **4** (1960), 228 pp.

[9] F. Halter-Koch, *Ideal systems*: *An introduction to multiplicative ideal theory*, Monographs and Textbooks in Pure and Applied Mathematics, 211, Marcel Dekker, Inc., New York, 1998.

[10] F. Halter-Koch, *Multiplicative ideal theory in the context of commutative monoids*, Commutative algebra–Noetherian and non-Noetherian perspectives, 203–231, Springer, 2011.

[11] S. C. Han, W.-J. Han, and W.-S. Pae, *Properties of the subtractive prime spectrum of a semimodule*, Hacet. J. Math. Stat. **52** (2023), no. 3, 546–559.

[12] S.-C. Han, W.-S. Pae, and J.-N. Ho, *Topological properties of the prime spectrum of a semimodule*, J. Algebra **566** (2021), 205–221. https://doi.org/10.1016/j.jalgebra.2020.08.033

[13] W. J. Heinzer, L. J. Ratliff, and D. E. Rush, *Strongly irreducible ideals of a commutative ring*, J. Pure Appl. Algebra **166** (2002), no. 3, 267–275. https://doi.org/10.1016/S0022-4049(01)00043-3

[14] K. Iséki, *Ideal theory of semiring*, Proc. Japan Acad. **32** (1956), 554–559. http://projecteuclid.org/euclid.pja/1195525272

[15] A. Khaksari, M. Ershad, and H. Sharif, *Strongly irreducible submodules of modules*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 4, 1189–1196. https://doi.org/10.1007/s10114-005-0681-7

[16] J. D. McKnight Jr., *On the characterisation of rings of functions*, Purdu doctoral thesis, 1953.

[17] R. Naghipour and M. Sedghi, *Some characterizations of strongly irreducible submodules in arithmetical and Noetherian modules*, arXiv:2101.01695v1/

[18] N. Schwartz, *Strongly irreducible ideals and truncated valuations*, Comm. Algebra **44** (2016), no. 3, 1055–1087. https://doi.org/10.1080/00927872.2014.999926

[19] G. Yeşilot, *On the prime spectrum of a module over noncommutative rings*, Int. J. Algebra **5** (2011), no. 9-12, 523–528.

[20] G. Zhang and W. Tong, *Spectral spaces of top right R-modules*, J. Nanjing Univ. Math. Biq. **17** (2000), no. 1, 15–20.

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