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COMPLEX MOMENTS AND THE DISTRIBUTION OF VALUES OF $L(1, \chi_u)$ IN EVEN CHARACTERISTIC

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Abstract. In this paper, we announce that the strategy of comparing the complex moments of $L(1, \chi_u)$ to that of a random Euler product $L(1, \mathbb{X})$ is also valid in even characteristic case. We give an asymptotic formulas for the complex moments of $L(1, \chi_u)$ in a large uniform range. We also give Ω-results for the extreme values of $L(1, \chi_u)$.

1. Introduction

The study of distribution of class numbers is an important problem in number theory. The case of quadratic number fields $\mathbb{Q}(\sqrt{d})$ has a long history of investigation that extends back to Gauss. According to the Dirichlet's class number formula, the distribution of class numbers h_d of $\mathbb{Q}(\sqrt{d})$ is equivalent to that of $L(1, \chi_d)$, where $L(s, \chi_d)$ is the Dirichlet L-function associated to a quadratic character χ_d. Recently some remarkable progressions on this problem have been done by Granville and Soundararajan [5] and Dahl and Lamzouri [4]. Their strategy is to compare the complex moment of $L(1, \chi_d)$ to that of a random Euler product $L(1, \mathbb{X})$.

Let $\mathbb{F}_q[T]$ be the polynomial ring over a finite field $\mathbb{F}_q$, where q is odd. For any square-free monic polynomial D in $\mathbb{F}_q[T]$, let $L(s, \chi_D)$ be the Dirichlet L-function associated to a quadratic character χ_D. Denote by $\mathcal{H}_n$ the set of square-free monic polynomials in $\mathbb{F}_q[T]$ of degree n. In [1], Andrade calculated the mean value of $L(1, \chi_D)$ averaging over $\mathcal{H}_{2g+1}$ by using an approximate functional equation for $L(1, \chi_D)$. The case of the mean value for $L(1, \chi_D)$ over $\mathcal{H}_{2g+2}$ was investigated by Jung [6]. This problem is also considered by the authors in [2] when q is even. In a recent paper [7], motivating by the work of Granville and Soundararajan [5], Lumley gave an asymptotic formula for the complex moments of $L(1, \chi_D)$ in a large uniform range by comparing with that of a random Euler product $L(1, \mathbb{X})$ and showed that the distribution function of

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$L(1,\chi_D)$ is very close to that of a corresponding probabilistic model. She also obtained Ω-results for the extreme values of $L(1,\chi_D)$. In this paper, we show that the strategy of comparing the complex moments of $L(1,\chi_u)$ to that of a random Euler product $L(1,\mathbb{X})$ is also valid in even characteristic case. Here, χ_u denotes the character defined by quadratic symbol $\{\frac{u}{\cdot}\}$ (see §1.2). We give an asymptotic formula for the complex moments of $L(1,\chi_u)$ in a large uniform range. We also give Ω-results for the extreme values of $L(1,\chi_u)$.

We fix some basic notations. Let $k=\mathbb{F}_q(T)$ be the rational function field with a constant field $\mathbb{F}_q$, where q is assumed to be even throughout the paper, and $\mathbb{A}=\mathbb{F}_q[T]$. Denote by $\mathbb{A}^+$ the set of monic polynomials in $\mathbb{A}$ and by $\mathcal{P}$ the set of monic irreducible polynomials in $\mathbb{A}$. Let $\mathbb{A}_n=\{f\in\mathbb{A}:\deg f=n\}$, $\mathbb{A}_n^+=\mathbb{A}^+\cap\mathbb{A}_n$ and $\mathcal{P}_n=\mathcal{P}\cap\mathbb{A}_n$ for any positive integer n. The zeta function $\zeta_{\mathbb{A}}(s)$ of $\mathbb{A}$ is defined to be the following infinite series:

$$\zeta_{\mathbb{A}}(s)=\sum_{f\in\mathbb{A}^+}\frac{1}{|f|^s}=\prod_{P\in\mathcal{P}}\left(1-\frac{1}{|P|^s}\right)^{-1},\quad \mathrm{Re}(s)>1,$$

where $|f|=q^{\deg f}$. It is well known that $\zeta_{\mathbb{A}}(s)=1/(1-q^{1-s})$. For $f\in\mathbb{A}^+$, let $\Phi(f)=|(\mathbb{A}/f\mathbb{A})^\times|$.

1.1. Quadratic function field in even characteristic

In this subsection, we recall some basic facts on quadratic function field in even characteristic. For more details, we refer to [2, §2.2, §2.3]. Any separable quadratic extension of k is of the form $K_u=k(x_u)$, where x_u is a zero of $X^2+X+u=0$ for some $u\in k$. Fix an element $\xi\in\mathbb{F}_q\setminus\wp(\mathbb{F}_q)$, where $\wp:k\to k$ is the additive homomorphism defined by $\wp(x)=x^2+x$. We say that $u\in k$ is normalized if it is of the form

$$u=\sum_{i=1}^{m}\sum_{j=1}^{e_i}\frac{A_{ij}}{P_i^{2j-1}}+\sum_{\ell=1}^{n}\alpha_\ell T^{2\ell-1}+\alpha,$$

where $P_i\in\mathcal{P}$ are distinct, $A_{ij}\in\mathbb{A}$ with $\deg A_{ij}<\deg P_i$, $A_{ie_i}\neq 0$, $\alpha\in\{0,\xi\}$, $\alpha_\ell\in\mathbb{F}_q$ and $\alpha_n\neq 0$ for $n>0$. Let $u\in k$ be a normalized one. The infinite prime $(1/T)$ of k splits, is inert or ramified in K_u according as $n=0$ and $\alpha=0$, $n=0$ and $\alpha=\xi$, or $n>0$. Then the field K_u is called real, inert imaginary, or ramified imaginary, respectively. The discriminant D_u of K_u is given by

$$D_u=\begin{cases}\prod_{i=1}^m P_i^{2e_i} & \text{if } n=0,\\ \prod_{i=1}^m P_i^{2e_i}\cdot(1/T)^{2n} & \text{if } n>0,\end{cases}$$

and the genus g_u of K_u is given by $g_u=\deg D_u/2-1$.

For $M\in\mathbb{A}^+$, write $r(M)=\prod_{P|M}P$ and $t(M)=M\cdot r(M)$. For $P\in\mathcal{P}$, let ν_P be the normalized valuation at P, that is, $\nu_P(M)=e$, where $P^e\|M$. Let $\mathcal{B}$ be the set of non-constant monic polynomials M such that $\nu_P(M)$ is zero or odd for any $P\in\mathcal{P}$, that is, $t(M)$ is a square, and $\mathcal{B}_n=\{M\in\mathcal{B}:\deg t(M)=2n\}$. The map $\mathcal{B}_n\to\mathbb{A}_n^+$ defined by $M\mapsto\tilde{M}=\sqrt{M}$ is a bijection with the inverse

$N \mapsto N^* = N^2/r(N)$. Hence, $|\mathcal{B}_n| = |\mathbb{A}_n^+| = q^n$. Let $\mathcal{E}$ be the set of rational functions $D/M \in k$ with $D \in \mathbb{A}$, $M \in \mathcal{B}$ and $\deg D < \deg M$ which can be written as

$$\frac{D}{M} = \sum_{P|M} \sum_{i=1}^{\ell_P} \frac{A_{P,i}}{P^{2i-1}},$$

where $\deg A_{P,i} < \deg P$ for any $P \mid M$ and $1 \leq i \leq \ell_P = (\nu_P(M) + 1)/2$. Note that for $D/M \in \mathcal{E}$, $\gcd(D, M) = 1$ if and only if $A_{P,\ell_P} \neq 0$ for all $P \mid M$. Let $\mathcal{F}$ be the subset of $\mathcal{E}$ consisting of all $D/M \in \mathcal{E}$ such that $A_{P,\ell_P} \neq 0$ for all $P \mid M$. Under the correspondence $u \mapsto K_u$, $\mathcal{F}$ corresponds to the set of all real separable quadratic extensions K_u of k. For $M \in \mathcal{B}$, let $\mathcal{E}_M$ be the set of rational functions $u \in \mathcal{E}$ whose denominator is M and $\mathcal{F}_M = \mathcal{F} \cap \mathcal{E}_M$. Then $\mathcal{F}$ is the disjoint union of $\mathcal{F}_M$ with $M \in \mathcal{B}$. For $u \in \mathcal{F}_M$, the discriminant D_u and the genus g_u of K_u are $D_u = t(M)$ and $g_u = \deg t(M)/2 - 1$. For $n \geq 1$, let $\mathcal{F}_n$ be the union of $\mathcal{F}_M$ with $M \in \mathcal{B}_n$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{F}_n$ corresponds to the set of all real separable quadratic extensions K_u of k with genus $n-1$. For $M \in \mathcal{B}_n$, there are $\Phi(\tilde{M})$ D's such that $D/M \in \mathcal{F}_n$, so that $|\mathcal{F}_M| = \Phi(\tilde{M})$ and

$$|\mathcal{F}_n| = \sum_{M \in \mathcal{B}_n} \Phi(\tilde{M}) = \sum_{\tilde{M} \in \mathbb{A}_n^+} \Phi(\tilde{M}) = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}.$$

For any subset U of k and $w \in k$, write $U + w = \{u + w : u \in U\}$. Under the correspondence $u \mapsto K_u$, $\mathcal{F}' = \mathcal{F} + \xi$ corresponds to the set of all inert imaginary separable quadratic extensions K_u of k, and for $n \geq 1$, $\mathcal{F}'_n = \mathcal{F}_n + \xi$ corresponds to the set of all inert imaginary separable quadratic extensions K_u of k with genus $n-1$. For a positive integer s, let $\mathcal{G}_s$ be the set of polynomials $F(T) \in \mathbb{A}$ of the form

$$F(T) = \alpha + \sum_{i=1}^{s} \alpha_i T^{2i-1},$$

where $\alpha \in \{0, \xi\}$, $\alpha_i \in \mathbb{F}_q$ and $\alpha_s \neq 0$. For any two subsets U, V of k and $w \in k$, write $U + V = \{u + v : u \in U, v \in V\}$. Let $\mathcal{I} = (\mathcal{F} \cup \{0\}) + \mathcal{G}$, where $\mathcal{G} = \bigcup_{s \geq 1} \mathcal{G}_s$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{I}$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k. For $w \in \mathcal{F}_M + \mathcal{G}_s$, the discriminant D_w and the genus g_w of K_w are $D_w = t(M) \cdot (1/T)^{2s}$ and $g_w = \deg t(M)/2 + s - 1$. Let $\mathcal{F}_0 = \{0\}$. For any $r \geq 0$ and $s \geq 1$, let $\mathcal{I}_{(r,s)} = \mathcal{F}_r + \mathcal{G}_s$. If $w \in \mathcal{I}_{(r,s)}$, the genus g_w of K_w is $r + s - 1$. For $n \geq 1$, let $\mathcal{I}_n$ be the union of all $\mathcal{I}_{(r,s)}$, where (r, s) runs over all pairs of non-negative integers such that $s > 0$ and $r + s = n$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{I}_n$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k with genus $n-1$. Since $|\mathcal{G}_s| = 2\zeta_{\mathbb{A}}(2)^{-1} q^s$ for $s \geq 1$, we have

$$|\mathcal{I}_n| = \sum_{s=1}^{n} |\mathcal{F}_{n-s}| \cdot |\mathcal{G}_s| = 2\zeta_{\mathbb{A}}(2)^{-1} q^{2n-1}.$$

1.2. Hasse symbol and L-functions

For any $u \in k$ whose denominator is not divisible by $P \in \mathcal{P}$, the Hasse symbol $[u, P)$ with values in $\mathbb{F}_2$ is defined by

$$[u,P) = \begin{cases} 0 & \text{if } X^2 + X \equiv u \bmod P \text{ is solvable in } \mathbb{A}, \\ 1 & \text{otherwise.} \end{cases}$$

For $N \in \mathbb{A}$ prime to the denominator of u, if $N = sgn(N) \prod_{i=1}^{s} P_i^{e_i}$, where $sgn(N)$ is the leading coefficient of N and $P_i \in \mathcal{P}$ are distinct and $e_i \geq 1$, the symbol $[u, N)$ is defined to be $\sum_{i=1}^{s} e_i [u, P_i)$.

For $u \in k$ and $0 \neq N \in \mathbb{A}$, the quadratic symbol $\{\frac{u}{N}\}$ is defined as follows:

$$\left\{\frac{u}{N}\right\} = \begin{cases} (-1)^{[u,N)} & \text{if } N \text{ is prime to the denominator of } u, \\ 0 & \text{otherwise.} \end{cases}$$

This symbol is clearly additive in its first variable, and multiplicative in the second variable.

For the field K_u, we associate a character χ_u on $\mathbb{A}^+$ which is defined by $\chi_u(f) = \{\frac{u}{f}\}$, and let $L(s, \chi_u)$ be the L-function associated to the character χ_u: for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 1$,

$$L(s, \chi_u) = \sum_{f \in \mathbb{A}^+} \frac{\chi_u(f)}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{\chi_u(P)}{|P|^s}\right)^{-1}.$$

It is known that $L(s, \chi_u)$ is a polynomial in q^{-s} of degree $2g_u + (1 + (-1)^{\varepsilon(u)})/2$, where $\varepsilon(u) = 1$ if K_u is ramified imaginary and $\varepsilon(u) = 0$ otherwise.

For any $z \in \mathbb{C}$, the generalized divisor function $d_z(f)$ is defined on its prime powers as

$$d_z(P^a) = \frac{\Gamma(z+a)}{\Gamma(z) a!},$$

and is extended to all monic polynomials multiplicatively. We have

$$L(s, \chi_u)^z = \sum_{f \in \mathbb{A}^+} \frac{d_z(f) \chi_u(f)}{|f|^s} = \prod_{P \in \mathcal{P}} \left(1 - \frac{\chi_u(P)}{|P|^s}\right)^{-z}.$$

1.3. A random Euler product $L(1, \mathbb{X})$

Let $\{\mathbb{X}(P)\}$ be a sequence of independent random variables indexed by $P \in \mathcal{P}$, and taking the values $0, \pm 1$ as follows:

$$\mathbb{X}(P) = \begin{cases} 0 & \text{with probability } \frac{1}{|P|+1}, \\ \pm 1 & \text{with probability } \frac{|P|}{2(|P|+1)}. \end{cases}$$

The reason for defining $\mathbb{X}(P)$ is different from odd characteristic case. There are $|P| + 1$ values modulo P including $\infty = (1/T)$. Among these values one value a including ∞ has $\{\frac{a}{P}\} = 0$, $|P|/2$ values have $\{\frac{a}{P}\} = 1$, and $|P|/2$ values have $\{\frac{a}{P}\} = -1$. We extend the definition of $\mathbb{X}$ multiplicatively as follows:

$\mathbb{X}(1)=1$ and $\mathbb{X}(f)=\mathbb{X}(P_1)^{e_1}\mathbb{X}(P_2)^{e_2}\cdots\mathbb{X}(P_r)^{e_r}$ if $f=P_1^{e_1}P_2^{e_2}\cdots P_r^{e_r}$ is the prime power factorization of non-constant polynomial $f\in\mathbb{A}^+$. The random Euler product $L(1,\mathbb{X})$ is defined as

$$L(1,\mathbb{X})=\sum_{f\in\mathbb{A}^+}\frac{\mathbb{X}(f)}{|f|}=\prod_{P\in\mathcal{P}}\left(1-\frac{\mathbb{X}(P)}{|P|}\right)^{-1}.$$

Aside from the reason of definition, the random variables $\mathbb{X}(P)$ have the same values with the same probability as those of odd characteristic case. Thus the random Euler product $L(1,\mathbb{X})$ in this paper shares the same properties with the ones in [7]. For example, it satisfies Lemma 3.6 in [7], that is, the mean value $\mathbb{E}(\mathbb{X}(f))$ of $\mathbb{X}(f)$ is given as follows:

$$\mathbb{E}(\mathbb{X}(f))=\begin{cases}0 & \text{if } f \text{ is not a square,}\\ \prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1} & \text{if } f \text{ is a square.}\end{cases}\tag{1.1}$$

Hence, we also have

$$\mathbb{E}(L(1,\mathbb{X})^z)=\sum_{f\in\mathbb{A}^+}\frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}.\tag{1.2}$$

For the remainder of this article, log denotes the base q logarithm, $\log_j$ represents the j-fold iterated logarithm and ln is the natural logarithm. Write

$$\mathbb{E}(L(1,\mathbb{X})^z)=\prod_{P\in\mathcal{P}}E_P(z),\quad\text{where}\quad E_P(z)=\mathbb{E}\left(\left(1-\frac{\mathbb{X}(P)}{|P|}\right)^{-z}\right)$$

and

$$\mathcal{L}(z)=\ln\mathbb{E}(L(1,\mathbb{X})^z)=\sum_{P\in\mathcal{P}}\ln E_P(z).$$

Let

$$f(t)=\begin{cases}\ln\cosh(t) & \text{if } 0\le t<1,\\ \ln\cosh(t)-t & \text{if } t\ge 1.\end{cases}$$

Then we have the following proposition.

Proposition 1.1 ([7, Proposition 4.2]). *Let $c_q\ge q$ be a positive constant depending on q and r be a real number such that $r\ge c_q$. Let $k\in\mathbb{Z}$ be the unique positive integer such that $q^k\le r<q^{k+1}$ and let $t=r/q^k$. Then we have*

$$\mathcal{L}(r)=r(\ln\log r+\gamma)+\frac{r}{\log r}G_1(t)+O\left(\frac{r\log_2 r}{(\log r)^2}\right),$$

where

$$G_1(t)=\frac{1}{2}-\log t+\sum_{\ell=-\infty}^{\infty}\frac{f(tq^\ell)}{tq^\ell}.$$

Furthermore, we have

$$\mathcal{L}'(r) = \ln\log r + \gamma + \frac{1}{\log r} G_2(t) + O\left(\frac{\log_2 r}{(\log r)^2}\right),$$

where

$$G_2(t) = \frac{1}{2} - \log t + \sum_{\ell=-\infty}^{\infty} f'(tq^\ell).$$

Moreover, for all real numbers x, y such that $|y| \geq c_q$ and $|y| \leq |x|$ we have

$$\mathcal{L}''(y) \asymp \frac{1}{|y|\ln|y|} \quad \text{and} \quad \mathcal{L}'''(y) \asymp \frac{1}{|y|^2 \ln|y|}.$$

For $\tau > 0$, define

$$\Phi_{\mathbb{X}}(\tau) = \mathbb{P}(L(1,\mathbb{X}) > e^\gamma \tau) \quad \text{and} \quad \Psi_{\mathbb{X}}(\tau) = \mathbb{P}\left(L(1,\mathbb{X}) < \frac{\zeta(2)}{e^\gamma \tau}\right).$$

Then we have the following theorem concerning the asymptotic behaviours of $\Phi_{\mathbb{X}}(\tau)$ and $\Psi_{\mathbb{X}}(\tau)$.

Theorem 1.2 ([7, Theorem 1.3])**.** *For any large τ we have*

$$\Phi_{\mathbb{X}}(\tau) = \exp\left(-C_1(q^{\log\kappa(\tau)}) \frac{q^{\tau - C_0(q^{\log\kappa(\tau)})}}{\tau}\left(1 + O\left(\frac{\log\tau}{\tau}\right)\right)\right),$$

where $\kappa(\tau)$ is the unique solution of $\mathcal{L}'(r) = \ln\tau + \gamma$, $C_0(t) = G_2(t)$ and $C_1(t) = G_2(t) - G_1(t)$. The same estimate also holds for $\Psi_{\mathbb{X}}(\tau)$. Moreover, if $0 < \lambda < e^{-\tau}$, then we have

$$\Phi_{\mathbb{X}}(e^{-\lambda}\tau) = \Phi_{\mathbb{X}}(\tau)\left(1 + O(\lambda e^\tau)\right) \quad \text{and} \quad \Psi_{\mathbb{X}}(e^{-\lambda}\tau) = \Psi_{\mathbb{X}}(\tau)\left(1 + O(\lambda e^\tau)\right).$$

1.4. Results

We have the following lower and upper bounds of $L(s,\chi_u)$, which is an even characteristic analogue of [7, Proposition 1.4].

Proposition 1.3. *Let $u \in k$ be normalized one and g_u be the genus of K_u. For any complex number $s \in \mathbb{C}$ with $\mathrm{Re}(s) = 1$, we have*

$$\frac{\zeta_{\mathbb{A}}(2)}{2e^\gamma}\left(\log g_u + O(1)\right)^{-1} \leq |L(s,\chi_u)| \leq 2e^\gamma \log g_u + O(1). \tag{1.3}$$

We have the following result concerning the complex moments of $L(1,\chi_u)$ as u varies over $\mathcal{F}_n$ or $\mathcal{I}_n$.

Theorem 1.4. *Let n be a positive integer and $z \in \mathbb{C}$ be such that $|z| \leq n/(260\log n \ln\log n)$. Then we have*

$$\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n} L(1,\chi_u)^z = \sum_{f\in\mathbb{A}^+} \frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1 + \frac{1}{|P|}\right)^{-1}\left(1 + O\left(\frac{1}{n^{11}}\right)\right)$$

and

$$\frac{1}{|\mathcal{I}_n|}\sum_{u\in\mathcal{I}_n} L(1,\chi_u)^z = \sum_{f\in\mathbb{A}^+}\frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\left(1+O\left(\frac{1}{n^{11}}\right)\right).$$

We can prove that the distribution of $L(1,\chi_u)$ is well-approximated by the distribution of $L(1,\mathbb{X})$ uniformly in a large range.

Theorem 1.5. *Let n be large. Uniformly in $1\le\tau\le\log n-2\log_2 n-\log_3 n$ we have*

$$\frac{1}{|\mathcal{F}_n|}|\{u\in\mathcal{F}_n : L(1,\chi_u)>e^{\gamma}\tau\}| = \Phi_{\mathbb{X}}(\tau)\left(1+O\left(\frac{e^{\tau}(\log n)^2\log_2 n}{n}\right)\right)$$

and

$$\frac{1}{|\mathcal{F}_n|}\left|\left\{u\in\mathcal{F}_n : L(1,\chi_u)<\frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}\right\}\right| = \Psi_{\mathbb{X}}(\tau)\left(1+O\left(\frac{e^{\tau}(\log n)^2\log_2 n}{n}\right)\right).$$

Furthermore, the same result also holds for $L(1,\chi_u)$ over $\mathcal{I}_n$.

Let $\mathcal{O}_u$ denote the integral closure of $\mathbb{A}$ in K_u and h_u be the ideal class number of $\mathcal{O}_u$. If $u\in\mathcal{I}_n$, since $g_u=n-1$, we have (see (2.8) in [2])

$$L(1,\chi_u)=q^{1-n}h_u. \tag{1.4}$$

Then from Theorem 1.4 with (1.4), we get the following complex moment of h_u over $\mathcal{I}_n$.

Corollary 1.6. *Let $z\in\mathbb{C}$ be such that $|z|\le n/(260\log n\ln\log n)$. Then we have*

$$\frac{1}{|\mathcal{I}_n|}\sum_{u\in\mathcal{I}_n} h_u^z = q^{(n-1)z}\sum_{f\in\mathbb{A}^+}\frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\left(1+O\left(\frac{1}{n^{11}}\right)\right).$$

For any $u\in\mathcal{I}_n$, by (1.4), we have that $h_u>e^{\gamma}\tau q^{n-1}$ if and only if $L(1,\chi_u)>e^{\gamma}\tau$, and $h_u<q^{n-1}\zeta_{\mathbb{A}}(2)/(e^{\gamma}\tau)$ if and only if $L(1,\chi_u)<\zeta_{\mathbb{A}}(2)/(e^{\gamma}\tau)$. Thus Theorem 1.5 together with the asymptotic behaviors of $\Phi_{\mathbb{X}}(\tau)$ and $\Psi_{\mathbb{X}}(\tau)$ in Theorem 1.2 implies the following corollary.

Corollary 1.7. *Let n be large and $1\le\tau\le\log n-2\log_2 n-\log_3 n$. The number of $u\in\mathcal{I}_n$ such that*

$$h_u>e^{\gamma}\tau q^{n-1}$$

equals

$$|\mathcal{I}_n|\cdot\exp\left(-C_1(q^{\log\kappa(\tau)})\frac{q^{\tau-C_0(q^{\log\kappa(\tau)})}}{\tau}\left(1+O\left(\frac{\log\tau}{\tau}\right)\right)\right),$$

where $\kappa(\tau)$ is the unique solution of $\mathcal{L}'(r)=\ln\tau+\gamma$, $C_1(q^{\log\kappa(\tau)})$ and $C_0(q^{\log\kappa(\tau)})$ are positive constants depending on τ given in Theorem 1.2. Similar estimate holds for the number of $u\in\mathcal{I}_n$ such that

$$h_u<\frac{\zeta_{\mathbb{A}}(2)}{e^{\gamma}\tau}q^{n-1}.$$

For any $u \in \mathcal{F}_n$, we have

$$L(1, \chi_u) = \frac{h_u R_u}{\zeta_{\mathbb{A}}(2) q^{n-1}}, \tag{1.5}$$

where R_u is the regulator of $\mathcal{O}_u$. From Theorem 1.4 with (1.5), we get the following complex moment of $h_u R_u$ over $\mathcal{F}_n$.

Corollary 1.8. *Let $z \in \mathbb{C}$ be such that $|z| \le n/(260 \log n \ln \log n)$. Then*

$$\frac{1}{|\mathcal{F}_n|} \sum_{u \in \mathcal{F}_n} (h_u R_u)^z = \left(\frac{q^n}{q-1}\right)^z \sum_{f \in \mathbb{A}^+} \frac{d_z(f^2)}{|f|^2} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} \left(1 + O\left(\frac{1}{n^{11}}\right)\right).$$

Finally, we also obtain Ω-results for the extreme values of $L(1, \chi_u)$, which is an even characteristic analogue of [7, Theorem 1.6].

Theorem 1.9. *Let n be a large positive integer. There are monic irreducible polynomials Q_1 and Q_2 of degree n such that*

$$L(1, \chi_u) \ge e^\gamma (\log n + \log \log n) + O(1)$$

for some $u \in \mathcal{F}_{Q_1}$, and

$$L(1, \chi_v) \le \frac{\zeta_{\mathbb{A}}(2)}{e^\gamma} (\log n + \log \log n + O(1))^{-1}$$

for some $v \in \mathcal{F}_{Q_2}$.

Note that $|\mathcal{F}_n| = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}$, which is the same as the number of square-free monic polynomials in $\mathbb{A}$ of degree $2n$. We can follow almost the same arguments of [7] in odd characteristic, replacing χ_D by χ_u, $D \in \mathcal{H}_{2n}$ by $u \in \mathcal{F}_n$, $\log |D|$ by n, etc, to get Theorems 1.4 and 1.5, which are even characteristic analogues of Theorems 1.1 and 1.2 of [7], respectively. The proofs for $\mathcal{I}_n$ are almost the same as those for $\mathcal{F}_n$. We will give a sketch of a proof of Proposition 1.3 and proofs of Theorem 1.4 for $\mathcal{F}_n$ and of Theorem 1.9 in §3. More care is needed for Theorem 1.9, because there does not exist reciprocity law for Hasse symbols.

2. Two key lemmas

In this section, we give two key lemmas which are necessary ones in proofs.

We first give the following orthogonality relations for character sums over $\mathcal{F}_n$ and $\mathcal{I}_n$, which are even characteristic analogue of [7, Lemma 2.4].

Lemma 2.1. *Let $f \in \mathbb{A}^+$. If f is a square in $\mathbb{A}$, then*

$$\sum_{u \in \mathcal{F}_n} \chi_u(f) = |\mathcal{F}_n| \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} + O\left(|\mathcal{F}_n|^{\frac{1}{2}(1+\epsilon)}\right) \tag{2.1}$$

and

$$\sum_{u \in \mathcal{I}_n} \chi_u(f) = |\mathcal{I}_n| \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} + O\left(|\mathcal{I}_n|^{\frac{1}{2}(1+\epsilon)}\right). \tag{2.2}$$

Furthermore, if f is not a square in $\mathbb{A}$, then

$$\sum_{u \in \mathcal{F}_n} \chi_u(f) \ll 2^{\deg f/2} \sqrt{|\mathcal{F}_n|} \tag{2.3}$$

and

$$\sum_{u \in \mathcal{I}_n} \chi_u(f) \ll 2^{\deg f/2} n \sqrt{|\mathcal{I}_n|}. \tag{2.4}$$

Proof. The case of f being non-square in $\mathbb{A}$ follows immediately from Proposition 3.15 and Proposition 3.20 in [2] since $|\mathcal{F}_n| = \zeta_{\mathbb{A}}(2)^{-1} q^{2n}$ and $|\mathcal{I}_n| = 2\zeta_{\mathbb{A}}(2)^{-1} q^{2n-1}$.

Now we consider the case of f being a square in $\mathbb{A}$. Since $\mathcal{F}_n$ is a disjoint union of the $\mathcal{F}_M$'s, where M runs over $\mathcal{B}_n$ and $|\mathcal{F}_M| = \Phi(\tilde{M})$, we have

$$\sum_{u \in \mathcal{F}_n} \chi_u(f) = \sum_{\substack{\tilde{M} \in \mathbb{A}_n^+ \\ (\tilde{M}, f)=1}} \Phi(\tilde{M}) = |\mathcal{F}_n| \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} + O\left(|\mathcal{F}_n|^{\frac{1}{2}(1+\epsilon)}\right),$$

where the second equality follows from Lemma 3.3 in [2]. To prove (2.2), write

$$\sum_{u \in \mathcal{I}_n} \chi_u(f) = \sum_{r=0}^{n-1} \sum_{u \in \mathcal{I}_{(r,n-r)}} \chi_u(f).$$

Note that $\mathcal{I}_{(0,n)} = \mathcal{G}_n$. For $M \in \mathcal{B}_r$ with $1 \le r \le n-1$, let $\mathcal{I}_M = \mathcal{F}_M + \mathcal{G}_{n-r}$. Then $\mathcal{I}_{(r,n-r)}$ is the disjoint union of the $\mathcal{I}_M$'s, where M runs over $\mathcal{B}_r$. Thus we have

$$\sum_{u \in \mathcal{I}_n} \chi_u(f) = \sum_{u \in \mathcal{G}_n} 1 + \sum_{r=1}^{n-1} \sum_{\substack{M \in \mathcal{B}_r \\ (M,f)=1}} \sum_{u \in \mathcal{I}_M} 1 = |\mathcal{G}_n| + \sum_{r=1}^{n-1} \sum_{\substack{M \in \mathcal{B}_r \\ (M,f)=1}} |\mathcal{I}_M|.$$

Since $|\mathcal{G}_n| = 2\zeta_{\mathbb{A}}(2)^{-1} q^n$ and $|\mathcal{I}_M| = |\mathcal{F}_M| \cdot |\mathcal{G}_{n-r}| = 2\zeta_{\mathbb{A}}(2)^{-1} q^{n-r} \Phi(\tilde{M})$ for $M \in \mathcal{B}_r$, we have

$$\begin{aligned} \sum_{u \in \mathcal{I}_n} \chi_u(f) &= 2\zeta_{\mathbb{A}}(2)^{-1} q^n + 2\zeta_{\mathbb{A}}(2)^{-1} \sum_{r=1}^{n-1} q^{n-r} \sum_{\substack{\tilde{M} \in \mathbb{A}_r^+ \\ (\tilde{M}, f)=1}} \Phi(\tilde{M}) \\ &= |\mathcal{I}_n| \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1} + O\left(|\mathcal{I}_n|^{\frac{1}{2}(1+\epsilon)}\right), \end{aligned}$$

which completes the proof. □

For any $f \in \mathbb{A}^+$ and $M \in \mathcal{B}$, define

$$\Gamma_{f,M} = \sum_{u \in \mathcal{E}_M} \left\{\frac{u}{f}\right\} \quad \text{and} \quad T_{f,M} = \sum_{u \in \mathcal{F}_M} \left\{\frac{u}{f}\right\}.$$

Note that if $\gcd(f,M)=1$ and $\{\frac{u}{f}\}=-1$ for some $u\in\mathcal{E}_M$, then $\Gamma_{f,M}=0$. Thus, we have that $\Gamma_{f,M}=0$ or $\Gamma_{f,M}=|\mathcal{E}_M|=|\tilde{M}|$ for any $f\in\mathbb{A}^+$. It is known [2, Lemma 3.8] that if $\deg f\le \deg t(M)$, $\gcd(f,M)=1$ and f is not a perfect square in $\mathbb{A}$, then $\Gamma_{f,M}=0$.

We have the following lemma, which is an even characteristic analogue of [7, (2.5)].

Lemma 2.2. *For any non-square $f\in\mathbb{A}^+$, we have*

$$\sum_{Q\in\mathcal{P}_n}\sum_{u\in\mathcal{F}_Q}\chi_u(f)\ll\frac{q^n}{n}\deg f.$$

Proof. For $Q\in\mathcal{P}_n$ with $Q\mid f$, since $\{\frac{u}{f}\}=0$ for all $u\in\mathcal{F}_Q$, we have

$$\sum_{u\in\mathcal{F}_Q}\chi_u(f)=0.$$

For $Q\in\mathcal{P}_n$ with $Q\nmid f$, if $\{\frac{u}{f}\}=-1$ for some $u\in\mathcal{E}_Q$, then $\Gamma_{f,Q}=0$, and $\Gamma_{f,Q}=|Q|=q^n$ otherwise. Let $Q_1,\ldots,Q_s\in\mathcal{P}_n$ be distinct primes such that $\Gamma_{f,Q_i}\neq 0$ for $1\le i\le s$. For $M=Q_1\cdots Q_s$, if $\deg f\le\deg t(M)=sn$, then $\Gamma_{f,M}=0$ by Lemma 3.8 in [2]. But $\Gamma_{f,M}=\Gamma_{f,Q_1}\cdots\Gamma_{f,Q_s}\neq 0$ by Lemma 3.10 in [2], so $s<\deg f/n$, that is, there are at most $\deg f/n$ Q's in $\mathcal{P}_n$ such that $\Gamma_{f,Q}\neq 0$. For $Q\in\mathcal{P}_n$ with $Q\nmid f$, we have $T_{f,Q}=q^n-1$ if $\Gamma_{f,Q}\neq 0$ and $T_{f,Q}=-1$ otherwise. Thus we have

$$\begin{aligned}\sum_{Q\in\mathcal{P}_n}\sum_{u\in\mathcal{F}_Q}\chi_u(f)&=\sum_{\substack{Q\in\mathcal{P}_n\\ Q\nmid f,\Gamma_{f,Q}\neq 0}}T_{f,Q}+\sum_{\substack{Q\in\mathcal{P}_n\\ Q\nmid f,\Gamma_{f,Q}=0}}T_{f,Q}\\ &\le\frac{\deg f}{n}(q^n-1)+\frac{q^n}{n}\ll\frac{q^n}{n}\deg f,\end{aligned}$$

which completes the proof. □

3. Proofs

For any positive real number r, let $\mathbb{A}^+_{\le r}=\{f\in\mathbb{A}^+:\deg f\le r\}$ and $\mathcal{P}_{\le r}=\mathcal{P}\cap\mathbb{A}^+_{\le r}$. We also let $\mathbb{A}^+_{>r}=\{f\in\mathbb{A}^+:\deg f>r\}$ and $\mathcal{P}_{>r}=\mathcal{P}\cap\mathbb{A}^+_{>r}$.

3.1. Sketch of proof of Proposition 1.3

Let $u\in k$ be normalized one and g_u be the genus of K_u. We have (see [3, (3.2)])

$$\sum_{P\in\mathcal{P}_n}\chi_u(P)\ll\frac{q^{\frac{n}{2}}}{n}g_u,\tag{3.1}$$

which is an even characteristic analogue of [7, (2.5)]. For a positive integer n and any complex number $s\in\mathbb{C}$ with $\mathrm{Re}(s)=1$, we follow the same argument as in the proof of [7, Lemma 2.2] with (3.1) to obtain that

$$\ln L(s,\chi_u) = -\sum_{P\in\mathcal{P}_{\leq n}} \ln\left(1-\frac{\chi_u(P)}{|P|^s}\right) + O\left(\frac{q^{\frac{n}{2}}}{n}\, g_u\right). \tag{3.2}$$

The rest are straightforward by taking $n = [2\log g_u]$ and using Lemma 2.3 in [7] to complete the proof.

3.2. Proof of Theorem 1.4

We will only give a proof for the complex moments of $L(1,\chi_u)$ over $\mathcal{F}_n$ since the case over $\mathcal{I}_n$ is almost the same.

Proposition 3.1. *For $f \in \mathbb{A}^+$, we have*

$$\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\chi_u(f) = \mathbb{E}(\mathbb{X}(f)) + O\left(q^{-n}|f|^{\frac{1}{2}}\right).$$

Proof. If f is a square in $\mathbb{A}$, by Lemma 2.1 and (1.1) with the fact that $|\mathcal{F}_n| = \zeta_{\mathbb{A}}(2)^{-1}q^{2n}$, we have

$$\begin{aligned}\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\chi_u(f) &= \prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1} + O\left(q^{n(-1+\epsilon)}\right)\\ &= \mathbb{E}(\mathbb{X}(f)) + O\left(q^{n(-1+\epsilon)}\right).\end{aligned}$$

If f is not a square in $\mathbb{A}$, by Lemma 2.1 and (1.1), we have

$$\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\chi_u(f) = O\left(q^{-n}2^{\deg f/2}\right) = O\left(q^{-n}|f|^{\frac{1}{2}}\right)$$

since $q^{-n}2^{\deg f/2} = q^{-n}|f|^{\ln 2/(2\ln q)} \leq q^{-n}|f|^{1/2}$. Since $q^{n(-1+\epsilon)} \ll q^{-n}|f|^{1/2}$, the result follows. □

Lemma 3.2 ([7, Lemma 3.1])**.** *Let $u \in \mathcal{F}_n$. Let $a > 4$ be a constant, $z \in \mathbb{C}$ such that $|z| \leq n/(10a\log n \ln\log n)$ and $m = a\log n$. Then we have*

$$L(1,\chi_u)^z = \left(1+O\left(\frac{1}{n^b}\right)\right)\sum_{\substack{f\in\mathbb{A}^+_{\leq 2n/3}\\ P|f\Rightarrow \deg P\leq m}}\frac{d_z(f)\chi_u(f)}{|f|},$$

where $b = a/2-2$.

Using Proposition 3.1 and Lemma 3.2, we get the following proposition.

Proposition 3.3. *Let a, b, m and $z \in \mathbb{C}$ satisfy the conditions of Lemma 3.2. We have*

$$\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}L(1,\chi_u)^z = \left(1+O\left(\frac{1}{n^b}\right)\right)\left(\sum_{\substack{f\in\mathbb{A}^+_{\leq 2n/3}\\ P|f\Rightarrow \deg P\leq m}}\frac{d_z(f)}{|f|}\mathbb{E}(\mathbb{X}(f)) + O\left(q^{n(-\frac{1}{3}+\epsilon)}\right)\right).$$

Proof. By Proposition 3.1 and Lemma 3.2, we have

$$\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n} L(1,\chi_u)^z = \left(1+O\left(\frac{1}{n^b}\right)\right)\left(\sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\chi_u(f)\right)$$

$$= \left(1+O\left(\frac{1}{n^b}\right)\right)\left(\sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\left(\mathbb{E}(\mathbb{X}(f))+O\left(q^{-n}|f|^{\frac12}\right)\right)\right).$$

Since $|f|\le q^{2n/3}$ for any $f\in\mathbb{A}^+_{\le 2n/3}$, we have

$$q^{-n}\sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|^{\frac12}} \ll q^{-\frac{n}{3}}\sum_{f\in\mathbb{A}^+}\frac{d_z(f)}{|f|^{\frac32}} \ll q^{-\frac{n}{3}}\left(\zeta_{\mathbb{A}}(\tfrac32)\right)^{|z|}.$$

Note that $\zeta_{\mathbb{A}}(3/2)=c$ for some constant c so that

$$\left(\zeta_{\mathbb{A}}(\tfrac32)\right)^{|z|} \ll c^{\frac{n}{10a\log n\ln\log n}} = q^{\frac{n\log c}{10a\log n\ln\log n}} \ll q^{n\epsilon}$$

for n large enough. Hence, we have the desired the result. □

Lemma 3.4. *Let a, z and m be as in Lemma 3.2. Then for c_0 some positive constant we have*

$$\sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\mathbb{E}(\mathbb{X}(f)) = \sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|M\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\mathbb{E}(\mathbb{X}(f)) + O\left(q^{-\frac{n}{c_0\log n}}\right).$$

Proof. By Proposition 3.1, we have

$$\sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\mathbb{E}(\mathbb{X}(f)) = \sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\left(\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\chi_u(f)+O\left(q^{-n}|f|^{\frac12}\right)\right)$$

$$= \frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)\chi_u(f)}{|f|} + O\left(q^{-n}\sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}}\frac{d_z(f)}{|f|^{\frac12}}\right).$$

By Lemma 3.2 in [7] and Proposition 3.1, we have

$$\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)\chi_u(f)}{|f|}$$

$$= \frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n}\sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)\chi_u(f)}{|f|} + O\left(q^{-\frac{n}{c_0\log n}}\right)$$

$$= \sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\left(\mathbb{E}(\mathbb{X}(f)) + O\left(q^{-n}|f|^{\frac{1}{2}}\right)\right) + O\left(q^{-\frac{n}{c_0\log n}}\right)$$

for some positive constant c_0. As in the proof of Proposition 3.3, we have

$$q^{-n}\sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|^{\frac{1}{2}}} \ll q^{n(-\frac{1}{3}+\epsilon)}.$$

Hence, we get the result. □

Proof of Theorem 1.4. From the random Euler product definition we have

$$\mathbb{E}(L(1,\mathbb{X})^z) = \prod_{P\in\mathcal{P}} E_P(z),$$

where

$$\begin{aligned}E_P(z) &= \mathbb{E}\left(\left(1-\frac{\mathbb{X}(P)}{|P|}\right)^{-z}\right)\\ &= \frac{1}{|P|+1} + \frac{|P|}{2(|P|+1)}\left(\left(1-\frac{1}{|P|}\right)^{-z} + \left(1+\frac{1}{|P|}\right)^{-z}\right).\end{aligned}$$

Now, we notice if $\deg P > m$, then we can use the following Taylor expansions

$$\left(1-\frac{1}{|P|}\right)^{-z} = 1 + \frac{z}{|P|} + O\left(\frac{z}{|P|^2}\right)$$

and

$$\left(1+\frac{1}{|P|}\right)^{-z} = 1 - \frac{z}{|P|} + O\left(\frac{z}{|P|^2}\right).$$

That is to say, for $P\in\mathcal{P}_{>m}$ we have $E_P(z) = 1 + O(z/|P|^2)$, so that

$$\prod_{P\in\mathcal{P}_{>m}} E_P(z) \ll \exp\left(|z|\sum_{P\in\mathcal{P}_{>m}}\frac{1}{|P|^2}\right) = 1 + O\left(\frac{1}{n^b}\right),$$

where the last equality follows from the relative sizes of $|z|$ and $m = a\log n$, and we choose a large enough to provide the desired error term above. Thus, by Lemma 3.4, we have

$$\begin{aligned}\mathbb{E}(L(1,\mathbb{X})^z) &= \left(1+O\left(\frac{1}{n^b}\right)\right)\left(\prod_{P\in\mathcal{P}_{\le m}} E_P(z)\right)\\ &= \left(1+O\left(\frac{1}{n^b}\right)\right)\left(\sum_{\substack{f\in\mathbb{A}^+\\ P|f\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\mathbb{E}(\mathbb{X}(f))\right)\end{aligned}$$

$$(3.3)\qquad = \left(1+O\left(\frac{1}{n^b}\right)\right)\left(\sum_{\substack{f\in\mathbb{A}^+_{\le 2n/3}\\ P|M\Rightarrow \deg P\le m}} \frac{d_z(f)}{|f|}\mathbb{E}(\mathbb{X}(f)) + O(q^{-\frac{n}{c_0\log n}})\right).$$

From Proposition 3.3 and (3.3) with (1.2), we get that

$$\begin{aligned}\frac{1}{|\mathcal{F}_n|}\sum_{u\in\mathcal{F}_n} L(1,\chi_u)^z &= \left(1+O\left(\frac{1}{n^b}\right)\right)\mathbb{E}(L(1,\mathbb{X})^z)\\ &= \left(1+O\left(\frac{1}{n^b}\right)\right)\sum_{f\in\mathbb{A}^+}\frac{d_z(f^2)}{|f|^2}\prod_{P|f}\left(1+\frac{1}{|P|}\right)^{-1}\end{aligned}$$

and completes the proof of Theorem 1.4 by taking $a = 26$ and $b = 11$. □

3.3. Proof of Theorem 1.9

For each $P \in \mathcal{P}$, let $\delta_P \in \{-1, 1\}$. Define $\mathcal{S}_N(n, \{\delta_P\})$ to be the set of $u \in \mathcal{F}_Q$ with $Q \in \mathcal{P}_N$ such that $\{\frac{u}{P}\} = \delta_P$ for all $P \in \mathcal{P}_{\le n}$. We also let $\mathcal{P}(n)$ denote the product of all monic irreducible polynomials P with $\deg P \le n$. Let $\pi_q(n) = |\mathcal{P}_n|$ and $\Pi_q(n) = \sum_{j=1}^{n}\pi_q(j)$. Note that $\deg \mathcal{P}(n) = \sum_{j=1}^{n} j\pi_q(j) \asymp q^n$.

Lemma 3.5. *Let N be large, and n be a positive integer such that $1 \le n \le (\log N)^2$. Then we have*

$$|\mathcal{S}_N(n,\{\delta_P\})| = \frac{\pi_q(N)(q^N-1)}{2^{\Pi_q(n)}} + O\left(q^{N+n}\right).$$

Proof. For $f \in \mathbb{A}^+$, let $\delta_f = \prod_{P|f}\delta_P$. For any $u \in \mathcal{F}_Q$ with $Q \in \mathcal{P}_N$, we have

$$(3.4)\qquad \sum_{f|\mathcal{P}(n)}\delta_f\left\{\frac{u}{f}\right\} = \prod_{P\in\mathcal{P}_{\le n}}\left(1+\delta_P\left\{\frac{u}{P}\right\}\right) = \begin{cases}2^{\Pi_q(n)} & \text{if } u\in\mathcal{S}_N(n,\{\delta_P\}),\\ 0 & \text{otherwise.}\end{cases}$$

Thus, we deduce that

$$\begin{aligned}|\mathcal{S}_N(n,\{\delta_P\})| &= \frac{1}{2^{\Pi_q(n)}}\sum_{f|\mathcal{P}(n)}\delta_f\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{f}\right\}\\ (3.5)\qquad &= \frac{1}{2^{\Pi_q(n)}}\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}1 + \frac{1}{2^{\Pi_q(n)}}\sum_{1\ne f|\mathcal{P}(n)}\delta_f\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{f}\right\}.\end{aligned}$$

Since $|\mathcal{F}_Q| = q^N - 1$ for all $Q \in \mathcal{P}_N$, we have

$$(3.6)\qquad \frac{1}{2^{\Pi_q(n)}}\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}1 = \frac{\pi_q(N)(q^N-1)}{2^{\Pi_q(n)}}.$$

Since all the divisors of $\mathcal{P}(n)$ are square-free, we obtain from Lemma 2.2 that

$$\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{f}\right\}\ll q^N\deg f\ll q^{N+n}$$

for all $1\neq f\mid\mathcal{P}(n)$ because $\deg f\leq\deg\mathcal{P}(n)\asymp q^n$. Hence, we have

$$\frac{1}{2^{\Pi_q(n)}}\sum_{1\neq f|\mathcal{P}(n)}\delta_f\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{f}\right\}\ll q^{N+n} \tag{3.7}$$

since the number of divisors of $\mathcal{P}(n)$ is $2^{\Pi_q(n)}$. Finally, by inserting (3.6) and (3.7) into (3.5), we complete the proof. □

Proposition 3.6. *Let N be large, and n be a positive integer such that $1\leq n\leq(\log N)^2$. We have*

$$\sum_{u\in\mathcal{S}_N(n,\{\delta_P\})}L(1,\chi_u)=\zeta_{\mathbb{A}}(2)\frac{\pi_q(N)(q^N-1)}{2^{\Pi_q(n)}}\prod_{P\in\mathcal{P}_{\leq n}}\left(1+\frac{\delta_P}{|P|}\right)+O\left(N^2q^{N+2n}\right). \tag{3.8}$$

Proof. Note that $L(s,\chi_u)$ is a polynomial in q^{-s} of degree $2N-1$ for $u\in\mathcal{F}_Q$ with $Q\in\mathcal{P}_N$. Thus for any $m\geq 2N$, we have

$$L(1,\chi_u)=\sum_{F\in\mathbb{A}^+_{\leq m}}\frac{\chi_u(F)}{|F|}.$$

Let $a=2N\deg\mathcal{P}(n)\ll Nq^n$. Then, from (3.4), we obtain

$$\sum_{u\in\mathcal{S}_N(n,\{\delta_P\})}L(1,\chi_u)=\frac{1}{2^{\Pi_q(n)}}\sum_{f|\mathcal{P}(n)}\delta_f\sum_{F\in\mathbb{A}^+_{\leq a}}\frac{1}{|F|}\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{Ff}\right\}. \tag{3.9}$$

If Ff is a square, that is, $F=fh^2$ for some $h\in\mathbb{A}^+$, then

$$\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{Ff}\right\}=(q^N-1)(\pi_q(N)+O(\omega(F)))=(q^N-1)(\pi_q(N)+O(a)), \tag{3.10}$$

where $\omega(F)$ is the number of monic irreducible divisors of F, and $\omega(F)\leq\deg F\leq a$. Furthermore, if Ff is not a square in $\mathbb{A}$, then by Lemma 2.2, we get

$$\sum_{Q\in\mathcal{P}_N}\sum_{u\in\mathcal{F}_Q}\left\{\frac{u}{Ff}\right\}\ll q^N\deg Ff\ll aq^N \tag{3.11}$$

because of $\deg Ff \leq a + \deg \mathcal{P}(n) = a + a/(2N) \ll a$. Inserting (3.10) and (3.11) into (3.9), we get

(3.12)
$$\sum_{u \in \mathcal{S}_N(n,\{\delta_P\})} L(1,\chi_u) = \frac{(q^N-1)\pi_q(N)}{2^{\Pi_q(n)}} \sum_{f|\mathcal{P}(n)} \frac{\delta_f}{|f|} \sum_{h \in \mathbb{A}^+_{\leq(a-\deg f)/2}} \frac{1}{|h^2|} + O\left(a^2 q^N\right),$$

since $\sum_{f|\mathcal{P}(n)} 1 = 2^{\Pi_q(n)}$ and $\sum_{F \in \mathbb{A}^+_{\leq a}} 1/|F| = a$. For any $f \mid \mathcal{P}(n)$, we have

$$\sum_{h \in \mathbb{A}^+_{\leq(a-\deg f)/2}} \frac{1}{|h^2|} = \zeta_{\mathbb{A}}(2) + O\left(q^{-N}\right), \tag{3.13}$$

which follows from that

$$\sum_{h \in \mathbb{A}^+_{>(a-\deg f)/2}} \frac{1}{|h^2|} \leq \sum_{h \in \mathbb{A}^+_{>a/4}} \frac{1}{|h^2|} \ll q^{-N}.$$

Inserting (3.13) into (3.12), we complete the proof. □

We remark that the condition $1 \leq n \leq (\log N)^2$ in Lemma 3.5 and Proposition 3.6 is necessary for $\pi_q(N) \gg q^n$.

To prove Theorem 1.9, we choose n such that

$$\frac{N \log N}{10\zeta_{\mathbb{A}}(2)q} \leq q^n < \frac{N \log N}{10\zeta_{\mathbb{A}}(2)},$$

and the rest are straightforward using Lemma 3.5 and Proposition 3.6.

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