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A COTORSION PAIR INDUCED BY THE CLASS OF GORENSTEIN (m, n)-FLAT MODULES

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Abstract. In this paper, we introduce the notion of Gorenstein (m, n)-flat modules as an extension of (m, n)-flat left R-modules over a ring R, where m and n are two fixed positive integers. We demonstrate that the class of all Gorenstein (m, n)-flat modules forms a Kaplansky class and establish that $(\mathcal{GF}_{m,n}(R)$,$\mathcal{GC}_{m,n}(R))$ constitutes a hereditary perfect cotorsion pair (where $\mathcal{GF}_{m,n}(R)$ denotes the class of Gorenstein (m, n)-flat modules and $\mathcal{GC}_{m,n}(R)$ refers to the class of Gorenstein (m, n)-cotorsion modules) over slightly (m, n)-coherent rings.

1. Introduction

The notion of G-dimension for finitely generated modules over two-sided Noetherian rings was initially introduced by Auslander and Bridger [1]. Enochs et al. [4] further expanded the scope of this theory by introducing and investigating Gorenstein projective, Gorenstein injective and Gorenstein flat modules over arbitrary rings. This development is known as Gorenstein homological algebra, which has been extensively studied by several authors [2,4,6,9,10,12,13]. Enochs and López-Ramos [7] established that the class of Gorenstein flat modules constitutes a Kaplansky class over a ring R. Moreover, Holm investigated the projective resolving property of the class of Gorenstein flat modules over coherent rings [9]. Bennis [2] discussed several homological properties of Gorenstein flat modules and their dimensions over GF-closed rings. Additionally, Yang and Liu [13] demonstrated that $(\mathcal{G}, \mathcal{G}^{\perp})$ forms a hereditary perfect cotorsion pair over the GF-closed ring, where $\mathcal{G}$ denotes the class of Gorenstein flat modules.

Let m and n be fixed positive integers. Following the terminology of [14], a left R-module M is said to be (m, n)-injective if $\mathrm{Ext}^1_R(P, M) = 0$ for any (m, n)-presented left R-module P. It is worth noting that (m, n)-injective modules

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encompass several well-known classes, such as f-injective, P-injective and FP-injective modules. In [14], Zhang et al. introduced and studied (m,n)-flat modules, which they used to provide equivalent characterizations of (m,n)-coherent rings using these modules. Building on this work, Zhao and Li [15] extended the concept of (m,n)-coherent rings by introducing and analyzing slightly (m,n)-coherent rings. They showed that R is a left slightly (m,n)-coherent ring if and only if the cotorsion pair $(\mathcal{P}_{m,n},\mathcal{I}_{m,n})$ is hereditary, which is in turn equivalent to R being left (m,n)-coherent and the cotorsion pair $(\mathcal{F}_{m,n},\mathcal{C}_{m,n})$ being hereditary. Based on these results, we will define and study Gorenstein (m,n)-flat modules.

This paper is organized as follows. In Section 2, we give some definitions and preliminary results for using throughout this paper. In Section 3, we investigate some homological properties relate to Gorenstein (m,n)-flat modules and prove the class $\mathcal{GF}_{m,n}(R)$ is closed under direct limits over slightly (m,n)-coherent rings. In Section 4, we establish that if R is a slightly (m,n)-coherent ring, then the class of all Gorenstein (m,n)-flat left modules forms a Kaplansky class. Furthermore, we prove $(\mathcal{GF}_{m,n}(R),\mathcal{GC}_{m,n}(R))$ is a hereditary perfect cotorsion pair over slightly (m,n)-coherent rings.

2. Preliminaries

Throughout this paper, R is an associative ring with identity, all modules are assumed to be unitary, m and n are two fixed positive integers. By $\mathcal{M}(R)$ we denote the class of all R-modules, and by $\mathcal{P}(R)$ and $\mathcal{F}(R)$ the classes of all projective and flat R-modules, respectively. The character module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ of the module M is denoted by M^+. Next, we introduce some basic notions and notation used throughout this paper. For more details, one may consult [3, 5, 8, 11, 14, 15].

Orthogonal and resolving classes. Let $\mathcal{X}$ be a class of R-modules. We write $\mathcal{X}^{\perp}$ to denote the *right orthogonal* class of $\mathcal{X}$ if $\mathcal{X}$ satisfying $\mathcal{X}^{\perp}=\{M\in\mathcal{M}(R)\mid \mathrm{Ext}^n_R(X,M)=0 \text{ for any } X\in\mathcal{X} \text{ and all } n>0\}$. Dually, the *left orthogonal* class is defined. Given a class $\mathcal{X}$ R-modules. The class X is called *projectively resolving* if $\mathcal{P}(R)\subseteq\mathcal{X}$, and for every short exact sequence $0\to X_1\to X_2\to X_3\to 0$ with $X_2\in\mathcal{X}$ the conditions $X_1\in\mathcal{X}$ and $X_3\in\mathcal{X}$ are equivalent. Dually, the *injectively resolving* class is defined.

Resolution. Let $\mathcal{X}$ be a class of R-modules. For any R-module M, we define the left $\mathcal{X}$-resolution of M. A *left $\mathcal{X}$-resolution* of M is an exact sequence $\mathbb{X}=\cdots\to X_1\to X_0\to M\to 0$ with $X_i\in\mathcal{X}$ for all $i\geqslant 0$. Dually, the *right $\mathcal{X}$-resolution* of M is defined. A sequence $\mathbb{X}$ of left R-modules is said to be $\mathcal{X}\otimes_R -$*exact* if $X\otimes_R\mathbb{X}$ is exact for any $X\in\mathcal{X}$.

Envelope and cover. Let $\mathcal{X}$ be a class of R-modules and M an R-module. A morphism $\varphi: M\to X$ with $X\in\mathcal{X}$ is called an *$\mathcal{X}$-preenvelope* of M if for any morphism $\phi: M\to X'$ with $X'\in X$, there is a morphism $\psi: X\to X'$

such that $\psi\varphi = \phi$. A $\mathcal{X}$-preenvelope $\varphi : M \to X$ is called a *$\mathcal{X}$-envelope* if every endomorphism $\psi : X \to X$ such that $\psi\varphi = \varphi$ is an isomorphism. Dually, the *$\mathcal{X}$-precover* and *$\mathcal{X}$-cover* of M are defined.

Cotorsion pair. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of R-modules. The pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion pair* if $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *perfect*, provided that $\mathcal{A}$ is a covering class and $\mathcal{B}$ is an enveloping class. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *hereditary* if $\mathrm{Ext}_R^{\geqslant 1}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This equivalents to knowing that $\mathcal{A}$ is resolving (that is, $\mathcal{P}(R) \subseteq \mathcal{A}$, and $\mathcal{A}$ is closed under extensions and under taking kernels of epimorphisms), which is also equivalent to knowing that $\mathcal{B}$ is coresolving (that is, $\mathcal{I}(R) \subseteq \mathcal{B}$, and $\mathcal{B}$ is closed under extensions and under taking cokernels of monomorphisms). For more details on cotorsion pairs the reader can consult [5,8].

Definition 2.1 ([14])**.** Recall that a right R-module M is *(m,n)-presented* if there exists a right R-module exact sequence $0 \to K \to R^m \to M \to 0$, where K is n-generated. A ring R is called left *(m,n)-coherent* in case each n-generated submodule of the left R-module R^m is finitely presented.

Definition 2.2 ([3,11,14])**.** Recall that M is called *(m,n)-injective* if $\mathrm{Ext}_R^1(P, M) = 0$ for any (m,n)-presented right R-module P; M is said to be *(m,n)-flat* if $\mathrm{Tor}_1^R(M, Q) = 0$ for any (m,n)-presented left R-module Q. An R-module M is defined to be *(m,n)-projective* (resp. *(m,n)-cotorsion*) if $\mathrm{Ext}_R^1(M, N) = 0$ (resp. $\mathrm{Ext}_R^1(N, M) = 0$) for any (m,n)-injective (resp. (m,n)-flat) right R-module N.

Definition 2.3 ([15])**.** A ring R is called left *slightly (m,n)-coherent*, provided that each n-generated submodule of ${}_RR^m$ is (m,n)-projective.

Remark 2.4. It is clear that every flat R-module is (m,n)-flat, and then every projective R-module is (m,n)-flat.

In what following, we denote by $\mathcal{P}_{m,n}(R)$, $\mathcal{I}_{m,n}(R)$, $\mathcal{F}_{m,n}(R)$, $\mathcal{C}_{m,n}(R)$ and $\mathcal{H}_{m,n}(R) = \mathcal{I}_{m,n}(R) \cap \mathcal{P}_{m,n}(R)$ the classes of (m,n)-projective, (m,n)-injective, (m,n)-flat, (m,n)-cotorsion and (m,n)-injective (m,n)-projective R-modules, respectively.

3. Gorenstein (m,n)-flat modules

In this section, we introduce the concept of Gorenstein (m,n)-flat modules, which are a generalization of (m,n)-flat modules. We then investigate their homological properties and provide equivalent characterizations over the slightly (m,n)-coherent rings.

Definition 3.1. A *complete $\mathcal{F}_{m,n}(R)$-resolution* is an exact sequence

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of R-modules satisfying the following:

(1) $\mathbb{F}$ is $\mathcal{H}_{m,n}(R) \otimes_R -$exact.

(2) F_i and F^i are (m,n)-flat.

A left R-module M is called *Gorenstein (m,n)-flat* if there exits a complete $\mathcal{F}_{m,n}(R)$-resolution $\mathbb{F}$ such that $M \cong \mathrm{Im}(F_0 \to F^0)$. We denote by $\mathcal{GF}_{m,n}(R)$ the class of Gorenstein (m,n) flat left R-modules.

Remark 3.2. (1) Every (m,n)-flat module is Gorenstein (m,n)-flat.

(2) The class of Gorenstein (m,n)-flat left R-modules is closed under direct sums.

Lemma 3.3. *Let R be a right slightly (m,n)-coherent ring. Then*

(1) *The class of Gorenstein (m,n)-flat is closed under direct products.*

(2) *If M is an (m,n)-injective left R-module, then M^+ is Gorenstein (m,n)-flat.*

(3) *If M is an (m,n)-flat left R-module, then M^{++} is Gorenstein (m,n)-flat.*

Proof. (1) Since R is a right slightly (m,n)-coherent ring, $\mathcal{F}_{m,n}(R)$ is closed under direct products by [14, Theorem 5.7], and then $\mathcal{GF}_{m,n}(R)$ is closed under the direct products by Remark 3.2(1).

(2) and (3) They follow from [14, Theorem 5.7] and Remark 3.2(1). □

Lemma 3.4. *If R is a slightly (m,n)-coherent ring, then $(\mathcal{F}_{m,n}(R), \mathcal{C}_{m,n}(R))$ and $(\mathcal{P}_{m,n}(R), \mathcal{I}_{m,n}(R))$ are complete hereditary cotorsion theories.*

Proof. $(\mathcal{F}_{m,n}(R), \mathcal{C}_{m,n}(R))$ is a perfect cotorion pair by [11, Theorem 2.3(2)], and any perfect cotorsion pair is complete by [8, p. 106]. Since R is a left slightly (m,n)-coherent ring, $(\mathcal{F}_{m,n}(R), \mathcal{C}_{m,n}(R))$ is complete hereditary by [15, Theorem 2.4]. Similarly, R is a left slightly (m,n)-coherent ring, so $(\mathcal{P}_{m,n}(R), \mathcal{I}_{m,n}(R))$ is complete hereditary by [15, Theorem 2.2] and [11, Theorem 2.3(1)]. □

Proposition 3.5. *Let R be a right slightly (m,n)-coherent ring. Then the following statements are equivalent*:

(1) *M is a Gorenstein (m,n)-flat left R-module.*

(2) $\mathrm{Tor}_i^R(E,M) = 0$ *for any $i \geqslant 1$, $E \in \mathcal{H}_{m,n}(R)$, and there is an $\mathcal{H}_{m,n}(R) \otimes_R$ $-$exact exact sequence $0 \to M \to F^0 \to F^1 \to \cdots$ with each F^i is (m,n)-flat.*

(3) *There exists a short exact sequence of left R-modules $0 \to M \to F \to G \to 0$, where $F \in \mathcal{F}_{m,n}(R)$ and $G \in \mathcal{GF}_{m,n}(R)$.*

Proof. (1) $\Rightarrow$ (2) This follows the definition of Gorenstein (m,n)-flat modules.

(2) $\Rightarrow$ (1) Since R is a right slightly (m,n)-coherent ring, by Lemma 3.4, there is a left (m,n)-flat resolution $\cdots \to F_1 \to F_0 \to M \to 0$ of M, where every $F_i \in \mathcal{F}_{m,n}(R)$. By hypothesis, $\mathrm{Tor}_i^R(E,M) = 0$ for any $i \geqslant 1$ and $E \in \mathcal{H}_{m,n}(R)$, so we have an $E \otimes_R -$exact exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ such that $M \cong \mathrm{Im}(F_0 \to F^0)$ with every F_i and F^i belonging to $\mathcal{F}_{m,n}(R)$. Therefore, M is Gorenstein (m,n)-flat.

(1) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (2) Assume that there is a short sequence of left R-modules $0 \to M \to F \to G \to 0$, where $F \in \mathcal{F}_{m,n}(R)$ and $G \in \mathcal{GF}_{m,n}(R)$. By the equivalence (1) $\Leftrightarrow$ (2) and Remark 3.2, we have $\mathrm{Tor}_i^R(E, G) = 0$ and $\mathrm{Tor}_i^R(E, F) = 0$ for any $i \geqslant 1$ and $E \in \mathcal{H}_{m,n}(R)$. Then there exists an exact sequence

$$\mathrm{Tor}_{i+1}^R(E, G) \to \mathrm{Tor}_i^R(E, M) \to \mathrm{Tor}_i^R(E, F).$$

Hence, $\mathrm{Tor}_i^R(E, M) = 0$ for any $i \geqslant 1$ and $E \in \mathcal{H}_{m,n}(R)$. Meanwhile, because $G \in \mathcal{GF}_{m,n}(R)$, there is an exact sequence

$$\mathbb{G} : 0 \to G \to F^0 \to F^1 \to \cdots$$

with every $F^i \in \mathcal{F}_{m,n}(R)$ and $E \otimes_R \mathbb{G}$ exact for any $E \in \mathcal{H}_{m,n}(R)$. Therefore, we have an $E \otimes_R -$ exact exact sequence $0 \to M \to F \to F^0 \to F^1 \to \cdots$, where F and F^i belong to $\mathcal{F}_{m,n}(R)$. □

By Proposition 3.5, we immediately have the following result:

Corollary 3.6. *Let R be a right slightly (m, n)-coherent ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence of left R-modules. Then the following holds*:

(1) *If $M_1, M_3 \in \mathcal{GF}_{m,n}(R)$, then $M_2 \in \mathcal{GF}_{m,n}(R)$.*

(2) *If $M_2, M_3 \in \mathcal{GF}_{m,n}(R)$, then $M_1 \in \mathcal{GF}_{m,n}(R)$.*

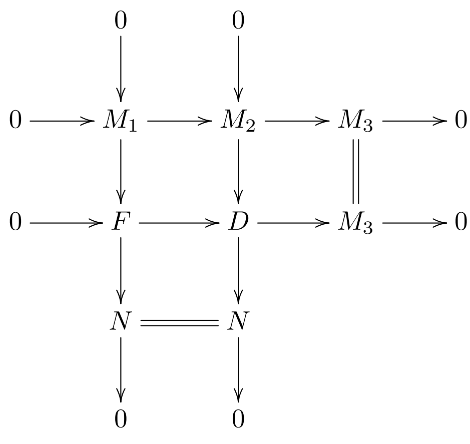
(3) *If $M_1, M_2 \in \mathcal{GF}_{m,n}(R)$, then $M_3 \in \mathcal{GF}_{m,n}(R)$ if and only if $\mathrm{Tor}_i^R(E, M_3) = 0$ for any $E \in \mathcal{H}_{m,n}(R)$.*

Corollary 3.7. *Let R be a right slightly (m, n)-coherent ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence of left R-modules. If M_1 is Gorenstein (m, n)-flat and M_3 is (m, n)-flat, then M_2 is Gorenstein (m, n)-flat.*

Proof. Since $M_1 \in \mathcal{GF}_{m,n}(R)$, there is an exact sequence

$$0 \to M_1 \to F \to N \to 0$$

with $F \in \mathcal{F}_{m,n}(R)$ and $N \in \mathcal{GF}_{m,n}(R)$ by Proposition 3.5. Consider the following pushout diagram:



It is clear that the class of $\mathcal{F}_{m,n}(R)$ is closed under extension, and so $D \in \mathcal{F}_{m,n}(R)$ in the exact sequence $0 \to F \to D \to M_3 \to 0$. Then in the exact sequence $0 \to M_2 \to D \to N \to 0$, since $D \in \mathcal{F}_{m,n}(R)$ and $N \in \mathcal{GF}_{m,n}(R)$, $M_2 \in \mathcal{GF}_{m,n}(R)$ by Proposition 3.5. □

We introduce Lemma 3.8 in order to make the proof of Theorem 3.9 clear.

Lemma 3.8 ([7, Proposition 2.3])**.** *Let $\mathfrak{M}$ be a class of R-modules. If $\mathfrak{M}$ is closed under well ordered direct limits, then it is closed under arbitrary direct limits.*

Proof. Let $((M_i),(\varrho_{ij}))_{i\in I}$ be a directed system. We will make transfinite induction on $\mathrm{Card}(I)$. If $\mathrm{Card}(I) = k < \aleph_0$, there is nothing to prove. If $\mathrm{Card}(I) = \aleph_0$, then there is a cofinal set $J \subseteq I$ with $J = \{j_0, j_1, \ldots\}$, where $j_0 < j_1 < \ldots$. In this way $\varinjlim_{i\in I} M_i = \varinjlim_{i\in J} M_j \in \mathfrak{M}$ by hypothesis.

We now assume that $\mathrm{Card}(I) > \aleph_0$. In this case there exists

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_\theta \subseteq J_{\theta+1} \subseteq \cdots \subseteq J_\alpha \subseteq \cdots$$

$\alpha < \lambda$ for λ an ordinal such that $\bigcup_{\alpha<\lambda} J_\alpha = I$, where each J_α is a right directed set and $\mathrm{Card}(J_\alpha) < \mathrm{Card}(I)$. Then

$$\varinjlim_{i\in I} M_i = \varinjlim_{\alpha<\lambda} (\varinjlim_{i\in J_\alpha} M_i).$$

Since by induction hypothesis $\varinjlim_{i\in J_\alpha} M_i \in \mathfrak{M}$, by the hypothesis in the statement we get that $\varinjlim_{i\in I} M_i \in \mathfrak{M}$. □

Theorem 3.9. *If R is a right slightly (m,n)-coherent ring, then the class $\mathcal{GF}_{m,n}(R)$ is closed under direct limits.*

Proof. Base on Lemma 3.8, we need only to prove that the class of Gorenstein (m,n)-flat modules is closed under well ordered direct limits. Therefore, we assume that $(M_\alpha)_{\alpha\in\lambda}$ is a well ordered direct system of Gorenstein (m,n)-flat modules. If $\lambda = k < \theta$, then $\varinjlim M_\alpha$ is a Gorenstein (m,n)-flat module.

Let $\lambda = \theta$. Then we start showing that $\varinjlim M_k (k < \theta)$ is Gorenstein (m,n)-flat. Since M_0 is Gorenstein (m,n)-flat, there is an exact sequence

$$F(0) = 0 \to M_0 \to F_0^0 \to F_0^1 \to F_0^2 \to \cdots$$

with each $F_0^i \in \mathcal{F}_{m,n}(R)$ and such that $E \otimes_R -$exact for any $E \in \mathcal{H}_{m,n}(R)$ by Proposition 3.5. Set $K_0^i = \mathrm{Ker}(F_0^i \to F_0^{i+1})$, where $i \geqslant 0$ and $K_0^0 = M_0$. Thus each K_0^i is Gorenstein (m,n)-flat by Proposition 3.5. Consider the following pushout diagram of morphisms $M_0 \to F_0^0$ and $M_0 \to M_1$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & F_0^0 & \longrightarrow & K_0^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & U & \longrightarrow & K_0^1 & \longrightarrow & 0 \end{array}$$

Since $M_1, K_0^1 \in \mathcal{GF}_{m,n}(R)$ and R is a right slightly (m,n)-coherent ring, $U \in \mathcal{GF}_{m,n}(R)$ by Corollary 3.6(1). Then there exists an exact sequence $0 \to U \to F_1^0 \to N \to 0$ with $F_1^0 \in \mathcal{F}_{m,n}(R)$, $N \in \mathcal{GF}_{m,n}(R)$ by Proposition 3.5(3). Again consider the following pushout diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & U & \longrightarrow & K_0^1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & F_1^0 & \longrightarrow & K_1^1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & N & = & N & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Then we obtain K_1^1 is Gorenstein (m,n)-flat since N and K_0^1 are so. Hence, there is the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & F_0^0 & \longrightarrow & K_0^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & F_1^0 & \longrightarrow & K_1^1 & \longrightarrow & 0 \end{array}$$

Using the same method above, we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0^1 & \longrightarrow & F_0^1 & \longrightarrow & K_0^2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_1^1 & \longrightarrow & F_1^1 & \longrightarrow & K_1^2 & \longrightarrow & 0 \end{array}$$

with $F_1^1 \in \mathcal{F}_{m,n}(R)$, $K_1^2 \in \mathcal{GF}_{m,n}(R)$. Thus there exists an exact sequence

$$F(1) = 0 \to M_1 \to F_1^0 \to F_1^1 \to F_1^2 \to \cdots$$

with each $F_1^i \in \mathcal{F}_{m,n}(R)$, each $K_1^i = \mathrm{Ker}(F_1^i \to F_1^{i+1})$ is Gorenstein (m,n)-flat for any $i \geqslant 0$ and $K_1^0 = M_1$. Continuing this process, we get a diagram:

$$\begin{array}{ccccccccccc}
F(0) = & & 0 \longrightarrow & M_0 & \longrightarrow & F_0^0 & \longrightarrow & F_0^1 & \longrightarrow & F_0^2 & \longrightarrow \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
F(1) = & & 0 \longrightarrow & M_1 & \longrightarrow & F_1^0 & \longrightarrow & F_1^1 & \longrightarrow & F_1^2 & \longrightarrow \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\vdots & & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

with exact rows and each $F_j^i \in \mathcal{F}_{m,n}(R)$, each $K_j^i = \mathrm{Ker}(F_j^i \to F_j^{i+1})$ is Gorenstein (m,n)-flat for all $i \geqslant 0$, all $j \geqslant 0$, and $K_j^0 = M_j$. Hence, each exact sequence $F(k)$, $k = 0,1,2,\dots$ remains exact after applying the functor $E \otimes_R -$ for any (m,n)-injective (m,n)-projective right R-modules E. Applying the exact functor $\varinjlim$ to the above commutative diagram, we have the following exact sequence

$$\varinjlim F(k) = 0 \to \varinjlim M_k \to \varinjlim F_k^0 \to \varinjlim F_k^1 \to \cdots$$

with each $\varinjlim F_k^i \in \mathcal{F}_{m,n}(R)$, $i = 0,1,2,\dots$. When $E \in \mathcal{H}_{m,n}(R)$, then $E \otimes_R \varinjlim F_k \cong \varinjlim (E \otimes_R F(k))$ is exact since $\varinjlim$ commutes with the homology functor. Since each $M_k \in \mathcal{GF}_{m,n}(R)$, $k = 0,1,2,\dots$, $\mathrm{Tor}_i^R(E, \varinjlim M_k) \cong \varinjlim \mathrm{Tor}_i^R(E, M_k) = 0$ for any $E \in \mathcal{H}_{m,n}(R)$ and $i > 0$. Therefore, $\varinjlim M_K$ is Gorenstein (m,n)-flat by Proposition 3.5.

Now, we reindex the modules $M_0, M_1, \dots, M_\theta, M_{\theta+1}, \dots$ such that $M_\theta = \varinjlim M_k$ and $M_{\theta+1}$ is the old version of M_θ. Consequently, we assume that the system $(M_\alpha)_{\alpha \in \lambda}$ is continuous, i.e., $M_\beta = \varinjlim M_\alpha (\alpha < \beta)$ if β is a limit ordinal with $\beta < \lambda$. Then using the transfinite induction, we obtain that $\varinjlim M_\alpha (\alpha < \lambda)$ is Gorenstein (m,n)-flat. □

The next result is motivated by [12, Corollary 4.10], which states that class $\mathcal{GF}_{m,n}(R)$ possess stability.

Proposition 3.10. *Let R be a slightly (m,n)-coherent ring and M a left R-module. Then there is a completely (m,n)-flat resolution of M if and only if there is a $\mathcal{GF}_{m,n}(R)$-resolution of M is $\mathcal{H}_{m,n}(R) \otimes_R -$exact.*

Proof. $(\Rightarrow)$ It follows from the definition of Gorenstein (m,n)-flat modules.

$(\Leftarrow)$ First we need to show that if there is a left $\mathcal{GF}_{m,n}(R)$-resolution of M is $\mathcal{H}_{m,n}(R) \otimes_R -$exact, then M has an $\mathcal{H}_{m,n}(R) \otimes_R -$exact left $\mathcal{F}_{m,n}(R)$-resolution. By hypothesis, there exists an $\mathcal{H}_{m,n}(R) \otimes_R -$exact exact sequence

$0 \to K \to G_0 \to M \to 0$, where $G_0 \in \mathcal{GF}_{m,n}(R)$ and there is a left $\mathcal{GF}_{m,n}(R)$-resolution of K is $\mathcal{H}_{m,n}(R) \otimes_R -$exact. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with $F_0 \in \mathcal{F}_{m,n}(R)$ and $G \in \mathcal{GF}_{m,n}(R)$. Since the third row and the middle column are $\mathcal{H}_{m,n}(R) \otimes_R -$exact, the middle row and first column are $\mathcal{H}_{m,n}(R) \otimes_R -$exact. Thus, for K, there is an $\mathcal{H}_{m,n}(R) \otimes_R -$exact exact sequence $0 \to L \to G_1 \to K \to 0$ with $G_1 \in \mathcal{GF}_{m,n}(R)$, and there exists an $\mathcal{H}_{m,n}(R) \otimes_R -$exact $\mathcal{GF}_{m,n}(R)$-resolution. Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L' & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G' & \longrightarrow & V & \longrightarrow & G_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G' & \longrightarrow & U & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $G', G_1 \in \mathcal{GF}_{m,n}(R)$, it follows that $V \in \mathcal{GF}_{m,n}(R)$ by Corollary 3.6(1). Then the middle row and column are $\mathcal{H}_{m,n}(R) \otimes_R -$exact. Hence, there is an $\mathcal{H}_{m,n}(R) \otimes_R -$exact $\mathcal{GF}_{m,n}(R)$-resolution of U and the sequence $0 \to U \to F_0 \to M \to 0$ is $\mathcal{H}_{m,n}(R) \otimes_R -$exact. By repeating the preceding process, we obtain an $\mathcal{H}_{m,n}(R) \otimes_R -$exact $\mathcal{F}_{m,n}$-resolution of M.

In addition, if the right Gorenstein $\mathcal{F}_{m,n}$-flat resolution of M is $\mathcal{H}_{m,n}(R)\otimes_R$ $-$exact, then M has an $\mathcal{H}_{m,n}(R)\otimes_R$ $-$exact right (m,n)-flat resolution due to symmetry, and the desired result follows. □

Setting $\mathcal{GF}^2_{m,n}(R) = \{\, M \mid$ there is an $\mathcal{H}_{m,n}(R)\otimes_R$ $-$exact exact sequence $\cdots \to G_1 \to G_0 \to G^1 \to G^1 \to \cdots$ such that $M \cong \mathrm{Im}(G_0 \to G^0)$, where each $G_i, G^i \in \mathcal{GF}_{m,n}(R)\}$.

From Proposition 3.10 and the above set, we have the following result:

Theorem 3.11. *If R is a slightly (m,n)-coherent ring, then*

$$\mathcal{GF}_{m,n}(R) = \mathcal{GF}^2_{m,n}(R).$$

4. A cotorsion pair induced by $\mathcal{GF}_{m,n}(R)$

In this section, the notion of Gorenstein (m,n)-injective modules is introduced and the class $\mathcal{GF}_{m,n}(R)$ is proven to form a Kaplansky by using these modules. In addition, it is proved that $(\mathcal{GF}_{m,n}(R), \mathcal{GC}_{m,n}(R))$ is a hereditary perfect cotorsion pair, where $\mathcal{GC}_{m,n}(R)$ is the class of Gorenstein (m,n)-cotorsion modules.

Definition 4.1. A right R-module is called *Gorenstein (m,n)-injective* if there is an exact sequence of (m,n)-injective right R-modules

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

such that $M \cong \mathrm{Im}(E_0 \to E^0)$ and $\mathrm{Hom}_R(E,-)$ leave the sequence exact whenever $E \in \mathcal{H}_{m,n}(R)$. We denote by $\mathcal{GI}_{m,n}(R)$ the class of Gorenstein (m,n)-injective right R-modules.

Remark 4.2. (1) If M is an (m,n)-injective right R-module, then $M\in\mathcal{GI}_{m,n}(R)$.

(2) If $M \in \mathcal{GI}_{m,n}(R)$, then there exists exact sequence $0 \to K \to E \to M \to 0$ with $E \in \mathcal{I}_{m,n}(R)$, $K \in \mathcal{GI}_{m,n}(R)$.

(3) Let R be a slightly (m,n)-coherent ring. If $M \in \mathcal{GI}_{m,n}(R)$ if and only if $\mathrm{Ext}^i_R(E,M) = 0$ for any $E \in \mathcal{H}_{m,n}(R)$, and there is a $\mathrm{Hom}_R(E,-)$-exact exact sequence $\cdots \to E_1 \to E_0 \to M \to 0$ with each $E_i \in \mathcal{H}_{m,n}(R)$ and $i \geqslant 1$.

Lemma 4.3. *Let R be a slightly (m,n)-coherent ring. Then the following holds.*

(1) *$\mathcal{GI}_{m,n}(R)$ is injectively resolving.*
(2) *$\mathcal{GI}_{m,n}(R)$ is closed under direct products.*
(3) *$\mathcal{GI}_{m,n}(R)$ is closed under direct summands.*

Proof. They follow from [10, Proposition 3.4] and [9, Proposition 1.4]. □

Proposition 4.4. *Let R be a slightly (m,n)-coherent ring. Then*

(1) *If $M \in \mathcal{GF}_{m,n}(R)$, then $M^+ \in \mathcal{GI}_{m,n}(R)$.*
(2) *If $M \in \mathcal{GI}_{m,n}(R)$, then $M^+ \in \mathcal{GF}_{m,n}(R)$.*

Proof. (1) Since $M \in \mathcal{GF}_{m,n}(R)$, there is an exact sequence of (m,n)-flat left R-modules

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and $E \otimes_R \mathbb{F}$ exact for any $E \in \mathcal{H}_{m,n}(R)$. By [14, Theorem 4.3], there exists an exact sequence of (m,n)-injective right R-modules

$$\mathbb{F}^+ = \cdots \to F^{1+} \to F^{0+} \to F_0^+ \to F_1^+ \to \cdots$$

such that $M \cong \mathrm{Im}(F^{0+} \to F_0^+)$. $\mathrm{Hom}_R(E, \mathbb{F}^+) \cong (E \otimes_R \mathbb{F})^+$ by adjoint isomorphism theorem, so $\mathrm{Hom}_R(E, \mathbb{F}^+)$ is exact. As a consequence, M^+ is Gorenstein (m,n)-injective.

(2) The proof is dual to that of (1). □

According to [7, Definition 2.1], there is the following definition of Kaplansky classes. Let $\mathcal{K}$ be a class of R-modules. Then $\mathcal{K}$ is said to be a *Kaplansky class* if there exists a cardinal $\mathcal{N}$ such that for every $M \in \mathcal{K}$ and for each $x \in M$, there exists a submodule F of M such that $x \in F \subseteq M$, $F, M/F \in \mathcal{K}$ and $\mathrm{Card}(F) \leqslant \mathcal{N}$.

Proposition 4.5. *Let R be a slightly (m,n)-coherent ring. Then the class of all Gorenstein (m,n)-flat left modules forms a Kaplansky class.*

Proof. Let $G \in \mathcal{GF}_{m,n}(R)$ and $F \subseteq G$ is a pure submodule of G. Then sequence $0 \to (G/F)^+ \to G^+ \to F^+ \to 0$ is split exact. Hence $G^+ \in \mathcal{GI}_{m,n}(R)$ by Proposition 4.4, and then F^+ and $(G/F)^+$ are Gorenstein (m,n)-injective by Lemma 4.3(3). Thus F and G/F are Gorenstein (m,n)-flat by Proposition 4.4. Therefore, the class of Gorenstein (m,n)-flat modules forms a Kaplansky class by [5, Lemma 5.3.12]. □

Next, the notion of Gorenstein (m,n)-cotorsion modules is introduced to obtain the main results of this paper.

Definition 4.6. A left R-modules N is called *Gorenstein (m,n)-cotorsion* if $\mathrm{Ext}^1_R(G, N) = 0$ for any Gorenstein (m,n)-flat module G. We denote by $\mathcal{GC}_{m,n}(R)$ the class of Gorenstein (m,n)-cotorsion left R-modules.

Using Definitions 2.2, 4.6 and Remark 4.2, $\mathcal{GC}_{m,n}(R) \subseteq \mathcal{C}_{m,n}(R)$. Since every (m,n)-cotorsion module is cotorsion according to [11, Remark 2.2(3)]. Every Gorenstein (m,n)-cotorsion module is cotorsion, that is, if R is a slightly (m,n)-coherent ring, then the class of Gorenstein (m,n)-cotorsion modules is injectively resolving.

Theorem 4.7. *Let R be a slightly (m,n)-coherent ring. Then $(\mathcal{GF}_{m,n}(R)$, $\mathcal{GC}_{m,n}(R))$ is a hereditary perfect cotorsion pair.*

Proof. By Remarks 2.4, 3.2, Proposition 4.5, Theorem 3.9 and [7, Theorem 2.9], $(\mathcal{GF}_{m,n}(R), \mathcal{GC}_{m,n}(R))$ is a perfect cotorsion pair. Since R is a slightly (m,n)-coherent ring, the class $\mathcal{GF}_{m,n}(R)$ is projectively resolving by Corollary

3.6 and Remark 2.4. Therefore, $(\mathcal{GF}_{m,n}(R), \mathcal{GC}_{m,n}(R))$ is a hereditary perfect cotorsion pair. □

By Theorem 4.7, we immediately have the following results.

Corollary 4.8. *Let R be a slightly (m,n)-coherent ring. Then*

(1) *All left modules have Gorenstein (m,n)-flat covers.*

(2) *Every module has a Gorenstein (m,n)-cotorsion envelope.*

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