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THE KERNELS OF THE LINEAR MAPS OF FINITE GROUP ALGEBRAS

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Abstract. Let G be a finite group, K a split field for G, and L a linear map from $K[G]$ to K. In our paper, we first give sufficient and necessary conditions for $\operatorname{Ker} L$ and $\operatorname{Ker} L \cap Z(K[G])$, respectively, to be Mathieu-Zhao spaces for some linear maps L. Then we give equivalent conditions for $\operatorname{Ker} L$ to be Mathieu-Zhao spaces of $K[G]$ in term of the degrees of irreducible representations of G over K if G is a finite Abelian group or G has a normal Sylow p-subgroup H and L are class functions of G/H. In particular, we classify all Mathieu-Zhao spaces of the finite Abelian group algebras if K is a split field for G.

1. Introduction

Throughout this paper, we will write K for a field without specific note and $K[G]$ for the group algebra of G over K. V_G is the K-subspace of the group algebra $K[G]$ consisting of all the elements of $K[G]$ whose coefficient of the identity element 1_G of G is equal to zero. It is easy to see that V_G is a subspace of $K[G]$ with codimension one. Let L be a linear map from $K[G]$ to K and $L|_H$ means restricting L to H, where H is a subgroup of G. We call H a p'-subgroup of G if $p \nmid |H|$. Let

$$\tau : K[H] \to K$$

such that $\tau(\sum a_x x) = \sum a_x$. Then $w(K[H]) := \operatorname{Ker} \tau$, which is called the augmentation ideal of $K[H]$. It's equal to $\sum_{h_i \in H}(h_i - 1)K[H]$ for any subgroup H of G and $w(K[H])K[G]$ is $\sum_{h_i \in H}(h_i - 1)K[G]$.

The Mathieu-Zhao space was introduced by W. Zhao in [7], which is a natural generalization of ideals, motivated by a conjecture of O. Mathieu. The term Mathieu-Zhao space was suggested and used by A. van den Essen. We recall

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the definitions of Mathieu-Zhao spaces of $K[G]$ and the radical of a subspace of $K[G]$. We say that a K-subspace M of $K[G]$ is called a Mathieu-Zhao space of $K[G]$ if for any $a, b \in K[G]$ with $a^m \in M$ for all $m \geq 1$, we have $ba^m \in M$ when $m \gg 0$. Let S be a K-subspace of $K[G]$. The radical of S is the set of all elements $a \in K[G]$ such that $a^m \in S$ when $m \gg 0$. We say that a subspace of $K[G]$ has MZ-property if it is a Mathieu-Zhao space of $K[G]$. In [1], J. J. Duistermaat and W. van der Kallen proved the Mathieu conjecture for the case of tori, which can be re-stated as follows.

Theorem 1.1. *Let $z = (z_1, z_2, \ldots, z_m)$ be m commutative free variables and V the subspace of the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ consisting of the Laurent polynomials with no constant term. Then V is a Mathieu-Zhao space of $\mathbb{C}[z^{-1}, z]$.*

Let G be the free Abelian group $\mathbb{Z}^m$ $(m \geq 1)$. Then the Laurent polynomial algebra $\mathbb{C}[z^{-1}, z]$ can be identified with the group algebra $\mathbb{C}[G]$. Under this identification, the subspace of V in the theorem is V_G. In [9], W. Zhao and R. Willems proved that V_G is a Mathieu-Zhao space of $K[G]$ if G is a finite group and $\operatorname{char} K = 0$ or $\operatorname{char} K = p > |G|$. For finite Abelian group, they proved that if K contains a primitive d-th root of unity and $\operatorname{char} K = p$, then V_G is a Mathieu-Zhao space of $K[G]$ if and only if $\operatorname{char} K = p > d$, where $|G| = p^a d$, $p \nmid d$. In [10], W. Zhao and the author give a sufficient and necessary condition for V_G to be a Mathieu-Zhao space of $K[G]$ if G is a finite group and K is a split field for G. Since V_G is just one subspace of $K[G]$ with codimension one, we first want to consider all subspaces of $K[G]$ with codimension one. Then we want to consider all subspaces of $K[G]$. Hence it is natural to ask the following question.

Problem 1.2. Let G be a finite group with $|G| = n$, $L = (L_1, L_2, \ldots, L_r)$ and L_i be a linear map from $K[G]$ to K such that $L_i(g_j) = l_{i,j}$ for all $1 \leq i \leq r$, $1 \leq j \leq n$. Suppose that $L_1, L_2, \ldots, L_r$ are linearly independent over K. Then under what conditions on L and K, $\operatorname{Ker} L$ forms a Mathieu-Zhao space of the group algebra $K[G]$?

It's easy to see that if $r \geq n$, then $\operatorname{Ker} L = 0$. If $r \leq n-1$, then $\dim_K \operatorname{Ker} L = n - r$ and every codimension r subspace of $K[G]$ is $\operatorname{Ker} L$ for some linear map L. Hence $\operatorname{Ker} L$ are all the codimension r subspaces of $K[G]$.

In our paper, we first prove some properties of $\operatorname{Ker} L$ and $\operatorname{Ker} L \cap Z(K[G])$ in Section 2. In Section 3, we give sufficient and necessary conditions for $\operatorname{Ker} L$ and $\operatorname{Ker} L \cap Z(K[G])$, respectively, to be Mathieu-Zhao spaces for some linear maps L. Then we classify all Mathieu-Zhao spaces of $K[G]$ if G is a finite Abelian group and K a split field for G in Section 4. Thus, we solve Problem 1.2 if G is a finite Abelian group. In Section 5, we give equivalent conditions for $\operatorname{Ker} L$ to be Mathieu-Zhao spaces of $K[G]$ in term of the degrees of irreducible representations of G over K if G has a normal Sylow p-subgroup H and L are class functions of G/H or $L_1, \ldots, L_{r-1}$ are class functions of G/H and

$L_r(\tilde{g}_j h_2) = L_r(\tilde{g}_j h_3) = \cdots = L_r(\tilde{g}_j h_{\tilde{t}})$ for all $1 \le j \le d$, where $G = \cup_{j=1}^d \tilde{g}_j H$, $H = \{1_H, h_2, \ldots, h_{\tilde{t}}\}$.

2. Some properties of Ker$L$ and Ker$L \cap Z(K[G])$

Proposition 2.1. *Let $L = (L_1, L_2, \ldots, L_r)$ and L_i be a linear map from $K[G]$ to K such that $L_i(g_j) = l_{i,j}$ for all $1 \le i \le r$, $1 \le j \le n$. K, G be as in Problem 1.2 and g_1 be the identity 1_G of G. Then we have the following statements*:

(1) *If all the $l_{i,j}$ are equal for all $1 \le i \le r$, $1 \le j \le n$, then* $\operatorname{Ker} L$ *is an ideal of $K[G]$.*

(2) *If* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of $K[G]$, then there exists $i_0 \in \{1, 2, \ldots, r\}$ such that $l_{i_0,1} \neq 0$.*

Proof. (1) Let $l := l_{i,j}$ for all $1 \le i \le r$, $1 \le j \le n$. Then $\operatorname{Ker} L = \{\sum_{j=1}^n c_j g_j \in K[G] \mid l \cdot \sum_{j=1}^n c_j = 0\}$. Since $l \neq 0$, we have that $\operatorname{Ker} L = \{\sum_{j=1}^n c_j g_j \in K[G] \mid \sum_{j=1}^n c_j = 0\}$. It is easy to check that $\operatorname{Ker} L$ is an ideal of $K[G]$.

(2) If $l_{1,1} = \cdots = l_{r,1} = 0$, then $1_G \in \operatorname{Ker} L$. If $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$, then $\operatorname{Ker} L = K[G]$. That is, $L = 0$, which is a contradiction. Then the conclusion follows. □

Remark 2.2. We can see from Proposition 2.1 that we can assume $l_{i_0,1} \neq 0$ for some $i_0 \in \{1, 2, \ldots, r\}$ in the following arguments. If $r = 1$ and $l_{1,2} = l_{1,3} = \cdots = l_{1,n} = 0$, $l_{1,1} \neq 0$, then $\operatorname{Ker} L = V_G$, which is discussed in [9] and [10].

Proposition 2.3. *Let R be any commutative ring and G any group. Suppose that $L = (L_1, L_2, \ldots, L_r)$ is a linear map from $R[G]$ to R. If* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of $R[G]$, then* $\operatorname{Ker}(L|_H)$ *is a Mathieu-Zhao space of $R[H]$, where H is any subgroup of G.*

Proof. Assume otherwise. Then there exist $u, v_1, v_2 \in R[H]$ such that $u^m \in \operatorname{Ker}(L|_H)$ for all $m \ge 1$ and $v_1 u^m v_2 \notin \operatorname{Ker}(L|_H)$ for infinitely many $m \ge 1$. Since $R[H] \subseteq R[G]$, we have $u, v_1, v_2 \in R[G]$ and $u^m \in \operatorname{Ker} L$ for all $m \ge 1$ and $v_1 u^m v_2 \notin \operatorname{Ker} L$ for infinitely many $m \ge 1$. Otherwise, $v_1 u^m v_2 \in \operatorname{Ker} L \cap R[H] = \operatorname{Ker}(L|_H)$, which is a contradiction. Hence $\operatorname{Ker} L$ is not a Mathieu-Zhao space of $R[G]$, which is a contradiction. Then the conclusion follows. □

Corollary 2.4. *Let L, G be as in Problem 1.2 and K a field of characteristic p, H a normal subgroup of G. If H is a p'-subgroup and* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of $K[G]$, then* $\operatorname{Ker}(L|_{G/H})$ *is a Mathieu-Zhao space of $K[G/H]$.*

Proof. Let φ be the natural surjective homomorphism from $K[G]$ to $K[G/H]$ and $E_H = \frac{1}{|H|} \sum_{j=1}^{|H|} h_j$. Then $(1 - E_H)K[G] = \operatorname{Ker} \varphi$ and $E_H K[G] \cong K[G/H]$. Thus, we have $K[G] \cong (1 - E_H)K[G] \oplus K[G/H]$. Therefore, $K[G/H]$ can be seen as a subalgebra of $K[G]$. It follows from the arguments of Proposition 2.3 that $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. □

Proposition 2.5. *Let K and L be as in Problem 1.2 and G a finite group with $G = \{g_1, g_2, \dots, g_n\}$, $g_1 = 1_G$. If there exists $\tilde{i} \in \{1, 2, \dots, r\}$ such that $\det M_{L_{\tilde{i}}} \neq 0$, then $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if all elements of $r(\operatorname{Ker} L)$ are nilpotent, where $M_{L_{\tilde{i}}} = (l_{\tilde{i},j_{1,2}})_{n\times n}$ and $l_{\tilde{i},j_{1,2}} = L_{\tilde{i}}(g_{j_1}^{-1} g_{j_2})$ for $1 \leq j_1,\ j_2 \leq n$.*

Proof. ($\Leftarrow$) It follows from the definition of Mathieu-Zhao spaces.

($\Rightarrow$) Let $u \in r(\operatorname{Ker} L)$. Replacing u by a positive power of u, if necessary, we may assume that $u^m \in \operatorname{Ker} L$ for all $m \geq 1$. Since G is finite, by definition of Mathieu-Zhao space, there exists $N \geq 1$ such that $g_{j_1}^{-1} u^m \in \operatorname{Ker} L$ for all $g_{j_1} \in G$ and $m \geq N$. Let $u^N = \sum_{j_2=1}^{n} d_{j_2} g_{j_2}$. Then we have $g_{j_1}^{-1} u^N \in \operatorname{Ker} L$ for all $1 \leq j_1 \leq n$. That is,

$$M_{L_i} \cdot \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = 0 \tag{2.1}$$

for all $1 \leq i \leq r$. Since there exists $\tilde{i} \in \{1, 2, \dots, r\}$ such that $\det M_{L_{\tilde{i}}} \neq 0$, we have that $d_1 = \cdots = d_n = 0$. That is, $u^N = 0$. Thus, u is nilpotent. □

Remark 2.6. If $l_{i,1} = 1$ and $l_{i,2} = \cdots = l_{i,n} = 0$ for some $i \in \{1, 2, \dots, r\}$, then M_{L_i} is the identity matrix. Thus, we have $\det M_{L_i} = 1$ in this case. It is easy to see that $\det M_{L_i}$ is the group determinant of G up to a sign for $1 \leq i \leq r$.

Corollary 2.7. *Let K and L be as in Problem 1.2 and G a finite group with $G = \{g_1, g_2, \dots, g_n\}$, $g_1 = 1_G$. If there exists $\tilde{i} \in \{1, 2, \dots, r\}$ such that $\det M_{L_{\tilde{i}}} \neq 0$, then $\operatorname{Ker} L \cap Z(K[G])$ is a Mathieu-Zhao space of $K[G]$ if and only if all elements of $r(\operatorname{Ker} L \cap Z(K[G]))$ are nilpotent, where $M_{L_{\tilde{i}}} = (l_{\tilde{i},j_{1,2}})_{n\times n}$ and $l_{\tilde{i},j_{1,2}} = L_{\tilde{i}}(g_{j_1}^{-1} g_{j_2})$ for $1 \leq j_1,\ j_2 \leq n$.*

Proof. The conclusion follows from the arguments of Proposition 2.5 by replacing $\operatorname{Ker} L$ with $\operatorname{Ker} L \cap Z(K[G])$. □

Proposition 2.8. *Let K, L and G be as in Problem 1.2. If there exists $\tilde{i} \in \{1, 2, \dots, r\}$ such that $\det M_{L_{\tilde{i}}} \neq 0$, then $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if $\operatorname{Ker} L$ contains no nonzero idempotent of $K[G]$.*

Proof. ($\Rightarrow$) Let $e \in \operatorname{Ker} L$ be an idempotent. Then $e^m = e \in \operatorname{Ker} L$ for all integers $m \geq 1$, whence $e \in r(\operatorname{Ker} L)$. It follows from Proposition 2.5 that e is nilpotent. Thus, we have $e = e^N = 0$ for some $N \in \mathbb{N}$. Thus, the conclusion follows.

($\Leftarrow$) Since G is finite, we have that $K[G]$ is algebraic over K. In particular, the radical $r(\operatorname{Ker} L)$ is algebraic over K. It follows from Theorem 4.2 in [8] that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$. □

Corollary 2.9. *Let K, L and G be as in Problem 1.2. If there exists $\tilde{i} \in \{1, 2, \dots, r\}$ such that $\det M_{L_{\tilde{i}}} \neq 0$, then $\operatorname{Ker} L \cap Z(K[G])$ is a Mathieu-Zhao* *space of* $K[G]$ *if and only if* $\operatorname{Ker} L \cap Z(K[G])$ *contains no nonzero idempotent of* $K[G]$.

Proof. The conclusion follows from the arguments of Proposition 2.8 by replacing $\operatorname{Ker} L$ with $\operatorname{Ker} L \cap Z(K[G])$. □

Remark 2.10. If $\operatorname{Ker} L$ ($\operatorname{Ker} L \cap Z(K[G])$) contains no nonzero idempotent of $K[G]$, then $\operatorname{Ker} L$ ($\operatorname{Ker} L \cap Z(K[G])$) is a Mathieu-Zhao space of $K[G]$ without the condition that $\det M_{L_{\tilde{i}}} \neq 0$ for some $\tilde{i} \in \{1, 2, \ldots, r\}$ in Proposition 2.8 (Corollary 2.9).

Corollary 2.11. *Let* K *be a field of characteristic* p *and* G *a* p*-group. Then* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

Proof. Note that $K[G]$ is a local K-algebra. Hence $K[G]$ does not contain nontrivial idempotent. Thus, $\operatorname{Ker} L$ contains no nonzero idempotent of $K[G]$. Then the conclusion follows from Proposition 2.8 and Remark 2.10. □

Remark 2.12. Corollary 2.11 can also be deduced from Theorem 7.6 in [8].

Lemma 2.13. *Let* L *and* G *be as in Problem 1.2. Then* $\operatorname{Ker} L = \{\beta \in K[G] \mid \operatorname{Tr} \beta\alpha_i = 0$ *for all* $1 \leq i \leq r\}$, *where* $\alpha_i = \sum_{j=1}^{n} l_{i,j} g_j^{-1}$ *for all* $1 \leq i \leq r$.

Proof. Let $\beta = \sum_{j=1}^{n} c_j g_j$. Then $L_i(\beta) = \sum_{j=1}^{n} c_j l_{i,j} = \operatorname{Tr} \beta\alpha_i$ for all $1 \leq i \leq r$. Hence the conclusion follows. □

Theorem 2.14. *Let* L *and* G *be as in Problem 1.2 and* K *a field of characteristic zero or a field of characteristic* p *and* $p \nmid |G|$. *If* K *is a split field for* G, *then*

$$\operatorname{Ker} L \cong \{(A_1, \ldots, A_s) \in A \mid \sum_{j=1}^{s} n_j \operatorname{Tr}(C_{i,j} A_j) = 0 \ \ \textit{for all} \ \ 1 \leq i \leq r\},$$

where $A = M_{n_1}(K) \times \cdots \times M_{n_s}(K)$ *is the product of matrices and* $C_{i,j} = \rho_j(\alpha_i) \in M_{n_j}(K)$, α_i *be as in Lemma 2.13,* ρ_j *is an irreducible representation of* G, $n_j = \rho_j(1)$ *for* $1 \leq j \leq s$, $1 \leq i \leq r$ *and* s *is the number of distinct (up to isomorphism) irreducible representations of* G.

Proof. Since $\operatorname{char} K = 0$ or $\operatorname{char} K = p$ and $p \nmid |G|$, we have that $K[G]$ is semi-simple. Since K is a split field for G, we have that

$$K[G] \cong M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K),$$

where $M_{n_j}(K)$ is the ring of $n_j \times n_j$ matrices over K for $1 \leq j \leq s$. Let $\tilde{\rho}$ be the regular representation of $K[G]$. Then $\operatorname{Tr}(\beta) = 0$ if and only if $\operatorname{Tr}(\tilde{\rho}(\beta)) = 0$ for all $\beta \in K[G]$. Let $\rho = (\rho_1, \rho_2, \ldots, \rho_s)$. Then ρ is a ring isomorphism from $K[G]$ to A. Let β be any element in $K[G]$. Then

$$\rho(\alpha_i\beta) = (\rho_1(\alpha_i\beta), \rho_2(\alpha_i\beta), \ldots, \rho_s(\alpha_i\beta)) = (\rho_1(\alpha_i)\rho_1(\beta), \ldots, \rho_s(\alpha_i)\rho_s(\beta)).$$

Suppose that

$$\rho(\alpha_i) = (\rho_1(\alpha_i), \dots, \rho_s(\alpha_i)) = \begin{pmatrix} C_{i,1} & 0 & \cdots & 0 \\ 0 & C_{i,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{i,s} \end{pmatrix} \in A$$

and

$$\rho(\beta) = (\rho_1(\beta), \dots, \rho_s(\beta)) = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix} \in A$$

for all $1 \le i \le r$. Then we have that

$$\rho(\alpha_i\beta) = \begin{pmatrix} C_{i,1}A_1 & 0 & \cdots & 0 \\ 0 & C_{i,2}A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{i,s}A_s \end{pmatrix} \in A.$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} K[G] & \xrightarrow{\cong} & M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K) \\ {\scriptstyle \tilde{\rho}(\alpha_i\beta)}\big\downarrow & & {\scriptstyle \phi(\rho(\alpha_i\beta))}\big\downarrow \\ K[G] & \xrightarrow{\cong} & \begin{pmatrix} M_{n_1}(K) & 0 & \cdots & 0 \\ 0 & M_{n_2}(K) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(K) \end{pmatrix}, \end{array}$$

where ϕ is the natural isomorphism between the two algebras. Thus, we have that $\operatorname{Tr}(\tilde{\rho}(\alpha_i\beta)) = 0$ if and only if $\operatorname{Tr}(\phi(\rho(\alpha_i\beta))) = 0$. Since $\operatorname{Tr}(\phi(\rho(\alpha_i\beta))) = n_1 \operatorname{Tr}(C_{i,1}A_1) + n_2 \operatorname{Tr}(C_{i,2}A_2) + \cdots + n_s \operatorname{Tr}(C_{i,s}A_s)$, we have that $\operatorname{Tr}(\alpha_i\beta) = 0$ if and only if $n_1 \operatorname{Tr}(C_{i,1}A_1) + n_2 \operatorname{Tr}(C_{i,2}A_2) + \cdots + n_s \operatorname{Tr}(C_{i,s}A_s) = 0$ for all $1 \le i \le r$. Thus, we have that $\operatorname{Ker} L \cong V$, where

$$V = \{(A_1, A_2, \dots, A_s) \in A \mid \sum_{j=1}^{s} n_j \operatorname{Tr} C_{i,j}A_j = 0 \ \text{ for all } \ 1 \le i \le r\}.$$

□

Corollary 2.15. *Let L and G be as in Problem 1.2 and K a field of characteristic zero or a field of characteristic p and $p \nmid |G|$. If K is a split field for G and $r = 1$, then* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$ *if and only if $n_1\lambda_1 d_1 + n_2\lambda_2 d_2 + \cdots + n_t\lambda_t d_t \neq 0$ for all non-zero vectors $\tilde{d} = (d_1, \dots, d_t)$, $d_j \in \{0, 1, \dots, n_j\}$ for $1 \le j \le t$, where $n_j\lambda_j = \operatorname{Tr} \rho_j(\alpha_1)$, α_1 is as in Lemma 2.13, ρ_j is an irreducible representation of G for $1 \le j \le s$ and s is the number of distinct* (*up to isomorphism*) *irreducible representations of G and $t \in \{1, 2, \dots, s\}$.*

Proof. It follows from Theorem 2.14 that $\operatorname{Ker} L \cong V$, where $V = \{(A_1, \dots, A_s) \in A \mid \sum_{j=1}^{s} n_j \operatorname{Tr}(C_{1,j}A_j) = 0\}$ and $C_{1,j} = \rho_j(\alpha_1) \in M_{n_j}(K)$ for $1 \le j \le s$. Let ρ be as in Theorem 2.14. Since $\alpha_1 \ne 0$ and ρ is an isomorphism, we have that $\rho(\alpha_1) \ne 0$. We can assume that $C_{1,1}, \dots, C_{1,t}$ are not equal to zero and $C_{1,t+1} = \cdots = C_{1,s} = 0$ for some $t \in \{1, 2, \dots, s\}$ by reordering the ρ_j for $1 \le j \le s$. It follows from Theorem 5.8.1 in [2] or Theorem 4.4 in [4] that V is a Mathieu-Zhao space of A if and only if $C_{1,j} = \lambda_j I_{n_j}$ and $n_1\lambda_1 d_1 + \cdots + n_t\lambda_t d_t \ne 0$ for all nonzero vectors $\tilde{d} = (d_1, \dots, d_t)$ and $d_j \in \{0, 1, \dots, n_j\}$ for $1 \le j \le t$. Then the conclusion follows. □

Proposition 2.16. *Let L and G be as in Problem 1.2 and K a field of characteristic zero or a field of characteristic p and $p \nmid |G|$. If K is a split field for G and $r = 1$, then the following two statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) *There exist $\mu_1, \dots, \mu_t \in K$ such that $L_1 = \mu_1\chi_1 + \mu_2\chi_2 + \cdots + \mu_t\chi_t$ and $\mu_1 d_1 + \cdots + \mu_t d_t \ne 0$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \dots, d_t)$, $d_j \in \{0, 1, \dots, n_j\}$ for $1 \le j \le t$, where $\chi_1, \chi_2, \dots, \chi_s$ are the non-isomorphic irreducible characters of G and $\mu_j = n^{-1}n_j\lambda_j$, $n_j = \chi_j(1)$, $n_j\lambda_j = \operatorname{Tr}\rho_j(\alpha_1)$, α_1 is as in Lemma 2.13 and ρ_j is an irreducible representation of G with character χ_j for $1 \le j \le s$, s is the number of distinct* (*up to isomorphism*) *irreducible representations of G and $t \in \{1, 2, \dots, s\}$. In particular, L_1 is a class function of G.*

Proof. (1) ⇒ (2) Since $L_1(\beta) = \operatorname{Tr}(\alpha_1\beta)$ for any $\beta \in K[G]$, where α_1 is as in Lemma 2.13, we have that

$$n \operatorname{Tr}(\alpha_1\beta) = \operatorname{Tr}\tilde{\rho}(\alpha_1\beta) = \operatorname{Tr}\phi(\rho(\alpha_1\beta))$$

by following the arguments of Theorem 2.14, where $\tilde{\rho}$ is as in Theorem 2.14. Since $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$, it follows from Corollary 2.15 that $C_{1,j} = \lambda_j I_{n_j}$ for $\lambda_j \in K$ and for all $1 \le j \le s$. We can assume that $\lambda_1 \cdots \lambda_t \ne 0$ and $\lambda_{t+1} = \cdots = \lambda_s = 0$ for some $t \in \{1, 2, \dots, s\}$ by reordering $\chi_1, \chi_2, \dots, \chi_s$.

Thus, it follows from Lemma 2.13 that

$$L_1(\beta) = \operatorname{Tr}(\alpha_1\beta) = n^{-1}(n_1\lambda_1 \operatorname{Tr} A_1 + n_2\lambda_2 \operatorname{Tr} A_2 + \cdots + n_t\lambda_t \operatorname{Tr} A_t).$$

Since $\operatorname{Tr} A_j = \chi_j(\beta)$ for all $1 \le j \le s$, we have that

$$L_1 = n^{-1}(n_1\lambda_1\chi_1 + n_2\lambda_2\chi_2 + \cdots + n_t\lambda_t\chi_t).$$

It follows from Corollary 2.15 that $n_1\lambda_1 d_1 + \cdots + n_t\lambda_t d_t \ne 0$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \dots, d_t)$, $d_j \in \{0, 1, \dots, n_j\}$ for $1 \le j \le t$. Let $\mu_j = n^{-1}n_j\lambda_j$ for all $1 \le j \le s$. Then the conclusion follows.

(2) ⇒ (1) Since $\operatorname{Ker} L = \{\beta \in K[G] \mid L_1(\beta) = 0\} = \{\beta \in K[G] \mid \mu_1\chi_1(\beta) + \cdots + \mu_t\chi_t(\beta) = 0\}$ and there exists $A_j \in M_{n_j}(K)$ such that $\operatorname{Tr} A_j = \chi_j(\beta)$ for

all $1 \le j \le t$, we have that

$$\operatorname{Ker} L = \{(A_1, \ldots, A_t) \in M_{n_1}(K) \times \cdots \times M_{n_t}(K) \mid \sum_{j=1}^{t} \mu_j \operatorname{Tr} A_j = 0\}.$$

Then the conclusion follows from Theorem 5.8.1 in [2] or Theorem 4.4 in [4]. □

Remark 2.17. To prove that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ for $r = 1$ if $n_1\lambda_1 d_1 + n_2\lambda_2 d_2 + \cdots + n_t\lambda_t d_t \neq 0$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \ldots, d_t)$ and $d_j \in \{0, 1, \ldots, n_j\}$ for $1 \le j \le t$, we don't need the condition that K is a split field for G in Corollary 2.15 by following the arguments Theorem 5.8.1 in [2], because an idempotent matrix can be conjugated to a diagonal matrix with only 0 and 1 on the diagonal over division rings.

If $L = \mu_j \chi_j$ for some $j \in \{1, 2, \ldots, t\}$, $\mu_j \in K^*$, then it follows from the arguments of Proposition 2.16 that the condition $n_1\lambda_1 d_1 + n_2\lambda_2 d_2 + \cdots + n_t\lambda_t d_t \neq 0$ in Theorem 2.14 is equivalent to $n_j d_j \neq 0$ for all $1 \le d_j \le n_j$, which is clearly true if $\operatorname{char} K = 0$. If $\operatorname{char} K = p$, then the condition is equivalent to $p > n_j$. To see this, we can assume that $p \mid n_j d_j$ for some $d_j \in \{1, 2, \ldots, n_j\}$, then $p \mid n_j$ or $p \mid d_j$, which contradicts with $p > n_j$. Thus, if $p > n_j$, then $n_j d_j \neq 0 \bmod p$ for all $1 \le d_j \le n_j$. Conversely, suppose that $p \le n_j$. Then let $d_j = p \in \{1, 2, \ldots, n_j\}$, we have that $n_j p = 0 \bmod p$, which is a contradiction. Thus, if $n_j d_j \neq 0 \bmod p$ for all $1 \le d_j \le n_j$, then $p > n_j$. Therefore, the conclusion is the same as Theorem 5.1 in [8] in this situation.

3. The MZ-property of Ker$\boldsymbol{L}$ and Ker$\boldsymbol{L} \cap \boldsymbol{Z(K[G])}$

Condition 1: Let L and G be as in Problem 1.2 and K a field of characteristic p, H a normal p-subgroup of G, $G = \cup_{j=1}^{k} \tilde{g}_j H$, $H = \{h_1, h_2, \ldots, h_{\tilde{t}}\}$ for $\tilde{t} = p^{\tilde{r}}$ for some $\tilde{r} \in \mathbb{N}$ and $L_i(\tilde{g}_j h_2) = \cdots = L_i(\tilde{g}_j h_{\tilde{t}})$ for all $1 \le i \le r$, $1 \le j \le k$.

Proposition 3.1. *Let L, G, K, H be as in Condition 1 and $L_i(\tilde{g}_j h_1) = L_i(\tilde{g}_j h_2)$ for $1 \le i \le r$, $1 \le j \le k$. Then* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$ *if and only if* $\operatorname{Ker}(L|_{G/H})$ *is a Mathieu-Zhao space of* $K[G/H]$.

Proof. Let φ be the natural surjective homomorphism from $K[G]$ to $K[G/H]$. Since $L_i(\tilde{g}_j h_1) = L_i(\tilde{g}_j h_2) = \cdots = L_i(\tilde{g}_j h_{\tilde{t}})$ for all $1 \le i \le r$, $1 \le j \le k$, there exists a linear map $\tilde{L}$ from $K[G/H]$ to K such that $L = \varphi^{-1}(\tilde{L})$, where $\tilde{L} = L|_{G/H}$. Since φ is surjective and $\operatorname{Ker} \varphi = w(K[H])K[G] = \sum_{l=1}^{\tilde{t}} (h_l - 1)K[G]$, we have $\operatorname{Ker} \varphi \subseteq \operatorname{Ker} L$. Then it follows from Theorem 5.2.19 in [2] that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. □

Corollary 3.2. *Let L, G, K, H be as in Condition 1, $|G| = p^a d$, $p \nmid d$, $\tilde{r} = a$, $k = d$ and H a normal Sylow p-subgroup of G. If $r = 1$, then the following two statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) *There exist* $\mu_1, \ldots, \mu_t \in K$ *such that* $L_1 = \mu_1\chi_1 + \mu_2\chi_2 + \cdots + \mu_t\chi_t$ *and* $\mu_1 d_1 + \cdots + \mu_t d_t \neq 0$ *for all nonzero vectors* $\tilde{d} = (d_1, d_2, \ldots, d_t)$, $d_j \in \{0, 1, \ldots, n_j\}$ *for* $1 \leq j \leq t$, *where* $\chi_1, \chi_2, \ldots, \chi_s$ *are the distinct (up to isomorphism) irreducible characters of* $K[G]$ *and* $\mu_j = d^{-1} n_j \lambda_j$, $n_j = \chi_j(1)$, $n_j\lambda_j = \operatorname{Tr}\rho_j(\alpha_1)$, $\alpha_1 = \sum_{j=1}^{d} l_{1,j}\tilde{g}_j^{-1}$ *and* ρ_j *is an irreducible representation of* $K[G]$ *with character* χ_j *for* $1 \leq j \leq t$ *and* $t \in \{1, 2, \ldots, s\}$.

Proof. It follows from Proposition 3.1 that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. Since $p \nmid |G/H|$, the conclusion follows from Proposition 2.16. □

Remark 3.3. Let the notations be the same as Corollary 3.2. Then $J(K[G]) = w(K[H])K[G] \subseteq \operatorname{Ker} L$ if and only if $L_i(\tilde{g}_j h_1) = L_i(\tilde{g}_j h_2) = \cdots = L_i(\tilde{g}_j h_{\tilde{t}})$ for all $1 \leq i \leq r$, $1 \leq j \leq d$.

Proposition 3.4. *Let* L, G, K, H *be as in Condition 1 and* $h_1 = 1_H$. *Then we have the following statements*:

(1) *If there exists* $\tilde{i} \in \{1, 2, \ldots, r\}$ *such that* $\det M_{L_{\tilde{i}}|_{G/H}} \neq 0$ *and* $\operatorname{Ker}(L|_{G/H})$ *is a Mathieu-Zhao space of* $K[G/H]$, *then* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$, *where* $M_{L_{\tilde{i}}|_{G/H}} = (\tilde{l}_{\tilde{i},j_{1,2}})_{k\times k}$ *and* $\tilde{l}_{\tilde{i},j_{1,2}} = L_{\tilde{i}}(\tilde{g}_{j_1}^{-1}\tilde{g}_{j_2})$ *for* $1 \leq j_1,\ j_2 \leq k$.

(2) *If there exists* $\hat{i} \in \{1, 2, \ldots, r\}$ *such that* $\det M_{L_{\hat{i}}} \neq 0$ *and* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$, *then* $\operatorname{Ker}(L|_{G/H})$ *is a Mathieu-Zhao space of* $K[G/H]$, *where* $M_{L_{\hat{i}}} = (l_{\hat{i},j_{1,2}})_{n\times n}$ *and* $l_{\hat{i},j_{1,2}} = L_{\hat{i}}(g_{j_1}^{-1}g_{j_2})$ *for* $1 \leq j_1,\ j_2 \leq n$.

Proof. Let φ be the natural surjective homomorphism from $K[G]$ to $K[G/H]$.

(1) Let E be an idempotent of $\operatorname{Ker} L$. Then

$$E = \tilde{g}_1 \cdot a_1(h) + \tilde{g}_2 \cdot a_2(h) + \cdots + \tilde{g}_k \cdot a_k(h),$$

where $a_i(h) \in K[H]$, $h = (h_1, h_2, \ldots, h_{\tilde{t}})$, $\tilde{g}_j \notin H$ for $2 \leq j \leq k$ and $\tilde{g}_1 = 1_{G/H}$. Let $b \in H$ and $b \neq 1_H$. Then b is a p-element. Thus, it follows from Lemma 2.7 in [6] that the sum of coefficients in E of the G-conjugacy class of b is equal to zero. Then $\varphi(E) = \tilde{g}_1 \cdot a_1(1) + \tilde{g}_2 \cdot a_2(1) + \cdots + \tilde{g}_k \cdot a_k(1)$. Let $a_j(h) = a_{j1}h_1 + a_{j2}h_2 + \cdots + a_{j\tilde{t}}h_{\tilde{t}}$ for $1 \leq j \leq k$. Then we have that $a_j(1) = a_{j1}$ and $L_i(\tilde{g}_j \cdot a_j(h)) = a_{j1}L_i(\tilde{g}_j)$ for all $1 \leq i \leq r$, $1 \leq j \leq k$. Thus, we have that $L_i(E) = a_{11}L_i(1) + a_{21}L_i(\tilde{g}_2) + \cdots + a_{k1}L_i(\tilde{g}_k) = L_i(\varphi(E))$ for all $1 \leq i \leq r$. Therefore, we have that $E \in \operatorname{Ker} L$ if and only if $\varphi(E) \in \operatorname{Ker}(L|_{G/H})$. That is, $\bar{E} = \varphi(E)$ is an idempotent of $\operatorname{Ker}(L|_{G/H})$. Since $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$, it follows from Proposition 2.8 that $\varphi(E) = 0$ in $K[G/H]$. That is, $E \in \operatorname{Ker}\varphi = w(K[H])K[G]$. It follows from Lemma 2.8 in [6] that E is nilpotent. Thus, we have $E = 0$. Hence it follows from Proposition 2.8 and Remark 2.10 that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$.

(2) Since $\operatorname{Ker}\varphi = w(K[H])K[G]$, it follows from Lemma 2.8 in [6] that $w(K[H])K[G]$ is a nilpotent ideal and $K[G/H] \cong K[G]/\operatorname{Ker}\varphi$. Let $\bar{u}$ be any idempotent of $\operatorname{Ker}(L|_{G/H})$. Then there exists a $u \in K[G]$ such that $\bar{u} = \varphi(u)$. It follows from Lemma 3.7(i) of Chapter 2 in [6] that there exists an idempotent

$e = u\tilde{b}u$ such that $\varphi(e) = \bar{u}$ for some $\tilde{b} \in K[G]$. We have that $e \in \operatorname{Ker} L$ by following the arguments of Proposition 3.4(1). Since $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$, it follows from Proposition 2.8 that $e = 0$. Thus, we have $\bar{u} = \varphi(e) = 0$. Hence it follows from Proposition 2.8 and Remark 2.10 that $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. □

Proposition 3.5. *Let L, G, K, H be as in Condition 1, $|G| = p^a d$, $p \nmid d$, $\tilde{r} = a$, $k = d$ and H a normal Sylow p-subgroup of G, $h_1 = 1_H$ and K is a split field for G/H. If there exist $\tilde{i}, \hat{i} \in \{1, 2, \ldots, r\}$ such that $\det M_{L_{\tilde{i}}|_{G/H}} \neq 0$ and $\det M_{L_{\hat{i}}} \neq 0$ and $r = 1$, then $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if $n_1\lambda_1 d_1 + n_2\lambda_2 d_2 + \cdots + n_t\lambda_t d_t \neq 0$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \ldots, d_t)$ and $d_j \in \{0, 1, \ldots, n_j\}$ for $1 \leq j \leq t$, where $n_j\lambda_j = \operatorname{Tr} \rho_j(\alpha_1)$ for $1 \leq j \leq s$ and $\alpha_1 = \sum_{j=1}^{d} l_{1,j}\tilde{g}_j^{-1}$, $\rho_1, \ldots, \rho_s$ are distinct (up to isomorphism) irreducible representations of $K[G]$ and $t \in \{1, 2, \ldots, s\}$, $M_{L_{\tilde{i}}|_{G/H}} = (\tilde{l}_{\tilde{i},j_{1,2}})_{d\times d}$ and $\tilde{l}_{\tilde{i},j_{1,2}} = L_{\tilde{i}}(\tilde{g}_{j_1}^{-1}\tilde{g}_{j_2})$ for $1 \leq j_1, j_2 \leq d$, $M_{L_{\hat{i}}} = (l_{\hat{i},j_{1,2}})_{n\times n}$ and $l_{\hat{i},j_{1,2}} = L_{\hat{i}}(\tilde{g}_{j_1}^{-1}\tilde{g}_{j_2})$ for $1 \leq j_1, j_2 \leq n$.*

Proof. It follows from Proposition 3.4 that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. Since $p \nmid |G/H|$, the conclusion follows from Corollary 2.15. □

Theorem 3.6. *Let L and G be as in Problem 1.2 and K a field of characteristic zero or a field of characteristic p, $p \nmid |G|$. Suppose that $G = \{g_1, \ldots, g_n\}$ with $g_1 = 1_G$ and $\chi_1, \ldots, \chi_s$ are the distinct (up to isomorphism) irreducible characters of $K[G]$. Then we have the following statements:*

(1) *If there exists $q_{i_1,\ldots,i_l} \in \{1, 2, \ldots, r\}$ such that $\sum_{i=1}^{n}(\sum_{j=1}^{l} \chi_{i_j}(1)\chi_{i_j}(g_i^{-1}))\cdot l_{q_{i_1,\ldots,i_l},i} \neq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_l \leq s$, $l \in \{1, 2, \ldots, s\}$, then $\operatorname{Ker} L \cap Z(K[G])$ is a Mathieu-Zhao space of $K[G]$.*

(2) *If there exists $\tilde{i} \in \{1, 2, \ldots, r\}$ such that $\det M_{L_{\tilde{i}}} \neq 0$ and $\operatorname{Ker} L \cap Z(K[G])$ is a Mathieu-Zhao space of $K[G]$, then there exists $q_{i_1,\ldots,i_l} \in \{1, 2, \ldots, r\}$ such that $\sum_{i=1}^{n}(\sum_{j=1}^{l} \chi_{i_j}(1)\chi_{i_j}(g_i^{-1}))l_{q_{i_1,\ldots,i_l},i} \neq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_l \leq s$.*

Proof. (1) Let $e_{\tilde{k}} = \frac{1}{n}\sum_{g\in G} \chi_{\tilde{k}}(1)\chi_{\tilde{k}}(g^{-1})g$ for $1 \leq \tilde{k} \leq s$. Then it follows from Theorem 2.12 in [3] that $e_1, e_2, \ldots, e_s$ are the primitive orthogonal idempotents of $Z(K[G])$. It follows from Theorem 3.11 in [5] that every idempotent of $Z(K[G])$ is some sum of $e_1, e_2, \ldots, e_s$. Since $\sum_{i=1}^{n}(\sum_{j=1}^{l} \chi_{i_j}(1)\cdot \chi_{i_j}(g_i^{-1}))l_{q_{i_1,\ldots,i_l},i} \neq 0$, we have that $L_{q_{i_1,\ldots,i_l}}(e_{i_1} + e_{i_2} + \cdots + e_{i_l}) \neq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_l \leq s$, $l \in \{1, 2, \ldots, s\}$. That is, any nonzero idempotent of $Z(K[G])$ is not in $\operatorname{Ker} L$. Thus, $\operatorname{Ker} L \cap Z(K[G])$ has no nonzero idempotent. It follows from Corollary 2.9 and Remark 2.10 that $\operatorname{Ker} L \cap Z(K[G])$ is a Mathieu-Zhao space of $K[G]$.

(2) It follows from Corollary 2.9 that $\operatorname{Ker} L \cap Z(K[G])$ has no nonzero idempotent. Hence there exists $q_{i_1,\ldots,i_l} \in \{1, 2, \ldots, r\}$ such that $L_{q_{i_1,\ldots,i_l}}(e_{i_1} + e_{i_2} +$

$\cdots + e_{i_l}) \neq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_l \leq s$, $l \in \{1, 2, \ldots, s\}$. That is, $\sum_{i=1}^{n}(\sum_{j=1}^{l} \chi_{i_j}(1)\chi_{i_j}(g_i^{-1})) \cdot l_{q_{i_1,\ldots,i_l},i} \neq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_l \leq s$, $l \in \{1, 2, \ldots, s\}$. □

Proposition 3.7. *Let L, G, K, H be as in Condition 1 and $h_1 = 1_H$. If there exists $\tilde{i} \in \{1, 2, \ldots, r\}$ such that* $\det M_{L_{\tilde{i}}|_{G/H}} \neq 0$ *and* $\mathrm{Ker}(L|_{G/H}) \cap Z(K[G/H])$ *is a Mathieu-Zhao space of $K[G/H]$, then* $\mathrm{Ker}\, L \cap Z(K[G])$ *is a Mathieu-Zhao space of $K[G]$, where $M_{L_{\tilde{i}}|_{G/H}} = (\tilde{l}_{\tilde{i},j_{1,2}})_{k\times k}$ and $\tilde{l}_{\tilde{i},j_{1,2}} = L_{\tilde{i}}(\tilde{g}_{j_1}^{-1}\tilde{g}_{j_2})$ for $1 \leq j_1,\ j_2 \leq k$.*

Proof. Let φ be the natural surjective homomorphism from $K[G]$ to $K[G/H]$. Then it's easy to check that if $E \in Z(K[G])$, then $\varphi(E) \in Z(K[G/H])$. Thus, the conclusion follows by following the arguments of Proposition 3.4(1). □

Corollary 3.8. *Let L, G, K, H be as in Condition 1, $|G| = p^a d$, $p \nmid d$, $\tilde{r} = a$, $k = d$ and H a normal Sylow p-subgroup of G and $h_1 = 1_H$. If there exists $\tilde{i} \in \{1, 2, \ldots, r\}$ such that* $\det M_{L_{\tilde{i}}|_{G/H}} \neq 0$ *and there exists $q_{i_1,\ldots,i_l} \in \{1, 2, \ldots, r\}$ such that $\sum_{\hat{i}=1}^{d}(\sum_{\hat{j}=1}^{l} \chi_{i_{\hat{j}}}(1)\chi_{i_{\hat{j}}}(\tilde{g}_{\hat{i}}^{-1})) \cdot l_{q_{i_1,\ldots,i_l},\hat{i}} \neq 0$ for all $1 \leq i_1 < i_2 < \cdots < i_l \leq s$, $l \in \{1, 2, \ldots, s\}$, then* $\mathrm{Ker}\, L \cap Z(K[G])$ *is a Mathieu-Zhao space of $K[G]$, where $\chi_1, \ldots, \chi_s$ are the distinct (up to isomorphism) irreducible characters of $K[G]$ and $G/H = \{\tilde{g}_1, \ldots, \tilde{g}_d\}$, $M_{L_{\tilde{i}}|_{G/H}} = (\tilde{l}_{\tilde{i},j_{1,2}})_{d\times d}$ and $\tilde{l}_{\tilde{i},j_{1,2}} = L_{\tilde{i}}(\tilde{g}_{j_1}^{-1}\tilde{g}_{j_2})$ for $1 \leq j_1,\ j_2 \leq d$.*

Proof. The conclusion follows from Theorem 3.6(1) and Proposition 3.7. □

4. Mathieu-Zhao spaces of finite Abelian group algebras

Proposition 4.1. *Let $B = K \times \cdots \times K$ be a K-algebra and*

$$V = \{(a_1, a_2, \ldots, a_n) \in B \mid \sum_{j=1}^{n} \gamma_{i,j} a_j = 0 \text{ for all } 1 \leq i \leq r\},$$

where $\gamma_{i,j} \in K$ for all $1 \leq i \leq r$, $1 \leq j \leq n$. If at least one of $\gamma_{i,j}$ is nonzero for all $1 \leq i \leq r$, $1 \leq j \leq n$, then V is a Mathieu-Zhao space of B if and only if $\gamma_{i,1}d_1 + \gamma_{i,2}d_2 + \cdots + \gamma_{i,t_i}d_{t_i} \neq 0$ for some $i \in \{1, 2, \ldots, r\}$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \ldots, d_{t_i})$ and $d_{j_i} \in \{0, 1\}$ for $1 \leq j_i \leq t_i$, $t_i \in \{1, \ldots, n\}$.

Proof. We can assume that $\gamma_{i,j} \neq 0$ for all $1 \leq j \leq t$ for some $i \in \{1, 2, \ldots, r\}$ and $\gamma_{i,j} = 0$ for all $1 \leq i \leq r$ and $t+1 \leq j \leq n$ by reordering $\gamma_{i,j}$ for $1 \leq i \leq r$, $1 \leq j \leq n$ and then we have

$$\overbrace{0 \times \cdots \times 0}^{t \text{ columns}} \times K \times \cdots \times K \subseteq V$$

and

$$0 \times \cdots \times K \times 0 \times \cdots \times \overbrace{0 \times \cdots \times 0}^{n-t \text{ columns}} \nsubseteq V,$$

where $t = \max\{t_1, t_2, \ldots, t_r\}$.

($\Rightarrow$) Suppose that $\gamma_{i,1}d_1 + \gamma_{i,2}d_2 + \cdots + \gamma_{i,t_i}d_{t_i} = 0$ for some nonzero vector $\tilde{d} = (d_1, d_2, \ldots, d_{t_i})$, $d_{j_i} = 0$ or 1 for $1 \le j_i \le t_i$ for all $1 \le i \le r$, then $e = (d_1, \ldots, d_t, 0, \ldots, 0)$ is an idempotent of V. Since V is a Mathieu-Zhao space of B, we have that $Be = Kd_1 \times \cdots \times Kd_t \times 0 \times \cdots \times 0 \subseteq V$, which is a contradiction. Then the conclusion follows.

($\Leftarrow$) Let $I = \overbrace{0 \times \cdots \times 0}^{t \text{ columns}} \times K \times \cdots \times K$. Then I is an ideal of B. We claim that V/I contains no nonzero idempotent. Suppose that e is a nonzero idempotent of V/I. Then we have $e = (e_1, e_2, \ldots, e_t)$, where $e_j = 0$ or 1 for $1 \le j \le t$. Let $\tilde{d} = (d_1, \ldots, d_t) = e \ne (0, \ldots, 0)$. Then $\gamma_{i,1}d_1 + \gamma_{i,2}d_2 + \cdots + \gamma_{i,t_i}d_{t_i} = 0$ for all $1 \le i \le r$, which is a contradiction. It follows from Theorem 4.2 in [8] that V/I is a Mathieu-Zhao space of B/I. Then it follows from Proposition 2.7 in [8] that V is a Mathieu-Zhao space of B. $\square$

Remark 4.2. In Proposition 4.1, if $\gamma_{i,j} = 0$ for all $1 \le i \le r$, $1 \le j \le n$, then $V = B$. Clearly, V is a Mathieu-Zhao space of B.

Corollary 4.3. *Let L and G be as in Problem 1.2 and K a field of characteristic zero or a field of characteristic p and $p \nmid |G|$. If K is a split field for G and G is Abelian, then* $\operatorname{Ker} L$ *is a Mathieu-Zhao space of $K[G]$ if and only if $\gamma_{i,1}d_1 + \gamma_{i,2}d_2 + \cdots + \gamma_{i,t_i}d_{t_i} \ne 0$ for some $i \in \{1, 2, \ldots, r\}$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \ldots, d_{t_i})$ and $d_{j_i} \in \{0, 1\}$ for $1 \le j_i \le t_i$, $t_i \in \{1, \ldots, n\}$, where $\gamma_{i,j} = \rho_j(\alpha_i)$ for all $1 \le i \le r$, $1 \le j \le n$ and ρ_j is an irreducible representation of G for $1 \le j \le n$ and α_i be as in Lemma 2.13 for $1 \le i \le r$.*

Proof. Since G is Abelian, we have that all the irreducible representations of G are of degree one. It follows from Theorem 2.14 that $\operatorname{Ker} L \cong \{(a_1, a_2, \ldots, a_n) \in A \mid \sum_{j=1}^{n} \gamma_{i,j}a_j = 0$ for all $1 \le i \le r\}$, where A is n times product of K, $\gamma_{i,j} = \rho_j(\alpha_i) = \operatorname{Tr} \rho_j(\alpha_i) \in K$ for all $1 \le i \le r$, $1 \le j \le n$. Since $L \ne 0$, we have that at least one of $\gamma_{i,j}$ is nonzero for $1 \le i \le r$, $1 \le j \le n$. Then the conclusion follows from Proposition 4.1. $\square$

Lemma 4.4. *Let R be an integral domain of characteristic p and G a finite Abelian group with $|G| = p^a d$, $p \nmid d$. Then every idempotent of $R[G]$ is also an idempotent of $R[\tilde{G}]$, where $G = H \times \tilde{G}$ and $|H| = p^a$. In particular, the idempotent elements of $R[G]$ are the same as the idempotent elements of $R[\tilde{G}]$.*

Proof. Since G is a finite Abelian group, we have that $G = H \times \tilde{G}$ and $|\tilde{G}| = d$. Let e be an idempotent of $R[G]$. Then e can be written as

$$e = \sum_{h \in H} \alpha_h h$$

with $\alpha_h \in R[\tilde{G}]$ for each $h \in H$. Since $|H| = p^a$, we have $h^{q^m} = 1$ for any $m \ge 1$, $h \in H$, where $q = p^a$. Thus, we have

$$e = e^{q^m} = \sum_{h \in H} \alpha_h^{q^m} \in R[\tilde{G}].$$

Then the conclusion follows. □

Theorem 4.5. *Let L and G be as in Problem 1.2 and K a field of characteristic p. If K is a split field for G and G is Abelian with $|G| = p^a d$, $p \nmid d$, then the following statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) $\gamma_{i,1}d_1 + \gamma_{i,2}d_2 + \cdots + \gamma_{i,t_i}d_{t_i} \neq 0$ *for some* $i \in \{1, 2, \ldots, r\}$ *for all nonzero vectors* $\tilde{d} = (d_1, d_2, \ldots, d_{t_i})$ *and* $d_{j_i} \in \{0, 1\}$ *for* $1 \leq j_i \leq t_i$, $t_i \in \{1, \ldots, d\}$, *where* $\gamma_{i,j} = \rho_j(\alpha_i) = \operatorname{Tr} \rho_j(\alpha_i)$ *for* $1 \leq i \leq r$, $1 \leq j \leq d$ *and* ρ_j *is an irreducible representation of* G/H *for* $1 \leq j \leq d$, H *is a Sylow* p*-subgroup of* G *and* α_i *is as in Lemma 2.13 by replacing* G *with* G/H *for* $1 \leq i \leq r$; $l_{i,1}, l_{i,2}, \ldots, l_{i,n}$ *satisfy the following equations*:

$$
\begin{cases}
\chi_j(\tilde{g}_1^{-1})l_{i,1} + \chi_j(\tilde{g}_2^{-1})l_{i,p^a+1} + \cdots + \chi_j(\tilde{g}_d^{-1})l_{i,(d-1)p^a+1} = 0, \\
\chi_j(\tilde{g}_1^{-1})l_{i,2} + \chi_j(\tilde{g}_2^{-1})l_{i,p^a+2} + \cdots + \chi_j(\tilde{g}_d^{-1})l_{i,(d-1)p^a+2} = 0, \\
\qquad \vdots \\
\chi_j(\tilde{g}_1^{-1})l_{i,p^a} + \chi_j(\tilde{g}_2^{-1})l_{i,2p^a} + \cdots + \chi_j(\tilde{g}_d^{-1})l_{i,dp^a} = 0
\end{cases}
\tag{4.1}
$$

for all $1 \leq i \leq r$ *and* $t+1 \leq j \leq d$, *where* χ_j *is the irreducible character according to* ρ_j *for* $t+1 \leq j \leq d$ *and* $G = \cup_{k=1}^{d} \tilde{g}_k H$ *with* $\tilde{g}_1 = 1_{G/H}$ *and* $H = \{h_1, h_2, \ldots, h_{p^a}\}$ *with* $h_1 = 1_H$, $L_i(h_k) = l_{i,k}$ *and* $L_i(\tilde{g}_k h_q) = l_{i,(k-1)p^a+q}$ *for all* $1 \leq i \leq r$, $1 \leq k \leq d$, $1 \leq q \leq p^a$ *and* $t = \max\{t_1, t_2, \ldots, t_r\}$.

Proof. Since G is Abelian, we have that $G = H \times \tilde{G}$, where $\tilde{G} \cong G/H$ and $|\tilde{G}| = d$.

Note that

$$\gamma_{i,j} = \operatorname{Tr} \rho_j(\alpha_i) = \sum_{k=1}^{d} \operatorname{Tr} \rho_j(\tilde{g}_k^{-1}) l_{i,(k-1)p^a+1} = \sum_{k=1}^{d} \chi_j(\tilde{g}_k^{-1}) l_{i,(k-1)p^a+1}$$

for all $1 \leq i \leq r$, $1 \leq j \leq d$. Let $e_j = d^{-1} \sum_{k=1}^{d} \chi_j(\tilde{g}_k^{-1})\tilde{g}_k$ for $1 \leq j \leq d$. Then it follows from Theorem 2.12 in [3] that $e_1, e_2, \ldots, e_d$ are the primitive orthogonal idempotents of $K[\tilde{G}]$. Without loss of generality, we can assume that $\gamma_{i,j} = 0$ for all $1 \leq i \leq r$, $t+1 \leq j \leq d$ and $\gamma_{i,j} \neq 0$ for all $1 \leq j \leq t$ for some $i \in \{1, 2, \ldots, r\}$ by reordering $\rho_j(\alpha_i)$ for all $1 \leq i \leq r$, $1 \leq j \leq d$.

(1) ⇒ (2) It's easy to see that if $\gamma_{i,j} = 0$ for all $1 \leq i \leq r$, $t+1 \leq j \leq d$, then $e_{t+1}, \ldots, e_d$ belong to $\operatorname{Ker}(L|_{\tilde{G}}) \subseteq \operatorname{Ker} L$. Thus, the ideal I generated by $e_{t+1}, \ldots, e_d$ belongs to $\operatorname{Ker} L$. Since $\tilde{G}$ is Abelian, it is easy to check that $e_j\tilde{g}_k = \chi_j(\tilde{g}_k)e_j$ for all $1 \leq j, k \leq d$. Hence we have $e_j\tilde{g}_k \in \operatorname{Ker} L$ for all $t+1 \leq j \leq d$, $1 \leq k \leq d$. Note that $e_j h_q \in \operatorname{Ker} L$ for all $t+1 \leq j \leq d$, $1 \leq q \leq p^a$. Then we have equations (4.1) for all $1 \leq i \leq r$, $t+1 \leq j \leq d$. It follows from Proposition 2.3 that $\operatorname{Ker}(L|_{\tilde{G}})$ is a Mathieu-Zhao space of $K[\tilde{G}]$. That is, $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. Since $p \nmid |G/H|$, the conclusion follows from Corollary 4.3.

(2) ⇒ (1) If $\gamma_{i,j} = 0$ for all $1 \le i \le r$, $t+1 \le j \le d$, then $e_{t+1}, \dots, e_d \in \operatorname{Ker}(L|_{\tilde{G}}) \subseteq \operatorname{Ker} L$. It is easy to check that $e_j \tilde{g}_k = \chi_j(\tilde{g}_k) e_j$ and $e_j \tilde{g}_k h_q = \chi_j(\tilde{g}_k) e_j h_q$ for all $t+1 \le j \le d$, $1 \le k \le d$, $1 \le q \le p^a$. Therefore, we have $I \subseteq \operatorname{Ker} L$, where I is an ideal generated by $e_{t+1}, \dots, e_d$. Since $e_1, \dots, e_d$ are the primitive orthogonal idempotent elements of $K[\tilde{G}]$ and there are 2^d idempotent elements in $K[\tilde{G}]$, we have that any idempotent of $K[\tilde{G}]$ is a sum of some of the e_j for $1 \le j \le d$. Note that the condition that $\gamma_{i,1} d_1 + \gamma_{i,2} d_2 + \cdots + \gamma_{i,t_i} d_{t_i} \neq 0$ for some $i \in \{1, 2, \dots, r\}$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \dots, d_{t_i})$ and $d_{j_i} \in \{0, 1\}$ is equivalent to that any sum of some of the e_j is not in $\operatorname{Ker}(L|_{\tilde{G}})$ except zero for all $1 \le j \le t$. Hence any sum of some of the e_j is not in $\operatorname{Ker}(L|_{\tilde{G}})$ for all $1 \le j \le d$ if it contains e_{j_0} for some $j_0 \in \{1, 2, \dots, t\}$. Thus, any sum of some of the e_j is not in $\operatorname{Ker} L$ for all $1 \le j \le d$ if it contains e_{j_0} for some $j_0 \in \{1, 2, \dots, t\}$. Otherwise, the sum of e_j belong to $\operatorname{Ker} L \cap K[\tilde{G}] = \operatorname{Ker}(L|_{\tilde{G}})$ for $1 \le j \le d$, which is a contradiction. It follows from Lemma 4.4 that $K[G]$ and $K[\tilde{G}]$ have the same idempotents. Hence $\operatorname{Ker} L/I$ has no nonzero idempotent. It follows from Theorem 4.2 in [8] that $\operatorname{Ker} L/I$ is a Mathieu-Zhao space of $K[G]/I$. Hence it follows from Proposition 2.7 in [8] that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$. □

Remark 4.6. If G is cyclic in Theorem 4.5, then all the primitive orthogonal idempotent elements of $K[G]$ are $e_j = d^{-1}(1 + (\xi^{d-1})^{j-1}\tilde{g} + \cdots + \xi^{j-1}\tilde{g}^{d-1})$ for $1 \le j \le d$, where ξ is a d-th root of unity and $\tilde{G}$ is generated by $\tilde{g}$, where $\tilde{G}$ be as in Theorem 4.5.

5. The kernels of the class functions of finite group algebras

Condition 2: Let L and G be as in Problem 1.2 and K a field of characteristic zero or a field of characteristic p, $p \nmid |G|$, $L_2, \dots, L_r$ are class functions of G and K is a split field for G

Proposition 5.1. *Let L, G, K be as in Condition 2 and L_1 is class functions of G. Then the following statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) $a_{i,1} d_1 + a_{i,2} d_2 + \cdots + a_{i,t_i} d_{t_i} \neq 0$ *for some* $i \in \{1, 2, \dots, r\}$ *for all nonzero vectors* $\tilde{d} = (d_1, d_2, \dots, d_{t_i})$ *and* $d_{j_i} \in \{0, 1, \dots, n_{j_i}\}$ *for* $1 \le j_i \le t_i$, $t_i \in \{1, \dots, s\}$, *where* $L_i = \sum_{j=1}^{s} a_{i,j} \chi_j$ *and* $\chi_1, \dots, \chi_s$ *are the distinct (up to isomorphism) irreducible characters of* G *and* $n_j = \chi_j(1)$, $a_{i,j} \in K$ *for all* $1 \le i \le r$, $1 \le j \le s$.

Proof. Since $L_1, \dots, L_r$ are class functions of G, we have $L_i = \sum_{j=1}^{s} a_{i,j} \chi_j$, where $a_{i,j} \in K$ for all $1 \le i \le r$, $1 \le j \le s$. Hence we have

$$\operatorname{Ker} L = \{\beta \in K[G] \mid \sum_{j=1}^{s} a_{i,j} \chi_j(\beta) = 0 \text{ for all } 1 \le i \le r\}.$$

Since $K[G]$ is semi-simple and K is a split field for G, $K[G]$ can be written as the product of matrices over K. That is, $K[G] \cong M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K) := A$. It's easy to see that there exists $A_j \in M_{n_j}(K)$ such that $\operatorname{Tr} A_j = \chi_j(\beta)$ for $1 \le j \le s$. Then we have

$$\operatorname{Ker} L = \{(A_1, \dots, A_s) \in A \mid \sum_{j=1}^{s} a_{i,j} \operatorname{Tr} A_j = 0 \text{ for all } 1 \le i \le r\}.$$

We can assume that $a_{i,j} \neq 0$ for all $1 \le j \le t$ for some $i \in \{1, 2, \dots, r\}$ and $a_{i,j} = 0$ for all $1 \le i \le r$, $t+1 \le j \le s$ by reordering χ_j for $1 \le j \le s$. Then we have $0 \times \cdots \times 0 \times M_{n_{t+1}}(K) \times \cdots \times M_{n_s}(K) \subseteq \operatorname{Ker} L$ and

$$0 \times \cdots \times M_{n_k}(K) \times 0 \times \cdots \times \overbrace{0 \times \cdots \times 0}^{n-t \text{ columns}} \nsubseteq \operatorname{Ker} L,$$

where $t = \max\{t_1, t_2, \dots, t_r\}$.

(1) $\Rightarrow$ (2) Suppose that $a_{i,1}d_1 + a_{i,2}d_2 + \cdots + a_{i,t_i}d_{t_i} = 0$ for some nonzero vectors $\tilde{d} = (d_1, d_2, \dots, d_{t_i})$ and $d_{j_i} \in \{0, 1, \dots, n_{j_i}\}$ for $1 \le j_i \le t_i$ for all $1 \le i \le r$. Then $e = (A_1, \dots, A_t, 0, \dots, 0)$ is an idempotent of $\operatorname{Ker} L$, where

$$A_k = \begin{pmatrix} I_{d_k} & 0 \\ 0 & 0 \end{pmatrix}$$

and $\operatorname{Tr} A_k = d_k$ for all $1 \le k \le t$. Since $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$, we have $K[G]eK[G] \subseteq \operatorname{Ker} L$. That is, $(M_{n_1}(K)A_1M_{n_1}(K), \dots, M_{n_t}(K)A_tM_{n_t}(K), 0, \dots, 0) \subseteq \operatorname{Ker} L$. Since $M_{n_k}(K)A_kM_{n_k}(K)$ is a submodule of $M_{n_k}(K)$ and $M_{n_k}(K)$ is simple, we have $M_{n_k}(K)A_kM_{n_k}(K) = 0$ or $M_{n_k}(K)$. Without loss of generality, we can assume that $A_1 \neq 0$. Then we have $M_{n_1}(K)A_1M_{n_1}(K) = M_{n_1}(K)$. That is, $M_{n_1}(K) \times 0 \times \cdots \times 0 \subseteq \operatorname{Ker} L$, which is a contradiction. Then the conclusion follows.

(2) $\Rightarrow$ (1) Let $I = 0 \times \cdots \times 0 \times M_{n_{t+1}}(K) \times \cdots \times M_{n_s}(K)$. Then I is an ideal of A. We claim that $\operatorname{Ker} L/I$ has no nonzero idempotent. Suppose that e is a nonzero idempotent of $\operatorname{Ker} L/I$. Then we have $e = (\tilde{A}_1, \dots, \tilde{A}_t, 0, \dots, 0)$ and $\tilde{A}_k$ is similar to A_k for all $1 \le k \le t$, where A_k is defined as above. Thus, we have $\operatorname{Tr} \tilde{A}_k \in \{0, 1, \dots, n_k\}$ for all $1 \le k \le t$ and at least one of $\operatorname{Tr} \tilde{A}_k$ is nonzero for $1 \le k \le t$. Let $\tilde{d} = (d_1, d_2, \dots, d_t) = (\operatorname{Tr} \tilde{A}_1, \operatorname{Tr} \tilde{A}_2, \dots, \operatorname{Tr} \tilde{A}_t) \neq (0, 0, \dots, 0)$. Then $a_{i,1}d_1 + a_{i,2}d_2 + \cdots + a_{i,t_i}d_{t_i} = 0$ for all $1 \le i \le r$, which is a contradiction. Hence the claim follows. It follows from Theorem 4.2 in [8] that $\operatorname{Ker} L/I$ is a Mathieu-Zhao space of A/I. Then it follows from Proposition 2.7 in [8] that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$. □

Corollary 5.2. *Let K be a field and $V = \{(A_1, \dots, A_s) \in A \mid \sum_{j=1}^{s} a_{i,j} \operatorname{Tr} A_j = 0$ for all $1 \le i \le r\}$, where $A = M_{n_1}(K) \times \cdots \times M_{n_s}(K)$. Then V is a Mathieu-Zhao space of A if and only if $a_{i,1}d_1 + a_{i,2}d_2 + \cdots + a_{i,t_i}d_{t_i} \neq 0$ for some $i \in \{1, 2, \dots, r\}$ for all nonzero vectors $\tilde{d} = (d_1, d_2, \dots, d_{t_i})$ and $d_{j_i} \in \{0, 1, \dots, n_{j_i}\}$ for $1 \le j_i \le t_i$, $t_i \in \{1, \dots, s\}$.*

Proof. The conclusion follows from the proof of Proposition 5.1. □

Theorem 5.3. *Let L and G be as in Problem 1.2 and K a field of characteristic p. If G has a normal Sylow p-subgroup H and $L_1,\dots,L_r$ are class functions of G/H and K is a split field for G/H, then the following statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) $a_{i,1}d_1 + a_{i,2}d_2 + \cdots + a_{i,t_i}d_{t_i} \neq 0$ *for some* $i \in \{1,2,\dots,r\}$ *for all nonzero vectors* $\tilde{d} = (d_1, d_2, \dots, d_{t_i})$ *and* $d_{j_i} \in \{0,1,\dots,n_{j_i}\}$ *for* $1 \le j_i \le t_i$, $t_i \in \{1,\dots,s\}$, *where* $L_i = \sum_{j=1}^{s} a_{i,j}\chi_j$ *and* $\chi_1,\dots,\chi_s$ *are the distinct* (*up to isomorphism*) *irreducible characters of* G/H *and* $n_j = \chi_j(1)$, $a_{i,j} \in K$ *for all* $1 \le i \le r$, $1 \le j \le s$.

Proof. Let $|G| = p^a d$, $p \nmid d$ and $G = \cup_{j=1}^{d} \tilde{g}_j H$, $H = \{h_1, h_2, \dots, h_{\tilde{t}}\}$ with $\tilde{t} = p^a$. Then we have $L_i(\tilde{g}_j h_1) = L_i(\tilde{g}_j h_2) = \cdots = L_i(\tilde{g}_j h_{\tilde{t}})$ for all $1 \le i \le r$, $1 \le j \le d$. Hence the conclusion follows from Proposition 3.1 and Proposition 5.1. □

Remark 5.4. It's easy to see that $\operatorname{Ker} L = V_G$ if $L = n_1\chi_1 + n_2\chi_2 + \cdots + n_s\chi_s$ and $\chi_1,\dots,\chi_s$ are the distinct (up to isomorphism) irreducible characters of $K[G]$. If G has a normal Sylow p-subgroup H, then Theorem 5.3 implies Theorem 1.5 in [10].

Proposition 5.5. *Let L, G, K be as in Condition 2. Then the following two statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) *For all* $0 \neq b = (b_1,\dots,b_s) \in \{0,1,\dots,n_1\} \times \cdots \times \{0,1,\dots,n_s\}$ *with* $a_{i,1}b_1 + a_{i,2}b_2 + \cdots + a_{i,s}b_s = 0$ *for all* $1 \le i \le r-1$, *the following are true*:

(a) *there exists a* $\lambda_m \in K$ *such that* $C_m = \lambda_m I_{n_m}$ *for all* $m \in T_b$,

(b) $\sum_{m \in T_b} n_m \lambda_m b_m + \sum_{m \in S_b} n_m \operatorname{Tr}(C_m) \neq 0$,

where $L_i = \sum_{j=1}^{s} a_{i,j}\chi_j$, $\chi_1,\dots,\chi_s$ *are the distinct* (*up to isomorphism*) *irreducible characters of* $K[G]$ *and* $n_j = \chi_j(1)$, $C_j = \rho_j(\alpha_r)$, $\alpha_r = \sum_{j=1}^{n} l_{r,j} g_j^{-1}$, $G = \{g_1,\dots,g_n\}$, ρ_j *is an irreducible representation according to* χ_j, $a_{i,j} \in K^*$ *for all* $1 \le i \le r-1$, $1 \le j \le s$ *and* $T_b := \{1 \le m \le s \mid b_m \neq 0, n_m\}$, $S_b := \{1 \le m \le s \mid b_m = n_m\}$.

Proof. Since $L_1,\dots,L_{r-1}$ are class functions of G, we have

$$L_i = \sum_{j=1}^{s} a_{i,j}\chi_j$$

for all $1 \le i \le r-1$. Since $a_{i,j} \in K^*$ for all $1 \le i \le r-1$, $1 \le j \le s$, we have

$$\operatorname{Ker} L = \{\beta \in K[G] \mid \sum_{j=1}^{s} a_{i,j}\chi_j(\beta) = 0 \ \text{ and } \ L_r(\beta) = 0 \ \text{ for all } \ 1 \le i \le r-1\}.$$

Since $K[G]$ is semi-simple, we have $K[G] \cong M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K) := A$. It's easy to see that there exists $A_j \in M_{n_j}(K)$ such that $\operatorname{Tr} A_j = \chi_j(\beta)$ for all $1 \leq j \leq s$. It follows from Lemma 2.13 and Theorem 2.14 that $L_r(\beta) = 0$ if and only if

$$\sum_{j=1}^{s} n_j \operatorname{Tr}(C_j A_j) = 0,$$

where $A_j = \rho_j(\beta)$ and $C_j = \rho_j(\alpha_r)$, $\alpha_r = \sum_{j=1}^{n} l_{r,j} g_j^{-1}$ for all $1 \leq j \leq s$.

(2) $\Rightarrow$ (1) Since $\operatorname{Ker} L \cong V$ and

$$V = \{(A_1, \ldots, A_s) \in A \mid \sum_{j=1}^{s} a_{i,j} \operatorname{Tr} A_j = 0 \text{ and } \sum_{j=1}^{s} n_j \operatorname{Tr}(C_j A_j) = 0 \text{ for all } 1 \leq i \leq r-1\}$$

and for all $0 \neq b = (b_1, b_2, \ldots, b_s) \in \{0, 1, \ldots, n_1\} \times \cdots \times \{0, 1, \ldots, n_s\}$ with $a_{i,1}b_1 + a_{i,2}b_2 + \cdots + a_{i,s}b_s = 0$ for all $1 \leq i \leq r-1$, we have that:

(a) there exists a $\lambda_m \in K$ such that $C_m = \lambda_m I_{n_m}$ for all $m \in T_b$,

(b) $\sum_{m \in T_b} n_m \lambda_m b_m + \sum_{m \in S_b} n_m \operatorname{Tr}(C_m) \neq 0$.

Now suppose that V contains a nonzero idempotent $(E_1, \ldots, E_s)$ and $b_j = \operatorname{Tr}(E_j)$ for $1 \leq j \leq s$. Then we have that $a_{i,1}b_1 + \cdots + a_{i,s}b_s = 0$ for all $1 \leq i \leq r-1$ and (a), (b) hold. Hence we have

$$\sum_{m \in T_b} n_m \lambda_m b_m + \sum_{m \in S_b} n_m \operatorname{Tr}(C_m) \neq 0,$$

which contradicts with $(E_1, \ldots, E_s) \in V$. Thus, V does not contain any nonzero idempotent and hence is Mathieu-Zhao space of $K[G]$. Then the conclusion follows.

(1) $\Rightarrow$ (2) Suppose that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ and there exists a $0 \neq b = (b_1, \ldots, b_s) \in \{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_s\}$ with $a_{i,1}b_1 + \cdots + a_{i,s}b_s = 0$ for all $1 \leq i \leq r-1$ such that (a) does not hold. Then there is an $m \in T_b$ such that C_m is not a multiple of the identity matrix. Let E_j be the matrix with ones on the first b_j diagonal entries and zeros on all other entries for all $1 \leq j \leq s$ with $j \neq m$. Then E_j is an idempotent of rank b_j. It follows from Lemma 4.6 in [4] that there exists an idempotent E_m of rank $b_m \neq 0, n_m$ such that

$$\operatorname{Tr}(C_m E_m) = -\frac{1}{n_m} \sum_{j \neq m} n_j \operatorname{Tr}(C_j E_j).$$

Since $\operatorname{Tr} E_j = \operatorname{rank} E_j$ for all $1 \leq j \leq s$, we have that $(E_1, E_2, \ldots, E_s)$ is a nonzero idempotent which contained in V. This contradicts with that V is a Mathieu-Zhao space of A.

Suppose that there exists a $0 \neq b = (b_1, \ldots, b_s) \in \{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_s\}$ with $a_{i,1}b_1 + \cdots + a_{i,s}b_s = 0$ for all $1 \leq i \leq r-1$ such that (1) does hold but (2) does not hold. Let E_j be the matrix with ones on the first b_j diagonal entries and zero on all other entries. Then E_j is an idempotent of rank b_j. Since $\operatorname{Tr} E_j = \operatorname{rank} E_j$ for all $1 \le j \le s$, we have

$$\sum_{m\in T_b} n_m\lambda_m b_m + \sum_{m\in S_b} n_m \operatorname{Tr}(C_m) = 0,$$

which exactly means that $(E_1,\dots,E_s)$ is contained in V. As $b \neq 0$, we have that V contains a nonzero idempotent, which contradicts with that V is a Mathieu-Zhao space of A. Then the conclusion follows. □

We can remove the condition that $a_{i,j} \in K^*$ for all $1 \le i \le r-1$, $1 \le j \le s$ in Proposition 5.5 by introducing a new set $X := \{a_{i,j} \mid \text{there exists } i_j \in \{1,2,\dots,r-1\} \text{ such that } a_{i_j,j} \neq 0 \text{ for } 1 \le i \le r-1, 1 \le j \le s\}$. Then we have the following theorem.

Theorem 5.6. *Let L, G, K be as in Condition 2. Then the following two statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) *For all* $0 \neq b = (b_{k_1}, b_{k_2},\dots,b_{k_t}) \in \{0,1,\dots,n_{k_1}\}\times\cdots\times\{0,1,\dots,n_{k_t}\}$ *with* $a_{i,k_1}b_{k_1} + a_{i,k_2}b_{k_2} + \cdots + a_{i,k_t}b_{k_t} = 0$ *for all* $1 \le i \le r-1$, *the following are true*:

(a) *there exists a* $\lambda_m \in K$ *such that* $C_m = \lambda_m I_{n_m}$ *for all* $m \in T_b \cap X$,

(b) $\sum_{m\in T_b\cap X} n_m\lambda_m b_m + \sum_{m\in S_b\cap X} n_m \operatorname{Tr}(C_m) \neq 0$,

where $L_i = \sum_{j=1}^s a_{i,j}\chi_j$, $\chi_1,\dots,\chi_s$ *are the distinct* (*up to isomorphism*) *irreducible characters of* $K[G]$, $a_{i,j} \in K$ *and* $n_j = \chi_j(1)$, $C_j = \rho_j(\alpha_r)$, $\alpha_r = \sum_{j=1}^n l_{r,j}g_j^{-1}$, $G = \{g_1,\dots,g_n\}$, ρ_j *is an irreducible representation according to* χ_j *for all* $1 \le i \le r-1$, $1 \le j \le s$ *and* $T_b := \{1 \le m \le s \mid b_m \neq 0, n_m\}$, $S_b := \{1 \le m \le s \mid b_m = n_m\}$, $X = \{a_{i,j} \mid \text{there exists } i_j \in \{1,2,\dots,r-1\} \text{ such that } a_{i_j,j} \neq 0 \text{ for } 1 \le i \le r-1, 1 \le j \le s\} = \{a_{i,k_1},\dots,a_{i,k_t} \text{ for } 1 \le i \le r-1\}$.

Proof. The conclusion follows by following the arguments of Proposition 5.5. □

Proposition 5.7. *Let L and G be as in Problem 1.2 and K a field of characteristic p, $|G| = p^a d$, H a normal Sylow p-subgroup of G and $G = \cup_{j=1}^d \tilde{g}_j H$, $H = \{1_H, h_2,\dots,h_{\tilde{t}}\}$ for $\tilde{t} = p^a$. Suppose that $L_r(\tilde{g}_j h_2) = L_r(\tilde{g}_j h_3) = \cdots = L_r(\tilde{g}_j h_{\tilde{t}})$ for all $1 \le j \le d$ and K is a split field for G/H. If there exist $\tilde{i}, \hat{i} \in \{1,2,\dots,r\}$ such that* $\det M_{L_{\tilde{i}}|_{G/H}} \neq 0$ *and* $\det M_{L_{\hat{i}}} \neq 0$ *and $L_1,\dots,L_{r-1}$ are class functions of G/H, then the following two statements are equivalent*:

(1) $\operatorname{Ker} L$ *is a Mathieu-Zhao space of* $K[G]$.

(2) *For all* $0 \neq b = (b_{k_1}, b_{k_2},\dots,b_{k_t}) \in \{0,1,\dots,n_{k_1}\}\times\cdots\times\{0,1,\dots,n_{k_t}\}$ *with* $a_{i,k_1}b_{k_1} + a_{i,k_2}b_{k_2} + \cdots + a_{i,k_t}b_{k_t} = 0$ *for all* $1 \le i \le r-1$, *the following are true*:

(a) *there exists a* $\lambda_m \in K$ *such that* $C_m = \lambda_m I_{n_m}$ *for all* $m \in T_b \cap X$,

(b) $\sum_{m\in T_b\cap X} n_m\lambda_m b_m + \sum_{m\in S_b\cap X} n_m \operatorname{Tr}(C_m) \neq 0$,
where $L_i = \sum_{j=1}^{s} a_{i,j}\chi_j$, $\chi_1,\dots,\chi_s$ *are the distinct (up to isomorphism) irreducible characters of* $K[G]$, $a_{i,j}\in K$ *and* $n_j=\chi_j(1)$, $C_j=\rho_j(\alpha_r)$, $\alpha_r = \sum_{j=1}^{d} l_{r,j}\tilde{g}_j^{-1}$, $G/H=\{\tilde{g}_1,\dots,\tilde{g}_d\}$, ρ_j *is an irreducible representation according to* χ_j *for all* $1\le i\le r-1$, $1\le j\le s$ *and* $M_{L_{\hat{i}}|_{G/H}}$, $M_{L_{\hat{i}}}$ *be as in Proposition 3.4*; T_b, S_b, X *be as in Theorem 5.6.*

Proof. Since $L_1,\dots,L_{r-1}$ are class functions of G/H, we have $L_i(\tilde{g}_j 1_H) = L_i(\tilde{g}_j h_2) = \cdots = L_i(\tilde{g}_j h_{\tilde{t}})$ for all $1\le i\le r-1$. Then it follows from Proposition 3.4 that $\operatorname{Ker} L$ is a Mathieu-Zhao space of $K[G]$ if and only if $\operatorname{Ker}(L|_{G/H})$ is a Mathieu-Zhao space of $K[G/H]$. Since $p\nmid |G/H|$, the conclusion follows from Theorem 5.6. □

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