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REGULAR t-BALANCED CAYLEY MAPS ON SPLIT METACYCLIC 2-GROUPS

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Abstract. A regular t-balanced Cayley map on a group Γ is an embedding of a Cayley graph on Γ into a surface with certain special symmetric properties. We completely classify regular t-balanced Cayley maps for a class of split metacyclic 2-groups.

1. Introduction

Suppose Γ is a finite group and Ω is a generating set of Γ such that $\omega^{-1} \in \Omega$ whenever $\omega \in \Omega$, and the identity $1 \notin \Omega$. The *Cayley graph* $\mathrm{Cay}(\Gamma, \Omega)$ is the graph having the vertex set Γ and the arc set $\Gamma \times \Omega$, where for $\eta \in \Gamma$, $\omega \in \Omega$, the arc from η to $\eta\omega$ is denoted as (η, ω).

A cyclic permutation ρ on Ω canonically induces a permutation on the arc set via $(\eta, \omega) \mapsto (\eta, \rho(\omega))$, and this equips each vertex η with a "cyclic order", which means a cyclic permutation on the set of arcs emanating from η. This determines an embedding of $\mathrm{Cay}(\Gamma, \Omega)$ into a closed oriented surface, which is characterized by the property that each connected component of the complement of the Cayley graph is a disk. Such an embedding is called a *Cayley map* and denoted by $\mathcal{CM}(\Gamma, \Omega, \rho)$. An *isomorphism* of Cayley maps $\mathcal{CM}(\Gamma, \Omega, \rho) \to \mathcal{CM}(\Gamma', \Omega', \rho')$ is by definition an isomorphism $\mathrm{Cay}(\Gamma, \Omega) \to \mathrm{Cay}(\Gamma', \Omega')$ which can be extended to an orientation-preserving homeomorphism between their embedding surfaces.

A Cayley map is called *regular* if its automorphism group acts regularly on the arc set, i.e., for any two arcs, there exists an automorphism sending one arc to the other. It was shown in [10] that $\mathcal{CM}(\Gamma, \Omega, \rho)$ is regular if and only if there exist a *skew-morphism* which is a bijective function $\varphi : \Gamma \to \Gamma$, and a *power function* $\pi : \Gamma \to \{1, \ldots, \#\Omega\}$ (where $\#\Omega$ is the cardinality of Ω), such that $\varphi|_{\Omega} = \rho$, $\varphi(1) = 1$ and $\varphi(\eta\mu) = \varphi(\eta)\varphi^{\pi(\eta)}(\mu)$ for all $\eta, \mu \in \Gamma$.

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Let $d = \#\Omega$, and t be an integer with $t^2 \equiv 1 \pmod{d}$. A regular Cayley map $\mathcal{CM}(\Gamma, \Omega, \rho)$ is called *t-balanced* if

$$\rho(\omega^{-1}) = (\rho^t(\omega))^{-1} \qquad \text{for all } \omega \in \Omega; \tag{1.1}$$

in particular, it is called *balanced* if $t \equiv 1 \pmod{d}$ and *anti-balanced* if $t \equiv -1 \pmod{d}$. It is the residue modulo d rather than t itself, that plays a key role. From now on we assume $0 < t < d$, and abbreviate "regular t-balanced Cayley map" to "RBCM$_t$".

Recall some facts on RBCM$_t$ from [1] Proposition 1.2.

Proposition 1.1. (a) *A Cayley map $\mathcal{CM}(\Gamma, \Omega, \rho)$ is an RBCM$_1$ if and only if ρ can be extended to an automorphism of Γ.*

(b) *Suppose $t > 1$. A Cayley map $\mathcal{CM}(\Gamma, \Omega, \rho)$ is an RBCM$_t$ if and only if ρ can be extended to a skew-morphism of Γ, $\pi(\omega) = t$ for all $\omega \in \Omega$ and $\pi(\eta) \in \{1, t\}$ for all $\eta \in \Gamma$.*

(c) *When the conditions in* (b) *are satisfied, $\Gamma_+ := \{\eta \in \Gamma : \pi(\eta) = 1\}$ is a subgroup of index 2, consisting of elements which are products of an even number of generators, $\varphi(\Gamma_+) = \Gamma_+$, and $\varphi_+ := \varphi|_{\Gamma_+}$ is an automorphism.*

By (1.1), there is an involution ι on $\{1, \ldots, d\}$ with $\omega_i^{-1} = \omega_{\iota(i)}$ and $\iota(i+1) \equiv \iota(i) + t \pmod{d}$ for all i. Let $\ell = \iota(d)$, then $\iota(i) \equiv \ell + ti \pmod{d}$, and the condition $\iota^2 = \mathrm{id}$ is equivalent to $(t+1)\ell \equiv 0 \pmod{d}$, which together with $t^2 \equiv 1 \pmod{d}$ implies $(t-1, d) \mid 2\ell$.

Remark 1.2. Observe that $(t-1, d) \mid \ell$ if and only if Ω contains an element of order 2, so RBCM$_t$'s of different type cannot be isomorphic. On the other hand, according to Lemma 2.4 of [11], two RBCM$_t$'s of the same type $\mathcal{CM}(\Gamma_j, \Omega_j, \rho_j)$, $j = 1, 2$ are isomorphic if and only if there exists an isomorphism $\sigma : \Gamma_1 \to \Gamma_2$ such that $\sigma(\Omega_1) = \Omega_2$ and $\sigma \circ \rho_1 = \rho_2 \circ \sigma$.

When $\mathcal{CM}(\Gamma, \{\omega_1, \ldots, \omega_d\}, \rho)$ has *type I* (resp. *type II*), by re-indexing the ω_i's if necessary, we may assume $\ell = (t-1, d)/2$ (resp. $\ell = (t-1, d)$).

So far, people have completely classified RBCM$_t$'s for the following classes of groups: dihedral groups (Kwak, Kwon and Feng [11], 2006), dicyclic groups (Kwak and Oh [12], 2008), semi-dihedral groups (Oh [14], 2009), cyclic groups (Kwon [13], 2013). In 2017 the first author [1] reduced the classification of RBCM$_t$'s on abelian groups to a problem about polynomial rings, and gave a complete classification for RBCM$_t$'s on abelian 2-groups. In 2018 Yuan, Wang and Qu [15] classified RBCM$_1$'s for the so-called minimal nonabelian metacyclic groups. For results on more general regular Cayley maps, see [2, 4, 6, 7, 9].

It is still challenging to study regular Cayley maps on nonabelian groups. We propose a "reduction method", through which known results about RBCM$_t$'s on simpler groups may be applicable. A key ingredient is the following observation.

Lemma 1.3. *Let $\mathcal{CM}(\Gamma, \Omega, \rho)$ be an RBCM$_t$ with skew-morphism φ. Suppose Ξ is a normal subgroup of Γ which is contained in Γ_+ and invariant under φ_+.*

Let $\overline{\Gamma}=\Gamma/\Xi$, *and let* $\overline{\Omega}$ *denote the image of* Ω *under the quotient map* $\Gamma\to\overline{\Gamma}$. *Then* ρ *induces a permutation* $\overline{\rho}$ *on* $\overline{\Omega}$ *and gives rise to an RBCM*$_t$ $\mathcal{CM}(\overline{\Gamma},\overline{\Omega},\overline{\rho})$. *Furthermore, if* $\mathcal{CM}(\Gamma,\Omega,\rho)$ *has type II, then so does* $\mathcal{CM}(\overline{\Gamma},\overline{\Omega},\overline{\rho})$.

Proof. For $\eta\in\Gamma$, let $\overline{\eta}$ denote its image under the quotient map $\Gamma\to\overline{\Gamma}$.

The map $\overline{\varphi}:\overline{\Gamma}\to\overline{\Gamma}$, $\overline{\eta}\mapsto\overline{\varphi(\eta)}$ is well-defined, as $\varphi(\xi\eta)=\varphi(\xi)\varphi(\eta)$ for any $\xi\in\Xi$. Let π be the power function of $\mathcal{CM}(\Gamma,\Omega,\rho)$. It induces a function $\overline{\pi}:\overline{\Gamma}\to\{1,t\}$ in an obvious way. For all η,μ, we have

$$\overline{\varphi}(\overline{\eta\mu})=\overline{\varphi(\eta\mu)}=\overline{\varphi(\eta)\varphi^{\pi(\eta)}(\mu)}=\overline{\varphi}(\overline{\eta})\overline{\varphi}^{\overline{\pi}(\overline{\eta})}(\overline{\mu}).$$

So $\rho=\varphi|_{\Omega}$ induces a permutation $\overline{\rho}$ on $\overline{\Omega}$, building $\mathcal{CM}(\overline{\Gamma},\overline{\Omega},\overline{\rho})$ into an RBCM$_t$.

The assertion about type follows from the first sentence of Remark 1.2. □

The idea is, to understand an RBCM$_t$ $\mathcal{M}$ on Γ, we take a suitable subgroup Ξ, investigate the quotient RBCM$_t$ $\overline{\mathcal{M}}$ on Γ/Ξ, and use knowledge on $\overline{\mathcal{M}}$ to extract information about $\mathcal{M}$ as much as possible.

In this paper, we apply the reduction method to classify RBCM$_t$'s for a class of split metacyclic 2-groups.

A general *split metacyclic group* can be presented as

$$\Lambda(n,m;r)=\langle\alpha,\beta\mid\alpha^n=\beta^m=1,\ \beta\alpha\beta^{-1}=\alpha^r\rangle \tag{1.2}$$

for some positive integers n,m,r such that $r^m\equiv 1\pmod{n}$; see [8, p. 2]. We focus on $\Lambda(2^a,2^b;1+2^c)$, with

$$\max\{2,a-b\}\le c\le a-3\quad\text{and}\quad b\neq c. \tag{1.3}$$

These groups constitute a major part of split metacyclic 2-groups of Class A, as introduced on [5, p. 2]. The artificial restriction (1.3) is imposed for simplicity, so that the paper has a clear structure and a moderate length; if $b=c$ is allowed, then some annoying subtleties will arise, but nothing interesting will happen.

The main result is Theorem 3.10. As shown in [15], any metacyclic p-group for odd prime p does not admit an RBCM$_1$; (by Proposition 1.1, it does not admit an RBCM$_t$ for $t>1$). On the contrary, we shall see that the metacyclic 2-group $\Lambda(2^a,2^b;1+2^c)$ admits a rich family of RBCM$_t$'s, consisting of 2^{a-c-1} isomorphism classes. To some extent, we can say that the richness and complexity of RBCM$_t$'s on metacyclic groups are concentrated on metacyclic 2-groups.

Section 2 presents a preliminary on metacyclic groups. Section 3 comprises the main steps of classifying RBCM$_t$'s. First, we combine Lemma 1.3 and the previous work [1] on RBCM$_t$'s on abelian 2-groups to deduce several constraints on RBCM$_t$'s on metacyclic 2-groups, stated as Lemma 3.2. Second, based on the work [3] on automorphisms of metacyclic groups, we show that each RBCM$_t$ can be "normalized", in the sense that it is isomorphic to one with the property that φ_+ and ω_d are in certain special forms. Third, we solve a system of congruence equations which characterize conditions for given data to determine a normalized RBCM_t. Finally we state the classification as Theorem 3.10.

Notation.

For positive integers u, s, let $[u]_s = 1 + s + \cdots + s^{u-1}$; let $[0]_s = 0$.

For $u \neq 0$, let $\|u\|$ denote the largest k with $2^k \mid u$; set $\|0\| = +\infty$.

For an element θ of a finite group, let $|\theta|$ denote its order.

Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, which is a quotient ring of $\mathbb{Z}$.

For an abelian 2-group Γ, let $\mathrm{rk}(\Gamma)$ denote its rank.

Given a normal subgroup $\Xi \lhd \Gamma$, the image of $\eta \in \Gamma$ under the quotient $\Gamma \to \Gamma/\Xi$ is usually denoted by $\overline{\eta}$, (but for $u \in \mathbb{Z}$, its image under $\mathbb{Z} \to \mathbb{Z}_n$ is still denoted by u), and if an automorphism ϕ of Γ satisfies $\phi(\Xi) = \Xi$, then its induced automorphism on Γ/Ξ is denoted by $\overline{\phi}$.

An RBCM_t $\mathcal{CM}(\Gamma, \Omega, \rho)$ is shorten as $\mathcal{CM}(\Gamma, \Omega)$ if Ω can be written as $\{\omega_1, \ldots, \omega_d\}$ and $\rho(\omega_i) = \omega_{i+1}$. The subscript in ω_i is always understood as modulo d. Let $\mathrm{Aut}^+(\Gamma) = \{\tau \in \mathrm{Aut}(\Gamma) \colon \tau(\Gamma_+) = \Gamma_+\}$.

Since various congruences modulo powers of 2 will appear in the computations, to simplify the writing we use $A \equiv^{(k)} B$ to indicate $A \equiv B \pmod{2^k}$. Furthermore, abbreviate $A \equiv^{(a-1)} B$ to $A \equiv B$, and $A \equiv^{(b)} B$ to $A \equiv' B$.

2. Preliminary on metacyclic groups

A general element of $\Lambda = \Lambda(n, m; r)$ can be written as $\alpha^x \beta^y$. By (1.2) we have

$$
\begin{aligned}
\beta^y \alpha^x &= \alpha^{x r^y} \beta^y, \\
(\alpha^{x_1}\beta^{y_1})(\alpha^{x_2}\beta^{y_2}) &= \alpha^{x_1 + x_2 r^{y_1}} \beta^{y_1 + y_2}, \\
(\alpha^x \beta^y)^u &= \alpha^{x[u]_{r^y}} \beta^{yu}, \\
[\alpha^{x_1}\beta^{y_1}, \alpha^{x_2}\beta^{y_2}] &= \alpha^{x_1(1 - r^{y_2}) - x_2(1 - r^{y_1})}.
\end{aligned}
\tag{2.1}
$$

Here r^y is understood as $r^{y - m[y/m]}$ if $y < 0$, $[u]_{r^y}$ is understood as $[u - n[u/n]]_{r^y}$ if $u < 0$, and the commutator $[\eta, \mu] = \eta\mu\eta^{-1}\mu^{-1}$. Consequently, the commutator subgroup is generated by $\langle \alpha^{r-1} \rangle$, hence the abelianization

$$\Lambda^{\mathrm{ab}} := \Lambda/[\Lambda, \Lambda] \cong \mathbb{Z}_{(r-1,n)} \times \mathbb{Z}_m.$$

Lemma 2.1. *There are three index 2 subgroups of $\Lambda = \Lambda(n, m; r)$, namely, $\langle \alpha^2, \beta \rangle$, $\langle \alpha, \beta^2 \rangle$ and $\langle \alpha^2, \alpha\beta \rangle$.*

Proof. Each homomorphism $\Lambda \to \mathbb{Z}_2$ factors through Λ^{ab}, and there are exactly three epimorphisms $\kappa_j : \Lambda^{\mathrm{ab}} \cong \mathbb{Z}_{(r-1,n)} \times \mathbb{Z}_m \to \mathbb{Z}_2$, $j = 1, 2, 3$, given by

$$\kappa_1(u, v) = u, \qquad \kappa_2(u, v) = v, \qquad \kappa_3(u, v) = u + v.$$

Let $\widetilde{\kappa}_j$ denote the composite of the quotient $\Lambda \to \Lambda^{\mathrm{ab}}$ and κ_j. It is easy to see that $\ker \widetilde{\kappa}_1 = \langle \alpha^2, \beta \rangle$, $\ker \widetilde{\kappa}_2 = \langle \alpha, \beta^2 \rangle$, $\ker \widetilde{\kappa}_3 = \langle \alpha^2, \alpha\beta \rangle$. $\square$

The following is a special case of [3] Theorem 2.9, in which, $\Lambda_1 = \{2\}$, $\Lambda_2 = \Lambda' = \emptyset$, $a_2 = c_2 = a$, $b_2 = b$, $d_2 = c$, $t = 2^a$, $m = 2^b$, $m_0 = 1$.

If there is an automorphism σ of $\Lambda(n, m; r)$ sending α and β to $\alpha^{x_1}\beta^{y_1}$ and $\alpha^{x_2}\beta^{y_2}$, respectively, then we denote such automorphism σ by $\sigma^{x_1,y_1}_{x_2,y_2}$ in this paper.

Lemma 2.2. *Suppose $\|r-1\| = c \geq 2$. Each automorphism of $\Lambda(2^a, 2^b; r)$ is given by $\sigma^{x_1,y_1}_{x_2,y_2} : \alpha \mapsto \alpha^{x_1}\beta^{y_1}$, $\beta \mapsto \alpha^{x_2}\beta^{y_2}$ for some integers x_1, y_1, x_2, y_2 with*

$$2 \nmid x_1y_2 - x_2y_1, \qquad \|y_1\| \geq b - c, \qquad \|x_2\| \geq a - b,$$
$$y_2 \equiv^{(a-c)} \begin{cases} 1 + 2^{a-c-1}, & \text{if } b = a - c = \|y_1\| + c, \\ 1, & \text{otherwise.} \end{cases}$$

Actually, any x_1, y_1, x_2, y_2 satisfying these define an automorphism.

Acting on general elements,

$$\sigma^{x_1,y_1}_{x_2,y_2}(\alpha^u\beta^v) = \alpha^{x_1[u]_{r^{y_1}} + r^{y_1u}x_2[v]_{r^{y_2}}}\beta^{y_1u+y_2v}.$$

Given $\sigma^{x_1,y_1}_{x_2,y_2}$ and $\sigma^{p_1,q_1}_{p_2,q_2}$, the composite $\sigma^{p_1,q_1}_{p_2,q_2} \circ \sigma^{x_1,y_1}_{x_2,y_2}$ sends α to $\alpha^{h_1}\beta^{q_1x_1+q_2y_1}$ and sends β to $\alpha^{h_2}\beta^{q_1x_2+q_2y_2}$, with

$$h_j = p_1[x_j]_{r^{q_1}} + r^{q_1x_j}p_2[y_j]_{r^{q_2}}, \qquad j = 1, 2.$$

Let $r = 1 + 2^c$. Since $2(\|q_1\| + c) \geq b + c \geq a$, we have $r^{q_1u} \equiv^{(a)} 1 + 2^cq_1u$, so

$$[x_j]_{r^{q_1}} = \sum_{i=0}^{x_j-1} r^{iq_1} \equiv^{(a)} x_j + 2^{c-1}q_1x_j(x_j-1),$$
$$r^{q_1x_j}p_2 \equiv^{(a)} p_2 + 2^cq_1x_jp_2 \equiv^{(a)} p_2.$$

Suppose $c > b$ which will hold in the next section. Then $\|x_2\| > a - c$, $\|p_2\| > a - c$, so that $2^{c-1}x_2 \equiv^{(a)} 0$, and $p_2[y_j]_{r^{q_2}} \equiv^{(c)} p_2y_j$, implying

$$h_1 \equiv^{(a)} p_1x_1(1 + 2^{c-1}q_1(x_1-1)) + p_2y_1, \qquad h_2 \equiv^{(a)} p_1x_2 + p_2y_2.$$

Thus

$$\sigma^{p_1,q_1}_{p_2,q_2} \circ \sigma^{x_1,y_1}_{x_2,y_2} = \sigma^{p_1x_1(1+2^{c-1}q_1(x_1-1))+p_2y_1,\,q_1x_1+q_2y_1}_{p_1x_2+p_2y_2,\,q_1x_2+q_2y_2}. \tag{2.2}$$

3. Classifying regular t-balanced Cayley maps for a class of split metacyclic 2-groups

Let $\Delta = \Lambda(2^a, 2^b; 1 + 2^c)$ for (a, b, c) satisfying (1.3). In particular, $b \geq 3$, $c \geq 2$.

By [3] Lemma 2.1, $\|[u]_{(1+2^c)^y}\| = \|u\|$. Then by (2.1),

$$|\alpha^x\beta^y| = 2^{\max\{a-\|x\|,\, b-\|y\|\}}. \tag{3.1}$$

Suppose $\mathcal{CM}(\Delta, \{\omega_1, \ldots, \omega_d\})$ is an RBCM$_t$ with skew-morphism φ. As in Remark 1.2, we may further assume $\ell \in \{(t-1, d)/2, (t-1, d)\}$, so

$$\omega_{\ell+ti} = \omega_i^{-1}, \qquad i = 1, \ldots, d. \tag{3.2}$$

Let $\eta_j = \omega_j\omega_{j-1}^{-1} = \omega_j\omega_{\ell+t(j-1)}$. Then

$$\Delta_+ = \langle \eta_1, \ldots, \eta_d\rangle, \tag{3.3}$$
$$\omega_i\omega_d^{-1} = \eta_i\cdots\eta_1, \qquad i = 1,\ldots,d;$$

in particular,

$$\omega_d^{-2} = \omega_\ell\omega_d^{-1} = \eta_\ell\cdots\eta_1. \tag{3.4}$$

Moreover, $\varphi(\omega_j\omega_{\ell+t(j-1)}) = \omega_{j+1}\varphi^t(\omega_{\ell+t(j-1)}) = \omega_{j+1}\omega_{\ell+tj}$, i.e.,

$$\varphi_+(\eta_j) = \eta_{j+1}. \tag{3.5}$$

3.1. Constraints

Lemma 3.1. *Suppose* $\mathcal{CM}(\Gamma, \{\mu_1, \ldots, \mu_m\})$ *is an RBCM$_t$ with skew-morphism* ψ, *and* Γ *is an abelian* 2*-group such that* $\mathrm{rk}(\Gamma) = \mathrm{rk}(\Gamma_+) = 2$. *Then*

- (i) *there exists an isomorphism* $\Gamma_+ \cong \mathbb{Z}_{2^{k'}} \times \mathbb{Z}_{2^k}$ *for some* $k' \geq k \geq 1$, *sending* θ_1 *to* $(1,0)$ *and* θ_2 *to* $(-1,1)$, *where* $\theta_j = \mu_j - \mu_{j-1}$;
- (ii) $\mathcal{CM}(\Gamma, \{\mu_1, \ldots, \mu_m\})$ *has type I, and* $m = 2^{k+1} \mid t+1$;
- (iii) $\psi_+^2 = \mathrm{id}$.

Proof. RBCM$_t$'s on abelian 2-groups were completely classified in [1] Section 4.2, Corollary 4.3 and Corollary 4.7; obviously $\mathrm{rk}(\Gamma) = \mathrm{rk}(\Gamma_+) = 2$ only occurs in the last case of Section 4.2. The conditions (i)–(iii) can be easily verified. □

Lemma 3.2. *For our RBCM$_t$* $\mathcal{CM}(\Delta, \{\omega_1, \ldots, \omega_d\})$, *the following holds*:

- (i) *it has type I, with* $\Delta_+ = \langle \alpha^2, \beta\rangle \cong \Lambda(2^{a-1}, 2^b; 1 + 2^c)$;
- (ii) $c > b$, *and* $\|t+1\| > b$;
- (iii) $\varphi_+ = \sigma^{x_1;y_1}_{x_2;y_2}$ *for some* x_1, y_1, x_2, y_2 *with* $2 \nmid y_1$ *and*

$$x_1^2 + x_2y_1 \equiv^{(c-1)} 1, \qquad x_1 + y_2 \equiv' y_2^2 + x_2y_1 - 1 \equiv' 0.$$

Remark 3.3. As a consequence of (ii), $c \geq 4$.

Be careful: here $\varphi_+ = \sigma^{x_1;y_1}_{x_2;y_2}$ means that it sends α^2 to $\alpha^{2x_1}\beta^{y_1}$ and sends β to $\alpha^{2x_2}\beta^{y_2}$.

Proof of Lemma 3.2. The proof consists of three parts.

(1) Assume $\Delta_+ = \langle \alpha^2, \alpha\beta\rangle$, $\eta_j = \alpha^{u_j}\beta^{v_j}$ $(j = 1, \ldots, d)$, and

$$\varphi_+(\alpha^2) = (\alpha^2)^{x_1}(\alpha\beta)^{y_1}, \qquad \varphi_+(\alpha\beta) = (\alpha^2)^{x_2}(\alpha\beta)^{y_2}.$$

Since $|(\alpha^2)^{x_2}(\alpha\beta)^{y_2}| = |\varphi_+(\alpha\beta)| = |\alpha\beta|$, by (3.1) we have $2 \nmid y_2$; from

$$1 = \varphi_+\big((\alpha^2)^{2^{a-1}}\big) = ((\alpha^2)^{x_1}(\alpha\beta)^{y_1})^{2^{a-1}} = (\alpha^{2x_1 + [y_1]_{1+2^c}}\beta^{y_1})^{2^{a-1}}$$

we see $2 \mid y_1$. From (3.5) we see that all the v_j's have the same parity, which, by (3.3), must be odd. By (3.4), $2 \mid v_1 + \cdots + v_\ell$, so ℓ is even.

On the other hand, one can verify that the subgroup

$$\Xi = \big\langle \alpha^{2^c}, \beta^{2^c}\big\rangle = \big\langle \alpha^{2^c}, (\alpha\beta)^{2^c}\big\rangle$$

is normal in Δ and invariant under φ_+. By Lemma 1.3 there is a quotient RBCM$_t$ $\overline{\mathcal{M}}$ on $\Delta/\Xi \cong \mathbb{Z}_{2^c} \times \mathbb{Z}_{2^c}$. Clearly $\mathrm{rk}(\Delta/\Xi) = \mathrm{rk}(\Delta_+/\Xi) = 2$, hence by Lemma 3.1, $\overline{\mathcal{M}}$ has type I, and $4 \mid t+1$. By Lemma 1.3, our RBCM$_t$ has type I. Hence $\ell = (t-1, d)/2$, contradicting $2 \mid \ell$.

(2) Assume $\Delta_+ = \langle \alpha, \beta^2 \rangle$, $\eta_j = \alpha^{u_j}\beta^{2v_j}$ $(j = 1, \ldots, d)$ and $\varphi_+ = \sigma^{x_1,y_1}_{x_2,y_2}$. Since $\Delta_+ \cong \Lambda(2^a, 2^{b-1}; (1+2^c)^2)$, by Lemma 2.2 we have

$$\|y_1\| \geq b - c - 2, \quad \|x_2\| \geq a - b + 1, \quad \|y_2 - 1\| \geq a - c - 2. \tag{3.6}$$

The subgroup $\Xi' = \left\langle \alpha^{2^c}, \beta^{2^{b-1}} \right\rangle$ is normal in Δ and invariant under φ_+, with $\Delta/\Xi' \cong \mathbb{Z}_{2^c} \times \mathbb{Z}_{2^{b-1}}$ and $\Delta_+/\Xi' \cong \mathbb{Z}_{2^c} \times \mathbb{Z}_{2^{b-2}}$. In Δ_+/Ξ',

$$\overline{\eta_1} = u_1\overline{\alpha} + v_1\overline{\beta^2},$$
$$\overline{\eta_1} + \overline{\eta_2} = ((x_1+1)u_1 + x_2v_1)\overline{\alpha} + (y_1u_1 + (y_2+1)v_1)\overline{\beta^2}.$$

By Lemma 1.3 and Lemma 3.1, $4 \mid t+1$ and $\ell = (t-1,d)/2$, so that $2 \nmid \ell$.

If $2 \mid x_2$, then $2 \nmid x_1$, so that $u_j \equiv u_1 \pmod 2$; by (3.3), $2 \nmid u_1$, hence $\eta_\ell \cdots \eta_1 = \alpha^{u'}\beta^{v'}$ for some odd u', but this contradicts (3.4). Hence $2 \nmid x_2$, and consequently $b - 1 \geq a \geq c + 3$.

By Lemma 3.1(i), $|\overline{\eta_1}| = 2^{b-2}$ and $|\overline{\eta_1} + \overline{\eta_2}| = 2^c$. Hence $2 \nmid v_1$, and

$$c \geq b - 2 - \|y_1u_1 + (y_2+1)v_1\| = b - 3;$$

the inequality comes from (3.1), and the equality relies on (3.6) which implies $\|y_1\| \geq b - c - 2 \geq 2 > \|y_2 + 1\| = 1$. This contradicts $b - 1 \geq c + 3$.

(3) Therefore by Lemma 2.1, $\Delta_+ = \langle \alpha^2, \beta \rangle \cong \Lambda(2^{a-1}, 2^b; 1 + 2^c)$. Suppose $\varphi_+ = \sigma^{x_1,y_1}_{x_2,y_2}$, and $\eta_j = \alpha^{2u_j}\beta^{v_j}$.

The subgroup $\langle \alpha^{2^c} \rangle$ is normal in Δ and invariant under φ_+. Applying Lemma 3.1 to the quotient RBCM$_t$ on $\Delta/\langle \alpha^{2^c} \rangle \cong \mathbb{Z}_{2^c} \times \mathbb{Z}_{2^b}$, we obtain $4 \mid t+1$ and $\ell = (t-1,d)/2$. So $2 \nmid \ell$.

If $2 \mid y_1$, then $v_j \equiv v_1 \pmod 2$ for all j; by (3.3), $2 \nmid v_1$, and since $2 \nmid \ell$, we have $\eta_\ell \cdots \eta_1 = \alpha^{2u'}\beta^{v'}$ for some odd v', contradicting (3.4). Hence $2 \nmid y_1$, and then by Lemma 2.2, $b \leq c + \|y_1\| = c$. Since $b \neq c$, actually $c > b$.

By Lemma 3.1(iii), $\overline{\varphi_+}^2 = \mathrm{id}$. The expression for $\overline{\varphi_+}^2$ is

$$\overline{\alpha^2} \mapsto (x_1^2 + x_2y_1)\overline{\alpha^2} + y_1(x_1 + y_2)\overline{\beta},$$
$$\overline{\beta} \mapsto x_2(x_1 + y_2)\overline{\alpha^2} + (x_2y_1 + y_2^2)\overline{\beta}.$$

Thus, $x_1^2 + x_2y_1 \equiv^{(c-1)} 1$ and $x_1 + y_2 \equiv' y_2^2 + x_2y_1 - 1 \equiv' 0$. □

3.2. Normalization

Lemma 3.4. *Let $\sigma^{z_1,w_1}_{z_2,w_2} \in \mathrm{Aut}(\Delta_+)$. There exists $\tau \in \mathrm{Aut}^+(\Delta)$ with $\tau_+ = \sigma^{z_1,w_1}_{z_2,w_2}$ if and only if $2 \mid w_1$ and $\|w_2 - 1\| \geq a - c$.*

Proof. If $\tau=\sigma^{p_1,q_1}_{p_2,q_2}\in\mathrm{Aut}^+(\Delta)$, then by Lemma 2.2, $2\mid p_2$ and $\|q_2-1\|\geq a-c$. As an automorphism of Δ_+, $\tau_+(\alpha^2)=\alpha^{p_1(2+2^cq_1)}\beta^{2q_1}$, $\tau_+(\beta)=\alpha^{p_2}\beta^{q_2}$, hence

$$\tau_+=\sigma^{p_1(1+2^{c-1}q_1),2q_1}_{p_2/2,q_2}.$$

So $2\mid w_1$ and $\|w_2-1\|\geq a-c$ are necessary for there to exist $\tau\in\mathrm{Aut}^+(\Delta)$ with $\tau_+=\sigma^{z_1,w_1}_{z_2,w_2}$.

Conversely, suppose $2\mid w_1$ and $\|w_2-1\|\geq a-c$. Put

$$\tau=\sigma^{(1-2^{c-2}w_1)z_1,w_1/2}_{2z_2,w_2}.$$

It is clear that $\tau\in\mathrm{Aut}^+(\Delta)$ with $\tau_+=\sigma^{z_1,w_1}_{z_2,w_2}$. □

Lemma 3.5. *Suppose h is odd, $e>2$, and $s^2\equiv^{(e)}h$. There exists a sequence $\{\tilde{s}_k\}_{k=2}^{\infty}$ such that $\tilde{s}_k^2\equiv^{(k(e-1))}h$ and $\tilde{s}_{k+1}\equiv^{(k(e-1)-1)}\tilde{s}_k$ for each k. Consequently, for each $\tilde{e}>e$, there exists $\tilde{s}$ such that $\tilde{s}^2\equiv^{(\tilde{e})}h$ and $\tilde{s}\equiv^{(e-1)}s$.*

Proof. We construct $\tilde{s}_k$ recursively. Since s is odd, we may take $a_1\in\mathbb{Z}$ such that $h\equiv^{(2(e-1))}s^2+2^esa_1$. Set $\tilde{s}_2=s+2^{e-1}a_1$. Then clearly $\tilde{s}_2^2\equiv^{(2(e-1))}h$ and $\tilde{s}_2\equiv^{(e-1)}s$.

Assume $k\geq 2$ and $\tilde{s}_k$ has been obtained. Take $a_k\in\mathbb{Z}$ with

$$h\equiv^{((k+1)(e-1))}\tilde{s}_k^2+2^{k(e-1)}\tilde{s}_ka_k,$$

and set

$$\tilde{s}_{k+1}=\tilde{s}_k+2^{k(e-1)-1}a_k.$$

Then $\tilde{s}_{k+1}\equiv^{(k(e-1)-1)}\tilde{s}_k$ and $\tilde{s}_{k+1}^2\equiv^{((k+1)(e-1))}h$, due to $2k(e-1)-2\geq(k+1)(e-1)$. □

Lemma 3.6. *There exists $\tau_1\in\mathrm{Aut}^+(\Delta)$ such that $(\tau_1\varphi\tau_1^{-1})_+=\sigma^{z,1}_{0,-z}$ for some $z\equiv^{(c-2)}-1$.*

Proof. We are going to find u_1,v_1,u_2,v_2,z,w satisfying the following:

$$u_1x_1(1+2^{c-1}v_1(x_1-1))+u_2y_1\equiv zu_1(1+2^{c-1}(u_1-1)), \tag{3.7}$$

$$v_1x_1+v_2y_1\equiv' u_1+wv_1, \tag{3.8}$$

$$u_1x_2+u_2y_2\equiv zu_2, \tag{3.9}$$

$$v_1x_2+v_2y_2\equiv' u_2+wv_2. \tag{3.10}$$

In view of (2.2), these will ensure

$$\sigma^{u_1,v_1}_{u_2,v_2}\circ\varphi_+=\sigma^{z,1}_{0,w}\circ\sigma^{u_1,v_1}_{u_2,v_2}.$$

Take γ with $(1+2^{c-1}(y_1-1))\gamma\equiv 1$, and let

$$f(x)=(x-\gamma x_1)(x-y_2)-x_2y_1=\Big(x-\frac{\gamma x_1+y_2}{2}\Big)^2-x_2y_1-\Big(\frac{\gamma x_1+y_2}{2}\Big)^2.$$

Remember that $c\geq a-b>a-c\geq 3$ and $\|x_2\|\geq a-b$. Then $f(x_1)\equiv^{(a-b)}0$. By Lemma 3.5, there exists z with $f(z)\equiv 0$ and $z\equiv^{(a-b-1)}x_1$. Note that $\|z-y_2\|=\|x_1-y_2\|=1$.

Let $u_1 = y_1$, $v_1 = 0$, $v_2 = 1$, $w = y_2 - u_2$, and $u_2 = (1 + 2^{c-1}(y_1 - 1))z - x_1$. Then $f(z) \equiv 0$ is equivalent to

$$(z - y_2)u_2 \equiv x_2 y_1.$$

It is easy to verify that (3.7)–(3.10) all hold. Now it holds that

$$\|u_2\| = \|x_2\| + \|y_1\| - \|z - y_2\| = \|x_2\| - 1 \geq a - c - 1,$$

by Lemma 3.4, $\sigma^{u_1,v_1}_{u_2,v_2} = (\tau_1)_+$ for some $\tau_1 \in \mathrm{Aut}^+(\Delta)$.

Consider the automorphism of $\Delta_+/\langle\alpha^{2^c}\rangle$ induced by $\tau_1\varphi_+\tau_1^{-1}$. Similarly as the final part of the proof of Lemma 3.2, we have $z^2 \equiv^{(c-1)} 0$ and $z + w \equiv' w^2 - 1 \equiv' 0$. Thus $(\tau_1\varphi\tau_1^{-1})_+ = \sigma^{z,1}_{0,-z}$. Note that $w \equiv y_2 \equiv 1 \pmod 4$, implying $\|z + 1\| \geq c - 2$. □

Lemma 3.7. *Suppose $x \equiv^{(c-2)} z \equiv^{(c-2)} -1$. If $\tau \in \mathrm{Aut}^+(\Delta)$ with $\tau_+ = \sigma^{p_1,q_1}_{p_2,q_2}$, then $\tau_+ \circ \sigma^{z,1}_{0,-z} \circ \tau_+^{-1} = \sigma^{x,1}_{0,-x}$ is equivalent to*

$$\|p_2\| \geq a - 2, \qquad p_1 - q_2 \equiv' 2zq_1, \qquad x \equiv z + p_2. \tag{3.11}$$

In particular, $\tau_+ \circ \sigma^{z,1}_{0,-z} \circ \tau_+^{-1} = \sigma^{z,1}_{0,-z}$ if and only if $p_2 \equiv 0$ and $p_1 - q_2 \equiv' 2zq_1$.

Proof. By Lemma 3.4, $2 \mid q_1$ and $\|q_2 - 1\| \geq a - c$.

By (2.2), $\tau_+ \circ \sigma^{z,1}_{0,-z} = \sigma^{x,1}_{0,-x} \circ \tau_+$ is equivalent to

$$p_1 z\big(1 + 2^{c-1}q_1(z - 1)\big) + p_2 \equiv xp_1\big(1 + 2^{c-1}(p_1 - 1)\big), \tag{3.12}$$

$$q_1 z + q_2 \equiv' p_1 - xq_1, \tag{3.13}$$

$$-p_2 z \equiv xp_2, \tag{3.14}$$

$$-q_2 z \equiv' p_2 - xq_2. \tag{3.15}$$

Since $x \equiv^{(c-2)} z \equiv^{(c-2)} -1$, we have $\|x+z\| = 1$, hence by (3.14), $\|p_2\| \geq a-2$. Then (3.15) implies $x \equiv^{(a-2)} z$, and consequently by (3.13), $p_1 - q_2 \equiv' 2zq_1$. Now it holds that $c - 1 \geq b \geq a - c$, and one has

$$p_1 - 1 = (p_1 - q_2) + (q_2 - 1) \equiv^{(a-c)} 2zq_1 \equiv^{(a-c)} (z - 1)q_1,$$

which together with (3.12) implies

$$zp_1\big(1 + 2^{c-1}(z - 1)q_1\big) + p_2 \equiv xp_1\big(1 + 2^{c-1}(z - 1)q_1\big).$$

Since $p_2 \equiv p_2 \cdot p_1\big(1 + 2^{c-1}(z - 1)q_1\big)$, we have $x \equiv z + p_2$.

Conversely, assuming (3.11), it is rather easy to deduce (3.12)–(3.15). □

Lemma 3.8. (i) *There exists $\tau_2 \in \mathrm{Aut}^+(\Delta)$ such that $(\tau_2)_+ \circ \sigma^{z,1}_{0,-z} = \sigma^{z,1}_{0,-z} \circ (\tau_2)_+$ and $(\tau_2\tau_1)(\omega_d) = \alpha^{\tilde{u}}\beta$ for some odd $\tilde{u}$.*

(ii) *For any $\tilde{u}'$ with $\tilde{u}' \equiv^{(a-c)} \tilde{u}$, there exists $\tau \in \mathrm{Aut}^+(\Delta)$ such that $\tau_+ \circ \sigma^{z,1}_{0,-z} = \sigma^{z,1}_{0,-z} \circ \tau_+$ and $\tau(\alpha^{\tilde{u}}\beta) = \alpha^{\tilde{u}'}\beta$.*

Proof. (i) Suppose $\tau_1(\omega_d) = \alpha^{u_0}\beta^{v_0}$. Note that u_0 is odd: otherwise it is impossible for $\tau_1(\eta_1), \dots, \tau_1(\eta_d), \tau_1(\omega_d)$ to generate Δ.

Take y with $yu_0 \equiv' 1 - v_0$. Let $p = 1 + 4zy$, and let

$$\tilde{u} = (1 - 2^{c-1}y)pu_0(1 + 2^{c-1}y(u_0 - 1)).$$

Let $\tau_2 = \sigma_{0,1}^{(1-2^{c-1}y)p,y}$, so that $(\tau_2)_+ = \sigma_{0,1}^{p,2y}$. Then $(\tau_2\tau_1)(\omega_d) = \alpha^{\tilde{u}}\beta$, and by Lemma 3.7, $(\tau_2)_+ \circ \sigma_{0,-z}^{z,1} = \sigma_{0,-z}^{z,1} \circ (\tau_2)_+$.

(ii) Take y with $y\tilde{u} \equiv -2^{a-c}\overline{q}$, with $\overline{q}$ to be determined. Let $p' = 1+2^{a-c}\overline{q}+4zy$. Consider

$$\begin{aligned} u(\overline{q}) &= (1 - 2^{c-1}y)p'\tilde{u}(1 + 2^{c-1}y(\tilde{u} - 1)) \\ &= p'\tilde{u}\big((1 - 2^{c-1}y)2^{c-1}y\tilde{u} + 1 - 2^{c}y + 2^{2c-2}y^2\big) \\ &\equiv^{(a)} p'\tilde{u}(-2^{a-1}\overline{q} + 1) \\ &\equiv^{(a)} \big(1 + 4zy + (1 - 2^{c-1}(1 + 4zy))2^{a-c}\overline{q}\big)\tilde{u}. \end{aligned}$$

Obviously, we can find $\overline{q}$ such that $u(\overline{q}) \equiv^{(a)} \tilde{u}'$.

Let $\tau = \sigma_{0,1+2^{a-c}\overline{q}}^{p',0}$. Now $\tau_+ = \sigma_{0,1+2^{a-c}\overline{q}}^{p',0}$ commutes with $\sigma_{0,-z}^{z,1}$ and $\tau(\alpha^{\tilde{u}}\beta) = \alpha^{\tilde{u}'}\beta$. □

Concluding from the above lemmas, up to isomorphism we may just assume $\varphi_+ = \sigma_{0,-z}^{z,1}$ for a unique z with $0 \leq z < 2^{a-2}$ and $z \equiv^{(c-2)} -1$, and $\omega_d = \alpha^{\tilde{u}}\beta$ such that $\tilde{u}$ is an odd number whose residue modulo 2^{a-c} is unique.

3.3. Expressing necessary and sufficient conditions in terms of congruence equations

Remember that for each k,

$$(1 + 2^c)^k \equiv 1 + 2^c k, \qquad [k]_{1+2^c} \equiv k(1 + 2^{c-1}(k - 1)).$$

Implied by $z \equiv^{(c-2)} -1$,

$$\|4(z + 1)^2\| \geq 2c - 2 \geq a - 1. \tag{3.16}$$

Suppose $\eta_i = \alpha^{2u_i}\beta^{v_i}$. Then $\omega_i\omega_d^{-1} = \eta_i \cdots \eta_1 = \alpha^{2f_i}\beta^{g_i}$, where

$$f_i = u_i + (1 + 2^c v_i)u_{i-1} + \cdots + (1 + 2^c(v_i + \cdots + v_2))u_1, \tag{3.17}$$

$$g_i = v_i + \cdots + v_1. \tag{3.18}$$

So $\omega_i = \alpha^{2f_i+(1+2^c g_i)\tilde{u}}\beta^{g_i+1}$.

The condition (3.2) is equivalent to

$$f_{\ell+ti} + (1 - 2^c(g_i + 1))f_i + (1 + 2^{c-1}(g_{\ell+ti} - 1))\tilde{u} \equiv 0, \tag{3.19}$$

$$g_{\ell+ti} + g_i + 2 \equiv' 0. \tag{3.20}$$

Also the condition $1 = \omega_d\omega_d^{-1} = \alpha^{2f_d}\beta^{g_d}$ implies that

$$f_d \equiv 0, \qquad g_d \equiv' 0. \tag{3.21}$$

From (2.2) and $z^2 \equiv^{(c-1)} 1$ we see $\varphi_+^2 = \sigma_{0,1}^{s,0}$, with $s = z^2 + 2^{c-1}(z-1)$. Hence

$$u_{i+2} \equiv s u_i, \qquad v_{i+2} \equiv' v_i. \tag{3.22}$$

Clearly, $u_2 \equiv u_1 \pmod 2$. It follows from (3.3) that the u_i's are all odd.

Put

$$\overline{u} = u_2 + (1 + 2^c(u_1 + v_1))u_1, \qquad \overline{v} = v_2 + v_1.$$

Since

$$u_2 \equiv z[u_1]_{1+2^c} \equiv (z - 2^{c-1}(u_1 - 1))u_1, \qquad v_2 \equiv' u_1 - z v_1,$$

we have

$$\overline{u} \equiv \big(z + 1 + 2^{c-1}(u_1 + 2v_1 + 1)\big)u_1, \tag{3.23}$$

$$(s-1)\overline{u} \equiv (z+1)^2(z-1)u_1 \overset{(3.16)}{\equiv} 2(z+1)^2, \tag{3.24}$$

$$\overline{v} \equiv' u_1 + (1-z)v_1. \tag{3.25}$$

Obviously, $s \equiv^{(c-1)} 1$, so that for each n,

$$s^n \equiv 1 + n(s-1). \tag{3.26}$$

Now (3.17), (3.18), (3.22) imply

$$f_{2k} \equiv \overline{u} \cdot \sum\nolimits_{j=0}^{k-1} (1 + 2^c j(u_1 + 2v_1)) s^{k-1-j} \equiv \overline{u}[k]_s \equiv k\overline{u} - k(k-1)(z+1)^2,$$

$$f_{2k+1} \equiv s^k u_1 + (1 + 2^c v_1) f_{2k} \equiv (1 + k(s-1))u_1 + k\overline{u} - k(k-1)(z+1)^2,$$

$$g_{2k} \equiv' k\overline{v} \qquad \text{and} \qquad g_{2k+1} \equiv' k\overline{v} + v_1.$$

Lemma 3.9. *Let $h = (\ell - 1)/2$. The conditions* (3.19)–(3.21) *hold if and only if*

$$h\overline{v} + v_1 + 2 \equiv' 0, \tag{3.27}$$

$$u_1 + h\overline{u} \equiv h(h+1)(z+1)^2 + (3 \cdot 2^{c-1} - 1 + h(s-1))\tilde{u}, \tag{3.28}$$

$$2(z+1)^2 \equiv 2^{c-1}\overline{v} + (1-s), \tag{3.29}$$

$$\|t+1\| \geq \max\{a - c + 2, b + 1\}, \tag{3.30}$$

$$\|d\| \geq \max\{a - c + 2, b + 1\}. \tag{3.31}$$

Proof. Let $e = (t+1)/2$. Let $(3.20)_{i=2k}$ stand for (3.20) when $i = 2k$, and so forth.

The condition $(3.20)_{i=2k}$ reads

$$(h + kt)\overline{v} + v_1 + k\overline{v} + 2 \equiv' 0,$$

which, due to $\|t+1\| \geq b+1$, is equivalent to (3.27). Conversely, if (3.27) is satisfied, then $(3.20)_{i=2k+1}$ holds, too:

$$g_{\ell + t(2k+1)} + g_{2k+1} + 2 \equiv' (h + kt + e)\overline{v} + k\overline{v} + v_1 + 2 \equiv' 0.$$

In virtue of $2^c\overline{u} \equiv 0$ and (3.16), the condition $(3.19)_{i=2k}$ reads

$$\begin{aligned} & s^{h+kt}u_1 + h\overline{u} - (h^2 - h + (2h+2)k)(z+1)^2 \\ & + \big(1 + 2^{c-1}((h+kt)\overline{v} + v_1 - 1)\big)\tilde{u} \equiv 0. \end{aligned} \tag{3.32}$$

Then the difference between $(3.19)_{i=2k+2}$ and $(3.19)_{i=2k}$ is equal to

$$(1-s)u_1 + (2h+2)(z+1)^2 - 2^{c-1}\overline{v}\tilde{u} \equiv 0, \tag{3.33}$$

where (3.16), (3.26) have been used.

Setting $k = 0$ in (3.32), we obtain

$$\begin{aligned} & (1 + h(s-1))u_1 + h\overline{u} - h(h-1)(z+1)^2 \\ & + \big(1 + 2^{c-1}(h\overline{v} + v_1 - 1)\big)\tilde{u} \equiv 0. \end{aligned} \tag{3.34}$$

Clearly, $(3.19)_{i=2k}$ holds for all k if and only if (3.33) and (3.34) hold. With (3.24) referred to, (3.33), (3.34) are equivalent to

$$\begin{aligned} u_1 + h\overline{u} &\equiv h(h-1)(z+1)^2 - (1 + 2^{c-1}(v_1 - 1))\tilde{u}, \\ 2(z+1)^2 &\equiv (2^{c-1}\overline{v} + (1-s))\tilde{u}. \end{aligned} \tag{3.35}$$

Note that the second equation is equivalent to (3.29) and forces $\|s-1\| = c-1$. Hence $\|z+1\| = c-2$, and by (3.23), $\|\overline{u}\| = c-2$.

The condition $(3.19)_{i=2k+1}$ reads

$$\begin{aligned} & (h+e)\overline{u} - (h^2 - h - 2(h+1)k)(z+1)^2 + s^k u_1 - 2^c(k\overline{v} + v_1 + 1)u_1 \\ & + \big(1 + 2^{c-1}((h+kt)\overline{v} - 1)\big)\tilde{u} \equiv 0. \end{aligned} \tag{3.36}$$

So the difference between $(3.19)_{i=2k+3}$ and $(3.19)_{i=2k+1}$ equals

$$(s - 1 - 2^c\overline{v})u_1 + 2(h+1)(z+1)^2 - 2^{c-1}\overline{v}\tilde{u} \equiv 0,$$

which can be implied by (3.35), assuming (3.33).

Setting $k = 0$ in (3.36), we obtain

$$(h+e)\overline{u} - h(h-1)(z+1)^2 + (1 - 2^c(v_1+1))u_1 + (1 + 2^{c-1}(h\overline{v} - 1))\tilde{u} \equiv 0;$$

it combined with (3.35) implies $e\overline{u} \equiv 0$, which is equivalent to (3.30).

By (3.27), (3.29), $2^{c-1}v_1 \equiv h(1-s) - 2h(z+1)^2 - 2^c$, and hence (3.35) becomes (3.28).

Finally, (3.21) holds if and only if

$$\frac{d}{2}\overline{u} \equiv \frac{d}{2}\big(\frac{d}{2} - 1\big)(z+1)^2, \qquad \frac{d}{2}\overline{v} \equiv' 0,$$

which are equivalent to (3.31), as is easy to verify. □

Now since $\|z+1\| = c-2$ and $0 \le z < 2^{a-2}$, we may write

$$z = 2^{c-2}(2x-1) - 1, \qquad 1 \le x \le 2^{a-c-1}.$$

By (3.29), using $(2x-1)^2 \equiv 1 \pmod 4$ and $c-1 \geq b \geq a-c$, we obtain

$$\overline{v} \equiv^{(a-c)} \frac{z^2-1}{2^{c-1}} + z - 1 \equiv^{(a-c)} -2^{c-3} - 2x - 1,$$

$$v_1 \equiv^{(a-c)} -h\overline{v} - 2 \equiv^{(a-c)} -2^{c-3}h + h(2x+1) - 2, \tag{3.37}$$

$$u_1 \overset{(3.25)}{\equiv'} \overline{v} + (z-1)v_1 \equiv^{(a-c)} 4 - 2^{c-3} - (2h+1)(2x+1). \tag{3.38}$$

By (3.23), $\|\overline{u}\| = c-2$. By (3.28), $\tilde{u} \equiv^{(c-1)} h\overline{u} - u_1 \equiv^{(c-1)} 2^{c-2}h - u_1$, so that

$$\tilde{u} \equiv^{(a-c)} (2h+1)(2^{c-3} + 2x + 1) - 4.$$

According to the conclusion in the end of Section 3.2 we may just set

$$\tilde{u} = (2h+1)(2^{c-3} + 2x + 1) - 4.$$

By (3.37), (3.38),

$$u_1 + 2v_1 + 1 \equiv^{(a-c)} -2^{c-3}(2h+1) - 2x.$$

Hence by (3.23),

$$\overline{u} \equiv (z+1)u_1 - 2^{c-1}\big(2^{c-3}(2h+1) + 2x\big)\tilde{u}. \tag{3.39}$$

Using

$$s - 1 = z^2 - 1 + 2^{c-1}(z-1) \equiv -2^{2c-4} - 2^{c-1}(2x+1),$$

we convert (3.28) into

$$\begin{aligned}(1 + 2^{c-2}h(2x-1))u_1 &\equiv 2^{c-1}h(2^{c-3}(2h+1) + 2x)\tilde{u} + h(h+1)2^{2c-4} \\ &\quad + \big(3 \cdot 2^{c-1} - 1 - h(2^{2c-4} + 2^{c-1}(2x+1))\big)\tilde{u} \\ &\equiv h(h+1)2^{2c-4} + (2^{2c-3}h^2 + (3-h)2^{c-1} - 1)\tilde{u},\end{aligned}$$

implying

$$\begin{aligned}u_1 &\equiv (1 - 2^{c-2}h(2x-1) - 2^{2c-4}h^2)\big(h(h+1)2^{2c-4} \\ &\quad + (2^{2c-3}h^2 + (3-h)2^{c-1} - 1)\tilde{u}\big) \\ &\equiv h(h+1)2^{2c-4} + ((3 - h + hx)2^{c-1} - 1 + 2^{c-2}h - 2^{2c-4}h^2)\tilde{u}.\end{aligned} \tag{3.40}$$

So (3.39) becomes

$$\overline{u} \equiv (2^{2c-4}(h+1) - 2^{c-2}(6x-1))\tilde{u}.$$

Finally, (3.40) implies

$$u_1 \equiv' (2^{c-2}h - 1)\tilde{u} \equiv' 4 - 2^{c-3} - (2h+1)(2x+1).$$

Hence by (3.25) and (3.27),

$$v_1 \equiv' -\frac{hu_1 + 2}{1 + h(1-z)} \equiv' \frac{h(1 - 2^{c-2}h)\tilde{u} - 2}{2h + 1 - 2^{c-2}h(2x-1)} \equiv' h(2^{c-3} + 2x + 1) - 2,$$

where the meanings of fractions are self-evident. So

$$\overline{v} \equiv' u_1 + (2 + 2^{c-2})v_1 \equiv' -2^{c-3} - 2x - 1.$$

3.4. The result

Recall
$$h = \frac{\ell - 1}{2} = \frac{1}{2}\Big(\frac{(t-1,d)}{2} - 1\Big).$$
For each x with $1 \leq x \leq 2^{a-c-1}$, let
$$\begin{aligned}
\tilde{u} &= (2h+1)(2^{c-3} + 2x + 1) - 4,\\
\overline{u} &= (2^{2c-4}(h+1) - 2^{c-2}(6x-1))\tilde{u},\\
u &= h(h+1)2^{2c-4} + ((3 - h + hx)2^{c-1} - 1 + 2^{c-2}h - 2^{2c-4}h^2)\tilde{u},\\
f_{2k} &= k\overline{u} - k(k-1)2^{2c-4},\\
f_{2k+1} &= (1 + 2^c k)u + k\overline{u} - k(k-1)2^{2c-4},\\
g_{2k} &= -k(2^{c-3} + 2x + 1),\\
g_{2k+1} &= (h-k)(2^{c-3} + 2x + 1) - 2,
\end{aligned}$$
and put $\mathcal{M}(x) = \mathcal{CM}(\Delta, \{\omega_1, \ldots, \omega_d\})$ with $\omega_i = \alpha^{2f_i + (1+2^c g_i)\tilde{u}}\beta^{g_i+1}$.

Theorem 3.10. *If Δ admits d-valent RBCM$_t$'s, then necessarily $\|d\|$, $\|t+1\| \geq \max\{a-c+2, b+1\}$ and $c > b$. When these hold, each d-valent RBCM$_t$ on Δ has type I and is isomorphic to $\mathcal{M}(x)$ for a unique x with $1 \leq x \leq 2^{a-c-1}$.*

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