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ROTATIONALLY SYMMETRIC SOLUTIONS OF THE PRESCRIBED HIGHER MEAN CURVATURE SPACELIKE EQUATIONS IN MINKOWSKI SPACETIME

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Abstract. In this paper we consider the existence of rotationally symmetric entire solutions for the prescribed higher mean curvature spacelike equations in Minkowski spacetime. As a first step, we study the associated 0-Dirichlet problems on a ball, and then we prove that all possible solutions can be extended to $+\infty$. The proof of our main results are based upon the topological degree methods and the standard prolongability theorem of ordinary differential equations.

1. Introduction

Let $\mathbb{L}^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$ be the $(n+1)$-dimensional Minkowski spacetime endowed with its standard Lorentzian metric

$$\sum_{i=1}^{n}(dx_i)^2 - dt^2.$$

In this paper we are concerned with the mixed boundary value problem

$$\begin{cases} (r^{n-k}\phi^k(v'))' = nr^{n-1}H_k(r,v), & r \in (0,R), \\ |v'| < 1, & r \in (0,R), \\ v'(0) = v(R) = 0, \end{cases} \tag{1}$$

where $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, clearly, $\phi : (-1,1) \to \mathbb{R}$ is an increasing diffeomorphism with $\phi(0) = 0$, such an ϕ is called *singular*, $1 \le k \le n$ is an integer, and for each k, $H_k : [0,R] \times (-\alpha,\alpha) \to \mathbb{R}$ is a continuous function with $0 < \alpha \le \infty$. The aim of this paper is to investigate the existence of positive and negative solutions of problem (1) according to the different growth conditions of the H_k near 0 and α by using the topological degree methods, and consider the extendibility of

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solutions by using the standard prolongability theorem of ordinary differential equations.

This consideration is mainly motivated by the study of spacelike submanifolds of codimension one in $\mathbb{L}^{n+1}$ with prescribed higher mean curvature. The general problem of the curvature prescription is, for a given prescription function H_k, a spacelike hypersurface (i.e., spacelike submanifolds of codimension one) Σ in $\mathbb{L}^{n+1}$ which satisfies

$$S_k(p) = H_k(p) \quad \text{for all } p \in \Sigma, \tag{2}$$

where S_k is the k-th mean curvature of the hypersurface and $1 \leq k \leq n$ is an integer. (2) is called the prescribed k-th mean curvature spacelike equations in $\mathbb{L}^{n+1}$, this type of equations are of interest in special relativity and related aspects from Minkowski geometry. The equation with $k = 1$ is called directly the prescribed mean curvature spacelike equation, and the equations with $k \geq 2$ are called collectively the prescribed higher mean curvature spacelike equations. Following the method developed in [19], to investigate the rotationally symmetric solutions of (2), we know it is enough to find the solutions of the equations

$$(r^{n-k}\phi^k(v'))' = nr^{n-1}H_k(r, v), \quad |v'| < 1, \quad r \in (0, +\infty) \tag{3}$$

for a given prescription function $H_k : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, especially, $|v'| < 1$ is the spacelike condition.

In the recent years, most of the efforts have been directed to the prescribed mean curvature spacelike equation in $\mathbb{L}^{n+1}$ ($k = 1$), perhaps the fact that the mean curvature operator $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)$ and its radially symmetric form $\frac{1}{r^{n-1}}\left(r^{n-1}\frac{u'}{\sqrt{1-u'^2}}\right)'$ are explicit. In this context, we mention the seminal work of R. Bartnik and L. Simon [1], E. Calabi [8], S.-Y. Cheng and S.-T. Yau [10] and A. E. Treibergs [32], in these papers, the spacelike hypersurface having the property that their mean curvature is zero or constant are considered. More recently, Dirichlet problems for the prescribed mean curvature spacelike equation in $\mathbb{L}^{n+1}$ have been widely concerned by many scholars, and their attention is mainly focused on their positive solutions, we refer the reader to [3–6, 11–17, 20, 22, 27, 28, 30, 34–37] and the references therein. In particular, based on the detailed analysis of time map, some exact multiplicity of positive solutions have been obtained in [22,37], for the radially symmetric solutions on a ball, some existence, nonexistence and multiplicity results have been established in [4, 5], and some bifurcation results have been obtained in [13, 27] via bifurcation technique, and when the domain is a general domain in $\mathbb{R}^n$, some existence and bifurcation results have been obtained in the papers [12,14,15,30]. In addition to, these concern discrete problems associated with the prescribed mean curvature spacelike equation in $\mathbb{L}^{n+1}$, we refer the reader to [7, 9, 24, 25] and the references therein.

In comparison with the study in prescribed mean curvature spacelike equation in $\mathbb{L}^{n+1}$, the number of references devoted to the prescribed higher mean curvature spacelike equations in $\mathbb{L}^{n+1}$ is appreciably lower. The earliest studies on Dirichlet problem for prescribed higher mean curvature spacelike equations in $\mathbb{L}^{n+1}$ can be seen in [23], N. M. Ivochkina proved the existence of solutions by using the implicit function theorem, Leary-Schauder principle and some basic theories of second-order elliptic equations. Bayard [2] proved the existence of prescribed scalar curvature entire spacelike hypersurfaces in $\mathbb{L}^{n+1}$. On the Gauss-Kronecker curvature, we emphasize the work of Li [26] on constant Gauss curvature and Delanoë [18], in which the existence of entire spacelike hypersurfaces asymptotic to a lightcone with prescribed Gauss-Kronecker curvature function is proved. For the rotationally symmetric solutions of this equations, only in the recent years, de la Fuente, Romero and Torres [19] have got an existence and multiplicity results by using the Schauder fixed point theorem, see [1, Propositions 3.1-3.3] for the detail, Ma and Xu [29] have provided a geometric interpretation about the occurrence of the above solutions and got the existence of rotationally symmetric entire solutions via the global bifurcation theory.

Motivated by the interesting studies of [19] and some works in radially symmetric solutions of the prescribed mean curvature spacelike equations on a ball [4, 5, 13, 27], here we continue the investigations on the existence of solutions of problem (3). As a first step, we consider the associated 0-Dirichlet problems on the ball $B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$, by using the topological degree methods, we prove that when k is odd, the Dirichlet problem has at least one strictly decreasing positive solution, when k is even, it has at least two solutions, one is strictly decreasing and positive the other is strictly increasing and negative, provided that H_k is superlinear at 0 with respect to ϕ^k, that is

$$\lim_{s\to 0} \frac{H_k(r,s)}{\phi^k(s)} = \infty \quad \text{uniformly for } r \in [0,R] \tag{4}$$

and $R < \alpha$. When $R = \alpha = 1$, (4) is satisfied and H_k is sublinear at 1 with respect to ϕ^k, that is

$$\lim_{|s|\to 1^-} \frac{H_k(r,s)}{\phi^k(s)} = 0 \quad \text{uniformly for } r \in [0,1], \tag{5}$$

we prove the same conclusion (Theorem 2.6). Next, we prove that all possible solutions can also be extended to $+\infty$ based upon the standard prolongability theorem of ordinary differential equations (Theorems 3.1, 3.2). It is worth point out that, when $R < \alpha$, H_k can be singular at α, but condition (4) is sufficient to ensure the existence of solutions. The conditions (4) has been considered by many authors, we refer the readers to [21, 33] for the semilinear elliptic equations, and for the classical p-Laplacian case, for which $\phi_p(s) = |s|^{p-2}s$, to obtain the existence of positive solutions, (4) is considered together with the sublinear condition of H_k at infinity with respect to ϕ_p:

$$\lim_{s\to\infty}\frac{H_k(r,s)}{s^{p-1}}=0 \quad \text{uniformly for } r\in[0,R], \tag{6}$$

we refer the reader to see [4]. In our case, (6) naturally is replaced by (5).

The rest of the paper is arranged as follows. In Section 2, we analyze the associated 0-Dirichlet problem on a ball. In Section 3, we show that all possible solutions can be extended to $+\infty$.

2. Positive solutions and negative solutions: a topological degree approach

In this section, we show the existence of positive solutions and negative solutions of the mixed boundary value problem

$$\begin{cases}(r^{n-k}\phi^k(v'))'=nr^{n-1}H_k(r,v), & r\in(0,R),\\ |v'|<1, & r\in(0,R),\\ v'(0)=v(R)=0,\end{cases} \tag{7}$$

where $1\le k\le n$, $H_k:[0,R]\times(-\alpha,\alpha)\to\mathbb{R}$ are continuous functions and $0<\alpha\le\infty$.

First, we should realize that (7) has no solution if k is even and $H_k<0$. Therefore, if k is even, we will assume H_k is non-negative, and by this reason, for a more general prescription function of the curvature we need to distinguish two cases: k is odd or even.

We make the following main hypotheses:

- (A_1) For each k, $H_k(r,s):[0,R]\times(-\alpha,\alpha)\to\mathbb{R}$ is continuous with $0<\alpha\le\infty$, and $(-1)^kH_k(r,s)\ge 0$ for all $(r,s)\in[0,R]\times(-\alpha,\alpha)$ and $(-1)^kH_k(r,s)>0$ for all $(r,s)\in(0,R]\times(-\alpha,0)\cup(0,\alpha)$.

In the sequel, the space $C:=C[0,R]$ will be endowed with the usual supremum norm $\|\cdot\|_\infty$ and the corresponding open ball of center 0 and radius $\rho>0$ will be denoted by B_ρ. A solution of (7) we mean a function $v\in C^1[0,R]$ with $\|v'\|_\infty<1$ such that $r^{n-k}\phi^k(v')$ is differentiable and (7) is satisfied.

We need the following elementary result.

Proposition 2.1. *Assume that (A_1). Let v be a nontrivial solution of (7). Then*

- (i) *if k is odd, then $v>0$ on $[0,R)$ and v is strictly decreasing;*
- (ii) *if k is even, then either $v>0$ on $[0,R)$ and v is strictly decreasing or $v<0$ on $[0,R)$ and v is strictly increasing, and these two solutions must come in pairs.*

Proof. From (7), we have that

$$\phi^k(v')=\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}H_k(\tau,v(\tau))d\tau, \tag{8}$$

notice that (8) is well-defined thanks to condition (A_1).

If k is odd, then from

$$\phi(v') = \Big[\frac{n}{r^{n-k}}\int_0^r \tau^{n-1} H_k(\tau, v(\tau))d\tau\Big]^{\frac{1}{k}}, \tag{9}$$

it follows that $v' \leq 0$ because the definition of ϕ and the fact $(-1)^k H_k(r,s) \geq 0$ for each $1 \leq k \leq n$ and all $(r,s) \in [0,R] \times (-\alpha, \alpha)$, so v is decreasing. Since $v(R) = 0$, we have that $v \geq 0$ on $[0,R]$. As v is not identically zero, one has that $v(0) > 0$ and, from (9) we deduce that $v' < 0$ on $(0,R]$, which ensures that actually v is strictly decreasing and $v > 0$ on $[0,R)$.

If k is even, from (8), we have that

$$\phi(v') = \pm\Big[\frac{n}{r^{n-k}}\int_0^r \tau^{n-1} H_k(\tau, v(\tau))d\tau\Big]^{\frac{1}{k}}, \tag{10}$$

this fact together with hypothesis (A_1) and use the same way as above, we can conclude the conclusion. $\square$

Now we construct a fixed point operator $\mathcal{A}$ such that its fixed points are solutions of (7). Let us define

$$S : C \to C,$$
$$S(v)(r) = \frac{n}{r^{n-k}}\int_0^r \tau^{n-1} v(\tau)d\tau \quad (r \in (0,R]), \quad S(v)(0) = 0;$$
$$K : C \to C,$$
$$K(v)(r) = -\int_r^R v(\tau)d\tau \quad (r \in (0,R]).$$

An easy computation shows that, for any $h \in C$ (h is nonnegative if k is even), the mixed problem

$$(r^{n-k}\phi^k(v'))' = nr^{n-1}h, \quad |v'| < 1, \quad v'(0) = v(R) = 0$$

has a unique solution v given by

$$v = K \circ (\phi^{-1})^{\frac{1}{k}} \circ S \circ h,$$

where $(\phi^{-1})^{\frac{1}{k}}(y) := \phi^{-1}(y^{\frac{1}{k}})$. Moreover, from the compactness of S and K it follows that $K \circ (\phi^{-1})^{\frac{1}{k}} \circ S : C \to C$ is compact. Consider now the Nemytskii type operator associated to H_k,

$$N_{H_k} : B_\alpha \to C, \qquad N_{H_k}(v) = H_k(\cdot, v(\cdot)).$$

Clearly, N_{H_k} is continuous and $N_{H_k}(\bar{B}_\rho)$ is a bounded subset of C for any $\rho < \alpha$. So, we have the following fixed point reformulation of (7).

Proposition 2.2. *A function $v \in C$ is a solution of* (7) *if and only if it is a fixed point of the continuous nonlinear operator*

$$\mathcal{A} : B_\alpha \to C, \qquad \mathcal{A} = K \circ (\phi^{-1})^{\frac{1}{k}} \circ S \circ N_{H_k}.$$

Moreover, $\mathcal{A}$ is compact on $\bar{B}_\rho$ for all $\rho \in (0,\alpha)$.

Next we assume that H_k is superlinear with respect to ϕ^k at 0 for each $1 \leq k \leq n$, and we prove that the Leray-Schauder degree $d_{LS}[I - \mathcal{A}, B_\rho, 0]$ is zero for all sufficiently small ρ.

Proposition 2.3. *Assume that (A_1). If for each $1 \leq k \leq n$,*

$$\lim_{s\to 0} \frac{H_k(r,s)}{\phi^k(s)} = \infty \quad \textit{uniformly for } r \in [0,R], \tag{11}$$

then there exists $0 < \rho_0 < \alpha$ such that

$$d_{LS}[I - \mathcal{A}, B_\rho, 0] = 0 \ \ \textit{for all} \ \ 0 < \rho \leq \rho_0.$$

Proof. From the definition of ϕ, we have that

$$\lim_{s\to 0} \frac{\phi^k(\tau s)}{\phi^k(s)} = \tau^k < +\infty \quad \text{for all } \tau > 0, \tag{12}$$

therefore, if we let $\tau = \frac{3}{R}$ in (12), then there exists $m > 0$ such that

$$\lim_{s\to 0} \frac{\phi^k(3s/R)}{s^k} < \frac{m(R/3)^n}{(2R/3)^{n-k}}. \tag{13}$$

Case 1: k is odd.

In this case, we denote $\psi = \phi^k$, then $\psi : (-1,1) \to \mathbb{R}$ is also an odd, increasing diffeomorphism and such that $\psi(0) = 0$. First, we show that there exists $\rho_0 \in (0,\alpha)$ such that the perturbed problem

$$\begin{cases} (r^{n-k}\psi(v'))' = nr^{n-1}[H_k(r,v) - M], & r \in (0,R), \\ |v'| < 1, & r \in (0,R), \\ v'(0) = v(R) = 0 \end{cases} \tag{14}$$

has at most the trivial solution in $\bar{B}_{\rho_0}$, for any $M \geq 0$ and k. By contradiction, assume that there exist sequences $\{M_j\} \subset [0,\infty)$ and $\{v_j\} \subset C\backslash\{0\}$ with $\|v_j\|_\infty \to 0$, such that v_j is a solution of (14) with $M = M_j$, for all $j \in \mathbb{N}$. By virtue of Proposition 2.1 one has that $v_j > 0$ on $[0,R)$ and v_j is strictly decreasing.

For the same $m > 0$ in (13) and use (A_1) and (11), we can find $j_0 \in \mathbb{N}$ such that

$$-H_k(r, v_j(r)) \geq m(v_j(r))^k \quad \text{for all } r \in [0,R] \text{ and } j \geq j_0. \tag{15}$$

Integrating (14) with $M = M_j$ and $v = v_j$ over $[0,r]$ and taking into account (15), we get that

$$-\psi(v_j') \geq mS[(v_j)^k].$$

By the oddness and monotonicity of ψ, we infer that

$$-v_j' \geq \psi^{-1}(mS[(v_j)^k]).$$

Integrating the above inequality on $[\frac{R}{3}, \frac{2R}{3}]$ one obtains that

$$v_j\Big(\frac{R}{3}\Big) - v_j\Big(\frac{2R}{3}\Big) \geq \int_{\frac{R}{3}}^{\frac{2R}{3}} \psi^{-1}\Big(\frac{mn}{r^{n-k}}\int_0^r \tau^{n-1}(v_j(\tau))^k d\tau\Big)dr,$$

and, using that v_j is strictly decreasing on $[0,R]$ and $v_j > 0$ on $[0,R)$, it follows that

$$\begin{aligned} v_j\Big(\frac{R}{3}\Big) &\geq \int_{\frac{R}{3}}^{\frac{2R}{3}} \psi^{-1}\Big(\frac{mn}{(2R/3)^{n-k}}\int_0^{\frac{R}{3}} \tau^{n-1}(v_j(\tau))^k d\tau\Big)dr \\ &\geq \frac{R}{3}\psi^{-1}\Big(\frac{m(R/3)^n(v_j(R/3))^k}{(2R/3)^{n-k}}\Big) \end{aligned}$$

for all $j \geq j_0$. This implies that

$$\frac{\psi(v_j(R/3)3/R)}{(v_j(R/3))^k} \geq \frac{m(R/3)^n}{(2R/3)^{n-k}},$$

which together with $v_j(\frac{R}{3}) \to 0$ contradict (13).

Therefore, (14) has no solution in $\bar{B}_{\rho_0}$, for any $M > 0$.

Now, let $0 < \rho \leq \rho_0$ and consider the family of problems

$$\begin{cases} (r^{n-k}\psi(v'))' = nr^{n-1}[H_k(r,v) - \lambda], & r \in (0,R), \\ |v'| < 1, & r \in (0,R), \\ v'(0) = v(R) = 0, \end{cases} \tag{16}$$

where $\lambda \in [0,1]$. Let $\mathcal{H}(\lambda,\cdot) : B_\alpha \to C$ be the fixed point operator associated to (16). Note that $\mathcal{H}(0,\cdot) = \mathcal{A}$ and $\mathcal{H} : [0,1]\times\bar{B}_\rho \to C$ is a compact homotopy. Also the Leray-Schauder condition on the boundary

$$v \neq \mathcal{H}(\lambda, v) \quad \text{ for all } (\lambda, v) \in [0,1]\times\partial B_\rho$$

is fulfilled. This implies that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0].$$

From the previous arguments one has that

$$v \neq \mathcal{H}(1, v) \quad \text{ for all } \ v \in \bar{B}_\rho,$$

this implies that

$$d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = 0.$$

Consequently,

$$d_{LS}[I - \mathcal{A}, B_\rho, 0] = d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = 0.$$

Case 2: k is even

In this case, let $M \geq 0$, and we consider the perturbed problem

$$\begin{cases} (r^{n-k}\phi^k(v'))' = nr^{n-1}[H_k(r,v) + M], & r \in (0,R), \\ |v'| < 1, & r \in (0,R), \\ v'(0) = v(R) = 0. \end{cases} \tag{17}$$

According to the previous proof, the key is to prove that there exists $\rho_0 \in (0, \alpha)$ such that (17) has no solution in $\bar{B}_{\rho_0}$, for any $M > 0$. To do this, we show that there exists $\rho_0 \in (0, \alpha)$ such that the perturbed problem (17) has at most the trivial solution in $\bar{B}_{\rho_0}$, for any $M \geq 0$ and k.

By contradiction, assume that there exist sequences $\{M_j\} \subset [0, \infty)$ and $\{v_j\} \subset C \backslash \{0\}$ with $\|v_j\|_\infty \to 0$, such that v_j is a solution of (17) with $M = M_j$, for all $j \in \mathbb{N}$. By virtue of Proposition 2.1 one has that either $v_j > 0$ on $[0, R)$ and v_j is strictly decreasing or $v_j < 0$ on $[0, R)$ and v_j is strictly increasing.

For the same $m > 0$ in (13), and use (A_1) and (11) again, we can find $j_0 \in \mathbb{N}$ such that

$$H_k(r, v_j(r)) \geq m(v_j(r))^k \quad \text{for all } r \in [0, R] \text{ and } j \geq j_0. \tag{18}$$

Integrating (17) with $M = M_j$ and $v = v_j$ over $[0, r]$ and taking into account (18), we get that

$$\phi^k(v_j') \geq mS[(v_j)^k].$$

Since k is even, therefore by the oddness and monotonicity of ϕ, we have that

$$v_j' \geq \phi^{-1}\Big[\Big(mS[(v_j)^k]\Big)^{\frac{1}{k}}\Big] > 0 \tag{19}$$

or

$$-v_j' \geq \phi^{-1}\Big[\Big(mS[(v_j)^k]\Big)^{\frac{1}{k}}\Big] > 0. \tag{20}$$

It is easy to see that the v_j in (19) denote the strictly increasing negative solutions in $[0, R)$, and the v_j in (20) denote the strictly decreasing positive solutions in $[0, R)$.

Integrating (19) on $[\frac{R}{3}, \frac{2R}{3}]$, notice that v_j is strictly increasing on $[0, R]$ and $v_j < 0$ on $[0, R)$, and using the oddness and monotonicity of ϕ again, we get that

$$\begin{aligned}
-v_j\Big(\frac{R}{3}\Big) &\geq \int_{\frac{R}{3}}^{\frac{2R}{3}} \phi^{-1}\Big[\Big(\frac{mn}{r^{n-k}} \int_0^r \tau^{n-1}(v_j(\tau))^k d\tau\Big)^{\frac{1}{k}}\Big] dr \\
&\geq \int_{\frac{R}{3}}^{\frac{2R}{3}} \phi^{-1}\Big[\Big(\frac{mn}{(2R/3)^{n-k}} \int_0^{\frac{R}{3}} \tau^{n-1}(v_j(R/3))^k d\tau\Big)^{\frac{1}{k}}\Big] dr \\
&= \frac{R}{3}\phi^{-1}\Big(\frac{m(R/3)^n(v_j(R/3))^k}{(2R/3)^{n-k}}\Big)^{\frac{1}{k}}
\end{aligned}$$

for all $j \geq j_0$. This implies that

$$\frac{\phi^k(v_j(R/3)3/R)}{(v_j(R/3))^k} \geq \frac{m(R/3)^n}{(2R/3)^{n-k}},$$

which together with $v_j(\frac{R}{3}) \to 0$ contradict (13).

Similarly, let us integrate (20) on $[\frac{R}{3}, \frac{2R}{3}]$, we can obtain the same contradiction.

Finally, let $0 < \rho \le \rho_0$ and we consider the family of problems

$$\begin{cases} (r^{n-k}\phi^k(v'))' = nr^{n-1}[H_k(r,v)+\lambda], & r \in (0,R), \\ |v'| < 1, & r \in (0,R), \\ v'(0) = v(R) = 0, \end{cases} \tag{21}$$

where $\lambda \in [0,1]$, using the same way to (16), we can get the desired result. We complete the proof. □

Proposition 2.4. *If $R < \alpha$, then one has that*

$$d_{LS}[I - \mathcal{A}, B_R, 0] = 1.$$

Proof. Consider the compact homotopy

$$\mathcal{H} : [0,1] \times \bar{B}_R \to C, \quad \mathcal{H}(\lambda, v) = \lambda \mathcal{A}(v).$$

Notice that $\mathcal{H}(0,\cdot) = 0$ and $\mathcal{H}(1,\cdot) = \mathcal{A}$. Let $(\lambda, v) \in [0,1] \times \bar{B}_R$ be such that $\mathcal{H}(\lambda, v) = v$. It follows immediately that $\|v'\|_\infty < 1$, implying that $\|v\|_\infty < R$. So,

$$v \neq \mathcal{H}(\lambda, v) \quad \text{for all } (\lambda, v) \in [0,1] \times \partial B_R,$$

which implies that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_R, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_R, 0].$$

Consequently,

$$d_{LS}[I - \mathcal{A}, B_R, 0] = d_{LS}[I, B_R, 0] = 1,$$

and the proof is completed. □

Note that in Proposition 2.4, if $\alpha = 1$, then $R < 1$. In this case, H_k can be singular at ± 1. We consider now the case $R = \alpha = 1$, and assume H_k is sublinear with respect to ϕ^k at 1 and -1.

Proposition 2.5. *Assume that $R = \alpha = 1$. If for each $1 \le k \le n$,*

$$\lim_{|s| \to 1^-} \frac{H_k(r,s)}{\phi^k(s)} = 0 \quad \textit{uniformly for } r \in [0,1], \tag{22}$$

then there exists $0 < \delta_1 < 1$ such that

$$d_{LS}[I - \mathcal{A}, B_\delta, 0] = 1 \quad \textit{for all} \ \ \delta_1 \le \delta < 1.$$

Proof. Consider the family of problems

$$\begin{cases} (r^{n-k}\phi^k(v'))' = \lambda nr^{n-1}H_k(r,v), & r \in (0,1), \\ |v'| < 1, & r \in (0,1), \\ v'(0) = 0 = v(1), \end{cases} \tag{23}$$

where $\lambda \in [0,1]$. Let $\mathcal{H}(\lambda,\cdot) : B_1 \to C$ be the fixed point operator associated to (23). Notice that $\mathcal{H}(1,\cdot) = \mathcal{A}$ and $\mathcal{H} : [0,1] \times \bar{B}_\rho \to C$ is a compact homotopy for all $0 < \rho < 1$. We show that there exists $0 < \delta_1 < 1$ such that

$$v \neq \mathcal{H}(\lambda, v) \quad \text{for all } (\lambda, v) \in [0,1] \times (B_1 \backslash B_{\delta_1}). \tag{24}$$

By contradiction, assume that there exist sequences $\{\lambda_j\} \subset [0,1]$ and $\{v_j\} \subset C\backslash\{0\}$ such that $1 > \|v_j\|_\infty \to 1$ and $v_j = \mathcal{H}(\lambda_j, v_j)$. Clearly, $\lambda_j > 0$ for all $j \in \mathbb{N}$ and, from Proposition 2.1 one has that: if k is odd, then $v_j > 0$ on $[0,1)$ and v_j is strictly decreasing, if k is even, then either $v_j > 0$ on $[0,1)$ and v_j is strictly decreasing or $v_j < 0$ on $[0,1)$ and v_j is strictly increasing.

Case 1: k is odd

Let $\{\varepsilon_j\} \subset (0,\infty)$ be such that $\varepsilon_j \to 0$. From (A_1) and (22), we have that for each k,

$$\lim_{s\to 1^-} \frac{-H_k(r,s)}{\phi^k(s)} = 0 \quad \text{uniformly for } r \in [0,1],$$

and subsequently, there exists $\{c_j\} \subset (0,\infty)$ such that

$$-H_k(r,s) \le \varepsilon_j \phi^k(s) + c_j \quad \text{for all } (r,s) \in [0,1]\times[0,1). \tag{25}$$

As $\|v_j\|_\infty \to 1$, we can find a subsequence $\{v_{j_s}\}$ of $\{v_j\}$ satisfying

$$\varepsilon_j \phi^k(\|v_{j_s}\|_\infty) \ge c_j \quad \text{for all} \quad j \in \mathbb{N}. \tag{26}$$

Using the oddness, monotonicity of ϕ^{-1} and $v_j = \mathcal{H}(\lambda_j, v_j)$, we obtain that

$$v_j(0) \le \int_0^1 \phi^{-1}\Big[\Big(\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}(-H_k(\tau, v_j(\tau)))d\tau\Big)^{\frac{1}{k}}\Big]dr,$$

this together with (25) and (26), we have that

$$\|v_{j_s}\|_\infty \le \phi^{-1}[(2\,\varepsilon_{j_s}\phi^k(\|v_{j_s}\|_\infty))^{\frac{1}{k}}], \tag{27}$$

and then

$$\varepsilon_{j_s} \ge \frac{1}{2} \quad \text{for all} \quad j \in \mathbb{N}. \tag{28}$$

This contradicts $\varepsilon_{j_s} \to 0$ as $j \to \infty$.

Case 2: k is even and $v_j > 0$ on $[0,1)$

We also let $\{\varepsilon_j\} \subset (0,\infty)$ be such that $\varepsilon_j \to 0$. From (A_1) and (22), we know that for each k,

$$\lim_{s\to 1^-} \frac{H_k(r,s)}{\phi^k(s)} = 0 \quad \text{uniformly for } r \in [0,1],$$

and subsequently, there exists $\{c_j\} \subset (0,\infty)$ such that

$$H_k(r,s) \le \varepsilon_j \phi^k(s) + c_j \quad \text{for all } (r,s) \in [0,1]\times[0,1). \tag{29}$$

As $\|v_j\|_\infty \to 1$, we can find a subsequence $\{v_{j_s}\}$ of $\{v_j\}$ satisfying

$$\varepsilon_j \phi^k(\|v_{j_s}\|_\infty) \ge c_j \quad \text{for all} \quad j \in \mathbb{N}. \tag{30}$$

From $v_j = \mathcal{H}(\lambda_j, v_j)$, we have that

$$v_j'(r) = \phi^{-1}\Big[-\Big(\frac{n}{r^{n-k}}\int_0^r \lambda_j \tau^{n-1} H_k(\tau, v_j(\tau))d\tau\Big)^{\frac{1}{k}}\Big].$$

Integrating it on $[0,1]$ and using the oddness, monotonicity of ϕ^{-1}, (29) and (30), we can also obtain that (27) and (28). But we have already known this is a contradiction.

Case 3: k is even and $v_j < 0$ on $[0,1)$

Similarly, let $\{\varepsilon_j\} \subset (0,\infty)$ be such that $\varepsilon_j \to 0$. From (A_1) and (22), we know that for each k,

$$\lim_{s\to -1^+} \frac{H_k(r,s)}{\phi^k(s)} = 0 \quad \text{uniformly for } r \in [0,R],$$

and subsequently, there exists $\{c_j\} \subset (0,\infty)$ such that

$$H_k(r,s) \le \varepsilon_j \phi^k(s) + c_j \quad \text{for all } (r,s) \in [0,1]\times[0,1). \tag{31}$$

As $\|v_j\|_\infty \to 1$, we can find a subsequence $\{v_{j_s}\}$ of $\{v_j\}$ satisfying

$$\varepsilon_j \phi^k(\|v_{j_s}\|_\infty) \ge c_j \quad \text{for all } \ j \in \mathbb{N}. \tag{32}$$

From $v_j = \mathcal{H}(\lambda_j, v_j)$ we have that

$$v_j'(r) = \phi^{-1}\Big[\Big(\frac{n}{r^{n-k}}\int_0^r \lambda_j \tau^{n-1} H_k(\tau, v_j(\tau))d\tau\Big)^{\frac{1}{k}}\Big],$$

and using the monotonicity of ϕ^{-1} we get that

$$-v_j(0) \le \int_0^1 \phi^{-1}\Big[\Big(\frac{n}{r^{n-k}}\int_0^r \tau^{n-1} H_k(\tau, v_j(\tau))d\tau\Big)^{\frac{1}{k}}\Big]dr.$$

Similar to Case 1, and here using (31) and (32), we can also obtain the same contradiction.

Summarize the conclusions of above, we know that (24) is true. It follows that, for any $\delta_1 \le \delta < 1$, one has that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_\delta, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\delta, 0].$$

Consequently,

$$d_{LS}[I - \mathcal{A}, B_\delta, 0] = d_{LS}[I, B_\delta, 0] = 1,$$

and the proof is completed. □

Theorem 2.6. *Assume that (A_1) and (11) are fulfilled. Moreover,*

(i) *Assume $R < \alpha$. Then if k is odd, (7) has at least one strictly decreasing positive solution; if k is even, (7) has at least two solutions, one is strictly decreasing and positive the other is strictly increasing and negative.*

(ii) *Assume $R = \alpha = 1$ and (22) is fulfilled. Then if k is odd, (7) has at least one strictly decreasing positive solution; if k is even, (7) has at least two solutions, one is strictly decreasing and positive the other is strictly increasing and negative.*

Proof. (i) Assume that $R < \alpha$ and let ρ_0 be given in Proposition 2.3. We pick $\rho \in (0, \min\{\rho_0, R\})$. From Propositions 2.3, 2.4 it follows that

$$d_{LS}[I - \mathcal{A}, B_R \backslash \bar{B}_\rho, 0] = d_{LS}[I - \mathcal{A}, B_R, 0] - d_{LS}[I - \mathcal{A}, B_\rho, 0] = 1,$$

which ensures the existence of some $v \in B_R \backslash \bar{B}_\rho$, with $v = \mathcal{A}(v)$. Consequently, v is a nontrivial solution of (7). This together with Proposition 2.1, we have the result (i).

(ii) If $\alpha = R = 1$ and (22) is satisfied, then the proof follows exactly as above but with Proposition 2.5 instead of Proposition 2.4. $\square$

Remark 2.7. The literature [19] does not provide the information about the sign of the solutions of (7) when k is odd. If $H_k(r, 0) = 0$ for $r \in [0, R]$, then $v = 0$ is a solution of (7), and the result of [19] is invalid. However, our Theorem 2.6 get a positive solution in this case.

Remark 2.8. It is worth to point out that the existence result established when k is odd is consistent with the result of [4], but they only considered the case of $k = 1$. On the other hand, the existence result when k is even is completely different from the case of k is odd.

3. Existence of entire solutions

In this section, we provide some conditions to guarantee that every solution v given by Theorem 2.6, once R is fixed, can be continued until $+\infty$ as a solution of (3). The ideas of this section comes from the work of de la Fuente, Romero and Torres [19].

For each $k : 1 \le k \le n$, we shall make the following assumptions:

(A_2) $(-1)^k H_k(r, s) \ge 0$ for all $(r, s) \in \mathbb{R}^+ \times (-\alpha, \alpha)$ with $0 < \alpha \le \infty$;

(A_3) $(-1)^k H_k(r, s) > 0$ for all $(r, s) \in (0, +\infty) \times (-\alpha, 0) \cup (0, \alpha)$ with $0 < \alpha \le \infty$;

(A_4) $\lim\limits_{s \to 0} \frac{H_k(r,s)}{\phi^k(s)} = \infty$ uniformly for $r \in \mathbb{R}^+$;

(A_5) $\lim\limits_{|s| \to 1^-} \frac{H_k(r,s)}{\phi^k(s)} = 0$ uniformly for $r \in \mathbb{R}^+$.

And we have the following results.

Theorem 3.1. *Assume that* (A_2)-(A_4) (*or* (A_2)-(A_5)) *hold and k is odd. Then for each* $0 < R < \alpha$ (*or* $R = \alpha = 1$), (3) *has at least one entire solution whose k-th mean curvature equals to H_k, and the profile curve is decreasing.*

Theorem 3.2. *Assume that* (A_2)-(A_4) (*or* (A_2)-(A_5)) *hold and k is even. Then for each* $0 < R < \alpha$ (*or* $R = \alpha = 1$), (3) *has at least two entire solutions whose k-th mean curvature equal to H_k and the profile curve of one of them is decreasing and the other one is increasing.*

We first give the following result.

Proposition 3.3. *Let* (A_2) *and* (A_3) *hold. Let* $\rho \in (0, \infty)$ *be fixed. Assume that v is a nontrivial solution of*

$$\begin{cases} (r^{n-k}\phi^k(v'))' = nr^{n-1}H_k(r, v), & r \in (0, \rho), \\ |v'| < 1, & r \in (0, \rho), \\ v'(0) = 0 = v(\rho). \end{cases} \tag{33}$$

Then there exists a constant $\sigma \in (0,1)$ *such that*

$$|v'(r)| < 1-\sigma, \quad r \in [0,\rho]. \tag{34}$$

Proof. (33) is equivalent to

$$|v'(r)| = \Big|\,\phi^{-1}\Big(\Big[\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}\ H_k(\tau, v(\tau))d\tau\Big]^{\frac{1}{k}}\Big)\,\Big| \qquad \text{for } r \in (0,\rho].$$

This together with the definition of ϕ and the fact

$$\max\Big\{\frac{n}{r^{n-k}}\int_0^r \tau^{n-1}\ H_k(\tau, v(\tau))d\tau : r \in (0,\rho]\Big\} \ < \infty$$

imply that there exists $\sigma \in (0,1)$ such that (34) is valid. □

And, next we give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let v be a nontrivial solution of equation (3), and let $[0,b)$ be the maximal interval of definition of v. Suppose on the contrary that $b < +\infty$. We can rewrite equation (3) as a system of two ordinary differential equations of first order

$$\begin{cases} v'(r) = \phi^{-1}\Big[\Big(\dfrac{z(r)}{r^{n-k}}\Big)^{\frac{1}{k}}\Big], \\ z'(r) = nr^{n-1}H_k(r, v(r)), \end{cases}$$

to abbreviate, we denote

$$\begin{pmatrix} v' \\ z' \end{pmatrix} = \mathcal{F}(r, v, z),$$

where $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$.

By the standard prolongability theorem of ordinary differential equations [31], we have that the graph $\{(r, v(r), z(r)) : r \in [0,b)\}$ goes out of any compact subset of $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$.

However, by Proposition 3.3, $|v'(r)| < 1-\sigma$ for $r \in [0,b)$, then

$$|v(r)| < b(1-\sigma), \quad r \in [0,b).$$

Therefore, the graph can not go out of the compact subset $[0,b] \times [-b(1-\sigma), b(1-\sigma)] \times [-b^{n-k}\phi^k(1-\sigma),\, b^{n-k}\phi^k(1-\sigma)]$ contained in the domain of $\mathcal{F}$. This is a contradiction. Therefore, $b = +\infty$. □

Similarly, we have the proof of Theorem 3.2.

Proof of Theorem 3.2. We rewrite equation of (3) as the first order differential system

$$\begin{cases} v'(r) = \phi^{-1}\Big[\pm\Big(\dfrac{z(r)}{r^{n-k}}\Big)^{\frac{1}{k}}\Big], \\ z'(r) = nr^{n-1}H_k(r, v(r)), \end{cases}$$

use the prolongability theorem of ordinary differential equations [31], Proposition 3.3, and argue similarly to the proof of Theorem 3.1, we can obtain the conclusion. □

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