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S-VERSIONS AND *S*-GENERALIZATIONS OF IDEMPOTENTS, PURE IDEALS AND STONE TYPE THEOREMS

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Abstract. Let R be a commutative ring with nonzero identity and M be an R-module. In this paper, we first introduce the concept of S-idempotent element of R. Then we give a relation between S-idempotents of R and clopen sets of S-Zariski topology. After that we define S-pure ideal which is a generalization of the notion of pure ideal. In fact, every pure ideal is S-pure but the converse may not be true. Afterwards, we show that there is a relation between S-pure ideals of R and closed sets of S-Zariski topology that are stable under generalization.

1. Introduction

Throughout the paper, R denotes a commutative ring with identity, S denotes a multiplicatively closed subset (briefly, m.c.s.) of R and M denotes an R-module. $Spec_S(R)$, $Spec_S(M), Z(R)$ denote the set of all S-prime ideals of R, S-prime submodules of M, zero divisors of R, respectively. Consider $D_a^S = \{P \in Spec_S(R) : sa \notin P, \forall s \in S\}$. We have $D_1^S = Spec_S(R)$ and $D_a^S \cap D_b^S = D_{ab}^S$ for all $a, b \in R$. By a basic result in general topology, there exists a (unique) topology over $Spec_S(R)$ such that the collection of D_a^S with $a \in R$ forms a basis for the opens. We call it the S-Zariski topology. Also each D_a^S is called a principal S-Zariski open. Now it can be seen that every closed subset of $Spec_S(R)$ is of the form $V_S(I) = \{P \in Spec_S(R) : sI \subseteq P, \exists s \in S\}$, where I is an ideal of R [18].

Now we will look at the relation between the topology on a module and a topology on a ring. Suppose that R, R' are two rings, S is a m.c.s. of R' and M, M' are R'-modules. If $f : M' \to M$ is an epimorphism, then $\sigma' : Spec_S(M) \to Spec_S(M')$ is defined by $N \mapsto f^{-1}(N)$. If $g : R' \to R$ is an epimorphism with $0 \notin g(S)$, then $\sigma : Spec_{g(S)}(R) \to Spec_S(R')$ is defined by $P \mapsto g^{-1}(P)$. The map $\varphi : Spec_S(M) \to Spec_{g(S)}(R)$ is defined by

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$\varphi(N) = g((N : M))$. The map $\varphi' : Spec_S(M') \to Spec_S(R')$ is defined by $\varphi'(N') = (N' : M')$. The diagram below gives the relations between mentioned topological spaces:

$$\begin{array}{ccc} Spec_S(M) & \xrightarrow{\varphi} & Spec_{g(S)}(R) \\ \sigma' \big\downarrow & & \big\downarrow \sigma \\ Spec_S(M') & \xrightarrow{\varphi'} & Spec_S(R') \end{array}$$

Observe that the diagram above is commutative if g is an isomorphism.

Notice that φ and φ' are not reversible since if $(N : M)$ is an S-prime ideal, N is not necessarily to be S-prime submodule.

The notion of prime ideal and its generalizations have a significant place in Commutative Algebra and Algebraic Geometry. They are used to characterize a large of variety of rings and they have some applications to other areas such as General Topology. Recall from [4] that a proper ideal P of R is said to be a *prime ideal* if $ab \in P$ for some $a, b \in R$, then either $a \in P$ or $b \in P$. For many years, various authors constructed some topologies over algebraic structures and they investigated the relations between the algebraic properties of given algebraic structures (such as rings, modules, lattices and fuzzy structures) and topological properties of these topologies. See, for example, [1, 3, 5, 7, 8, 10, 11, 13, 17, 18]. In [5], the author constructed a topology on $Spec(M)$ which is the set of all prime submodules of M. He proved some results that are known for $Spec(R)$. Also he defined absolutely flat R-module. In 1995, Chin-Pi Lu investigated some properties of $Spec(M)$. She gave a relation between $Spec(M)$ and $Spec(S^{-1}M)$ [9]. In [13], the author rediscovered independently the flat topology on the prime spectrum which is the dual of the Zariski topology. He also determined connected subsets of prime spectrum with respect to the both Zariski and flat topologies. In 2020, same author showed the existence of a correspondence between pure ideals of R and Zariski closed subsets of $Spec(R)$ that are stable under generalization [15]. He studied on the projectivity of a finitely generated flat module over a commutative ring. He also obtained many new results [14]. In 2019, the Hamed and Malek defined the notion of S-prime ideal and investigated some properties of them [6]. A proper ideal P of R with $P \cap S = \varnothing$ is called S-prime if $ab \in P$ implies either $sa \in P$ or $sb \in P$ for some $s \in S$. At about the same time, Sevim et al. introduced S-prime submodules. Let N be a submodule of M such that $(N : M) \cap S = \varnothing$. Then N is called an S-prime submodule if there exists an $s \in S$ such that $am \in N$ for some $a \in R$, $m \in M$ implies that $sa \in (N : M)$ or $sm \in N$ [12]. They characterized some classical modules such as simple modules, S-Noetherian modules, and torsion-free modules. In [18], the authors constructed a topology on the set of S-prime ideals of R that is a generalization of Zariski topology. They gave many results related to this topology. Following year, Yildiz et al. constructed a topology on the set of S-prime submodules that is a generalization of S-Zariski topology on a ring [17].

In this paper, we first introduce the concepts of S-idempotent element of a commutative ring R. Then we give a theorem that proves a correspondence between S-idempotents of R and clopen sets of S-Zariski topology. After that we define S-pure ideal which is a generalization of the notion of pure ideal. In fact, every pure ideal is S-pure but the converse may not be true. Moreover, we give some properties of this class of ideals. Afterwards we show that there is a relation between S-pure ideals of R and closed sets of S-Zariski topology that are stable under generalization.

2. S-versions of Stone type theorems

Definition 2.1. Let S be a multiplicative subset of R and $f : R \to R'$ be a ring map. An element $a \in R'$ is called S-idempotent (with respect to f) if $a^2 = s \cdot a = f(s)a$ for some $s \in S$. Observe that every idempotent element is S-idempotent. But the following example shows that the converse may not be true.

Example 2.2. Let $R = \mathbb{Z}_6$ and $S = \{\bar{1}, \bar{5}\}$. Since $\bar{2}^2 = \bar{5} \cdot \bar{2}$, $\bar{2}$ is an S-idempotent element. But it is clear that $\bar{2}$ is not idempotent.

Definition 2.3. Let R be a ring. An ideal I of R is said to be an S-regular ideal if I is generated by some S-idempotent elements. A proper ideal I of R is called maximal S-regular if it is maximal among all S-regular ideals.

Note that every regular ideal is S-regular. The next example demonstrates that the converse may not hold.

Example 2.4. Let $R = \mathbb{Z}_6$ and $S = \{\bar{1}, \bar{5}\}$. Although $(\bar{2})$ is an S-regular ideal of $\mathbb{Z}_6$, it is not regular.

Definition 2.5. An element $x \in R$ is called S-zero if $sx = 0$ for some $s \in S$. Note that 0 is also an S-zero element.

Next we will give the S-version of [13, Theorem 3.19].

Theorem 2.6. *Given a ring R, a m.c.s. S of R and an S-regular ideal I of R, if I is a maximal S-regular ideal, then only S-idempotents of R/I are S-zero elements. The converse is true when $(I : s) = I$ for all $s \in S$.*

Proof. Let $I = (a_1, a_2, \ldots, a_n)$, where $s_i a_i = a_i^2$ for some $s_i \in S$ for all i. First note that by using Definition 2.1 for the canonical ring map $R \to R/I$ we get the right expression of the theorem. Assume that I is a maximal S-regular ideal of R and $a + I$ is S-idempotent in R/I. Then there exists $s \in S$ such that $sa + I = s(a + I) = (a + I)(a + I) = a^2 + I$. Thus we have $sa - a^2 \in I$. There are two cases: $I + (s - a) = R$ or $I + (s - a) \neq R$. Suppose $I + (s - a) = R$. Then $1 = r(s - a) + r_1 a_1 + r_2 a_2 + \cdots + r_n a_n$ for some $r, r_1, \ldots, r_n \in R$, where $I = (a_1, a_2, \ldots, a_n)$. This implies that $a =$

$r(s-a)a+r_1a_1a+r_2a_2a+\cdots+r_na_na \in I$. Hence $a \in I$ implying $a+I=0+I$. Now assume that $I+(s-a) \neq R$. Since $sa-a^2 \in I=(a_1,a_2,\ldots,a_n)$, we can write $sa-a^2=r_1a_1+r_2a_2+\cdots+r_na_n$. This equality implies that $(sa-a^2)\prod_{i=1}^n(s_i-a_i)=r_1a_1\prod_{i=1}^n(s_i-a_i)+\cdots+r_na_n\prod_{i=1}^n(s_i-a_i)=0$. In fact, $\prod_{i=1}^n(s_i-a_i)$ can be written as $s'-e$, where $s' \in S$ and e is an S-idempotent element of I such that $s'e=e^2$. Then we have $((s'-e)a)^2 = s'^2a^2-2s'ea^2+e^2a^2=s'^2sa-2s'esa+sas'e=s'^2sa-s'sea=s^*(s'-e)a$, where $s^*=s's \in S$. This shows that $(s'-e)a$ is S-idempotent. Now since $s'a=ea+(s'-e)a$, $s'a \in I+((s'-e)a)$. Here $I+((s'-e)a)$ is an S-regular ideal. We know that I is a maximal S-regular ideal and $I+((s'-e)a) \neq R$. Hence $I=I+((s'-e)a)$. In this case, we conclude $s'a \in I$ and this gives $s'(a+I)=I=0+I$, which shows that $a+I$ is S-zero. For the converse, assume that only S-idempotents of R/I are S-zero elements and $I \subset J$ for some S-regular ideal J of R. Then we can find an S-idempotent element $a \in J-I$ such that $sa=a^2$ for some $s \in S$. Thus $(a+I)^2=a^2+I=sa+I=s(a+I)$. Since $(I:s)=I$, $sa \notin I$ and this shows $a+I$ is non-S-zero S-idempotent of R/I, a contradiction. Therefore, I is maximal among all S-regular ideals of R. □

Proposition 2.7. *Let R be a ring and S be a multiplicatively closed subset of R. If $a \in R$ is S-idempotent with $sa=a^2$, then $\frac{a}{s}$ is idempotent in $S^{-1}R$. The converse is also true when $S \cap Z(R)=\varnothing$.*

Proof. Assume that a is S-idempotent. Then $sa=a^2$ for some $s \in S$. Then we have $(\frac{a}{s})^2=\frac{a^2}{s^2}=\frac{sa}{s^2}=\frac{a}{s}$, as needed. For the other direction, suppose that $\frac{a}{s}$ is idempotent in $S^{-1}R$. Then $(\frac{a}{s})^2=\frac{a}{s}$. This implies that $t(sa^2-as^2)=0$ for some $t \in S$. As $S \cap Z(R)=\varnothing$, we get $sa=a^2$ which shows a is S-idempotent. □

The following theorem gives a correspondence between idempotents of $S^{-1}R$ and clopen sets of S-Zariski topology.

Definition 2.8 ([18])**.** Let I be an ideal of R. Then S-radical of I is defined by

$$\sqrt[S]{I}=\{a \in R : sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{Z}^+\},$$

where S is a m.c.s. of R.

In particular, $\sqrt[S]{0}$ is denoted by $N_S(R)$. An element of $N_S(R)$ is said to be an S-nilpotent element.

Theorem 2.9 (Generalized Grothendieck Correspondence)**.** *Given a m.c.s. S of a ring R with $S \cap Z(R)=\varnothing$. Then there is a bijection ϕ from the set of idempotents of $S^{-1}R$ onto the set of clopens of the S-prime spectrum of R defined by $\frac{a}{s} \mapsto D_a^S$.*

Proof. Given S-idempotent a in R with $sa=a^2$ for some $s \in S$. Then $Spec_S(R)=V_S((a)) \cup V_S((s-a))$. So, $D_a^S=V_S((s-a))$ and this shows that D_a^S is clopen in $Spec_S(R)$. Let $\frac{a}{s}=\frac{a'}{s'}$. Then $s'a=sa'$, and this yields that $V_S(a) = V_S(s'a) = V_S(sa') = V_S(a')$. Thus $D_a^S = D_{a'}^S$. Therefore, the map is well defined. For injectivity, suppose $D_a^S = D_b^S$, where a and b are S-idempotents in R with $s_1 a = a^2$ and $s_2 b = b^2$ for some $s_1, s_2 \in S$. This gives $\sqrt[S]{(a)} = \sqrt[S]{(b)}$ by Proposition [18]. Since $a \in \sqrt[S]{(a)} = \sqrt[S]{(b)}$, $s^\star a = s'a^n = br_1$ for some $s^\star, s' \in S$, $r_1 \in R$, $n \in \mathbb{Z}^+$. Multiplying $(s_2 - b)$ both sides, we obtain $s^\star s_2 a = s^\star ab$. Also, since $b \in \sqrt[S]{(a)}$, $s^{\star\star} b = s'' b^m = ar_2$ for some $s^{\star\star}, s'' \in S$, $r_2 \in R$, $m \in \mathbb{Z}^+$. Similar argument shows that $s^{\star\star} s_1 b = s^{\star\star} ab$. Thus, we obtain $s_2 a = s_1 b$. Then this gives $\frac{a}{s_1} = \frac{b}{s_2}$. Therefore, the map is injective. Now we will show that the map is surjective. Suppose that K is clopen in $Spec_S(R)$. Since K is closed and $Spec_S(R)$ is compact, K is compact. Then $Spec_S(R) - K$ is also compact. So $K = \bigcup_{i=1}^n D_{a_i}^S$ and $Spec_S(R) - K = \bigcup_{j=1}^m D_{b_j}^S$ for some open sets $D_{a_i}^S$ and $D_{b_j}^S$ in $Spec_S(R)$. Here $D_{a_i}^S \cap D_{b_j}^S = \varnothing$ for all i, j. Thus we have $D_{a_i b_j}^S = D_{a_i}^S \cap D_{b_j}^S = \varnothing$. This implies that $a_i b_j \in N_S(R)$ for all i, j by [18, Proposition 6]. Note that $N_S(R) = N(R)$ since $S \cap Z(R) = \varnothing$. Now, consider $I = (a_1, a_2, \dots, a_n)$ and $J = (b_1, b_2, \dots, b_m)$. Then, $(IJ)^t = 0$, where t is large enough. Moreover, $V_S(I^t) = V_S(I) = Spec_S(R) - D_I^S = \bigcup_{j=1}^m D_{b_j}^S$ and $V_S(J^t) = V_S(J) = Spec_S(R) - D_J^S = \bigcup_{i=1}^n D_{a_i}^S$ and these give that $Spec_S(R) = V_S(I^t) \cup V_S(J^t)$. Hence, we get $V_S(I^t + J^t) = \varnothing$ showing $(I^t + J^t) \cap S \neq \varnothing$. Then $a + b = s$ for some $s \in S$, where $a \in I^t$ and $b \in J^t$. As $ab \in I^t J^t = (IJ)^t = 0$, we have

$$sa - a^2 = a(s - a) = ab = 0. \tag{1}$$

Thus, a is S-idempotent and $\phi(\frac{a}{s}) = D_a^S = \bigcup_{i=1}^n D_{a_i}^S = K$, as desired. $\square$

Let R be a ring and S be a multiplicatively subset of R. The set of all idempotent elements of $S^{-1}R$ is defined by $\mathcal{B}(S^{-1}R) = \{\frac{a}{s} \in S^{-1}R : (\frac{a}{s})^2 = \frac{a}{s}$ where $a \in R$, $s \in S\}$. Define an operation $+_S$ as $\frac{a}{s} +_S \frac{b}{t} = \frac{a}{s} + \frac{b}{t} - \frac{2ab}{st}$. Recall from [16] that $(\mathcal{B}(S^{-1}R), +_S, \cdot)$ is a ring. Moreover, if $S \cap Z(R) = \varnothing$, then note that $\frac{a}{s}$ is idempotent if and only if $sa = a^2$.

In the next theorem we will give the S-version of [16, Theorem 3.1].

Theorem 2.10. *Given a ring R and a m.c.s. S of R with $S \cap Z(R) = \varnothing$. Then the ring map $\phi : \mathcal{B}(S^{-1}R) \to Clop(Spec_S(R))$ defined by $\frac{a}{s} \mapsto D_a^S$ is an isomorphism.*

Proof. This map is bijective from Generalized Grothendieck Correspondence. Now, we will show that the map is a morphism. Here $\phi(\frac{a}{s} +_S \frac{b}{t}) = \phi(\frac{ta+sb-2ab}{st})$ $= D_{ta+sb-2ab}^S$ and $\phi(\frac{a}{s} \cdot \frac{b}{t}) = \phi(\frac{ab}{st}) = D_{ab}^S$. In fact, we have $\phi(\frac{a}{s}\frac{b}{t}) = D_{ab}^S = D_a^S \cap D_b^S = \phi(\frac{a}{s}) \cap \phi(\frac{b}{t})$, where $a, b \in R$, $s, t \in S$. We only need to show that $\phi(\frac{a}{s} +_S \frac{b}{t}) = \phi(\frac{a}{s}) \triangle \phi(\frac{b}{t})$, where $\triangle$ denotes the symmetric difference, that is, $A \triangle B = (A \cup B) - (A \cap B)$. Take $P \in \phi(\frac{a}{s} +_S \frac{b}{t}) = D_{ta+sb-2ab}^S$. Then $s'(ta + sb - 2ab) \notin P$ for all $s' \in S$. Now assume that $P \notin D_a^S$ and $P \notin D_b^S$. This gives $s_1 a \in P$ and $s_2 b \in P$ for some $s_1, s_2 \in S$. It implies that $s'(ta + sb - 2ab) \in P$, where $s' = s_1 s_2$, a contradiction. So $P \in D_a^S$ or

$P \in D_b^S$ giving $P \in D_a^S \cup D_b^S$. Also, since $\frac{a}{s}, \frac{b}{t}$ are idempotents in $S^{-1}R$ and $S \cap Z(R) = \varnothing$, we have $sa = a^2$ and $tb = b^2$. By using this fact, we can obtain that $st(ta + sb - 2ab) = (ta + sb - 2ab)^2$. Then $(ta + sb - 2ab)(st - (ta + sb - 2ab)) \in P$. As P is an S-prime ideal, we have $s'(ta + sb - 2ab) \in P$ or $s'(st - (ta + sb - 2ab)) \in P$ for some $s' \in S$. But former case is not possible by the assumption. Thus we must have $s'(st - (ta + sb - 2ab)) \in P$. This gives $s'stab = ab(s'st - s'ta - s'sb + 2s'ab) \in P$. Put $s^* = s'st$ and we get $s^*ab \in P$. Hence $P \notin D_{ab}^S$ which implies $P \in D_a^S \triangle D_b^S = \phi(\frac{a}{s}) \triangle \phi(\frac{b}{t})$. For the reverse inclusion, take $P \in (D_a^S \cup D_b^S) - D_{ab}^S$. Suppose that $P \notin \phi(\frac{a}{s} +_S \frac{b}{t}) = D_{ta+sb-2ab}^S$. Then we can find an $s' \in S$ providing $s'(ta + sb - 2ab) \in P$. As $P \notin D_{ab}^S$, we can find an $s'' \in S$ such that $s''ab \in P$. This gives $s's''ab \in P$ and so $s's''(ta + sb - ab) \in P$. As $s's''sta = s's''a(ta + sb - ab) \in P$, where $sa = a^2$, we obtain $P \notin D_a^S$. Similarly, $s's''stb = s's''b(ta + sb - ab) \in P$, where $tb = b^2$, we get $P \notin D_b^S$. This contradicts with the assumption and so it means $P \in \phi(\frac{a}{s} +_S \frac{b}{t}) = D_{ta+sb-2ab}^S$, as required. □

3. S-pure ideals

Definition 3.1. Let R be a commutative ring with 1 and S be a multiplicatively closed subset of R. An ideal I of R is called S-pure if for all $a \in I$ there exist $b \in I$ and $s \in S$ such that $sa = ab$.

Proposition 3.2. *Given a ring R and a m.c.s. S of R with $S \cap Z(R) = \varnothing$. Then an ideal I of R is S-pure if and only if $Ann(\frac{a}{s}) + S^{-1}I = S^{-1}R$ for all $\frac{a}{s} \in S^{-1}I$.*

Proof. Assume that I is an S-pure ideal of R. Then for all $a \in I$ there exist $b \in I$ and $t \in S$ such that $ta = ab$. This gives $\frac{a}{s} = \frac{ta}{ts} = \frac{ab}{ts}$. Then $\frac{a}{s}(\frac{1}{1} - \frac{b}{t}) = 0$. Thus $\frac{1}{1} - \frac{b}{t} \in Ann(\frac{a}{s})$. Since $\frac{1}{1} - \frac{b}{t} + \frac{b}{t} = \frac{1}{1} \in Ann(\frac{a}{s}) + S^{-1}I$, we conclude that $Ann(\frac{a}{s}) + S^{-1}I = S^{-1}R$. For the other direction, assume $Ann(\frac{a}{s}) + S^{-1}I = S^{-1}R$ for all $\frac{a}{s} \in S^{-1}I$. Then $\frac{r}{t} + \frac{b}{s'} = \frac{1}{1}$, where $\frac{r}{t} \in Ann(\frac{a}{s})$ and $\frac{b}{s'} \in S^{-1}I$. Multiplying each side by $\frac{a}{s}$, we obtain $\frac{ab}{ss'} = \frac{a}{s}$. So we can find $u \in S$ such that $u(sab - ss'a) = 0$. Since $S \cap Z(R) = \varnothing$, we get $ab = s'a$, as desired. □

Note that every pure ideal is S-pure but the converse is not true in general.

Example 3.3. Let $R = \mathbb{Z}$, $S = \mathbb{Z} - 2\mathbb{Z}$. Take $I = 3\mathbb{Z}$. Here $Ann(\frac{a}{s}) + S^{-1}I = S^{-1}R$ and so I is an S-pure ideal. Now choose $a = 3$. Then $Ann(3) + 3\mathbb{Z} \neq \mathbb{Z}$. Thus I is not a pure ideal.

Corollary 3.4. *Let R be a ring, S be a multiplicatively subset of R and I be an ideal of R. Then I is S-pure if and only if $S^{-1}I$ is pure.*

Definition 3.5. Let R be a commutative ring with 1 and S be a multiplicatively closed subset of R with $S \cap Z(R) = \varnothing$. An ideal I of R is strongly S-pure if and only if $Ann(\frac{a}{s}) + \frac{a}{s}S^{-1}R = S^{-1}R$ for all $\frac{a}{s} \in S^{-1}I$.

Recall from [2] that a multiplicatively closed subset S of R is said to satisfy the maximal multiple condition if there exists $s \in S$ such that s' divides s for all $s' \in S$. Then such an $s \in S$ is denoted by $\tilde{s}$.

Lemma 3.6. *Given a m.c.s. S of a ring R satisfying maximal multiple condition by $\tilde{s}$ and $S \cap Z(R) = \varnothing$, and an ideal I of R with $(I : \tilde{s}) = I$. Then $S^{-1}I = (\frac{a_1}{s_1}, \frac{a_2}{s_2}, \dots, \frac{a_n}{s_n})$ if and only if $I = (a_1, a_2, \dots, a_n)$.*

Proof. One can easily see that if $I = (a_1, a_2, \dots, a_n)$, then $S^{-1}I = (\frac{a_1}{s_1}, \frac{a_2}{s_2}, \dots, \frac{a_n}{s_n})$. On the other hand, assume that $S^{-1}I = (\frac{a_1}{s_1}, \frac{a_2}{s_2})$. Let $a \in I$. Then $\frac{a}{\tilde{s}} \in S^{-1}I$. So there exist $\frac{r_1}{t_1}, \frac{r_2}{t_2} \in S^{-1}R$ such that $\frac{a}{\tilde{s}} = \frac{r_1}{t_1}\frac{a_1}{s_1} + \frac{r_2}{t_2}\frac{a_2}{s_2}$. This gives $\frac{a}{1} = \frac{\tilde{s}r_1a_1}{t_1s_1} + \frac{\tilde{s}r_2a_2}{t_2s_2}$. Since S satisfies the maximal multiple condition by $\tilde{s}$, we get $\tilde{s} = k_1t_1s_1$ and $\tilde{s} = k_2t_2s_2$ for some k_1, k_2. Hence we obtain $a = (k_1r_1)a_1 + (k_2r_2)a_2$. Moreover, as $(I : \tilde{s}) = I$, $a_1, a_2 \in I$. Therefore $I = (a_1, a_2)$. By applying induction on the number of elements of the generator, the proof is completed. □

Proposition 3.7. *Let S be a multiplicatively closed subset of ring R satisfying maximal multiple condition by $\tilde{s}$. Then every strongly S-pure ideal I with $(I : \tilde{s}) = I$ is S-regular.*

Proof. Assume that I is a strongly S-pure ideal with $(I : \tilde{s}) = I$. Let $a \in I$. Then $\frac{a}{s} \in S^{-1}I$. So $Ann(\frac{a}{s}) + \frac{a}{s}S^{-1}R = S^{-1}R$. Then we can write $\frac{b}{v} + \frac{r}{t} = \frac{1}{1}$ for some $\frac{b}{v} \in Ann(\frac{a}{s})$ and $\frac{r}{t} \in \frac{a}{s}S^{-1}R$. Then $0 = \frac{r}{t}\frac{b}{v} = \frac{r}{t}(\frac{1}{1} - \frac{r}{t})$. This shows that $\frac{r}{t}$ is idempotent. Since $\frac{b}{v} + \frac{r}{t} = \frac{1}{1}$ and $\frac{b}{v} \in Ann(\frac{a}{s})$, we obtain $\frac{a}{s}\frac{r}{t} = \frac{a}{s}$. Thus $\frac{a}{s}S^{-1}R = \frac{r}{t}S^{-1}R$ and so $\frac{a}{s}S^{-1}R$ is a regular ideal. Hence $S^{-1}I$ is a regular ideal satisfying $S^{-1}I = \sum_{\frac{a}{s} \in S^{-1}I} \frac{a}{s}S^{-1}R$. This gives $I = \sum_{a \in I} aR$ by Lemma 3.6, where a is S-idempotent. Therefore, I is an S-regular ideal, as needed. □

Proposition 3.8. *If I is an S-pure ideal, then for each finite subset $\{a_1, a_2, \dots, a_n\}$ of I there exist $b \in I$ and $s \in S$ such that $a_i(s - b) = 0$.*

Proof. Consider the subset $\{a, a'\}$ of I. Since I is S-pure, there exist $x, x' \in I$ and $s, s' \in S$ such that $sa = ax$ and $s'a' = a'x'$. Choose $b = s'x + sx' - xx'$. Then we obtain $ab = ss'a$ and $a'b = s'sa'$. By choosing $s^\star = ss'$, we are done. By applying induction on n, we can get the result. □

Proposition 3.9. *Let I be an ideal of R and $\sqrt[S]{I}$ be S-pure. Then $S^{-1}I = S^{-1}\sqrt[S]{I}$.*

Proof. As $I \subseteq \sqrt[S]{I}$, $S^{-1}I \subseteq S^{-1}\sqrt[S]{I}$. For the other inclusion, choose $\frac{a}{s} \in S^{-1}\sqrt[S]{I}$. Then there exists $u \in S$ such that $ua \in \sqrt[S]{I}$. Since $\sqrt[S]{I}$ is S-pure, we can find $b \in \sqrt[S]{I}$ such that $t(ua) = (ua)b$ for some $t \in S$, where $s'b^n \in I$ for some $s' \in S$ and $n \in \mathbb{Z}^+$. Then $s't^n ua = s'tuab^{n-1} = s'uab^n$. As $s'b^n \in I$, we obtain $s^\star a \in I$, where $s^\star = s't^n u$. This gives $\frac{s^\star a}{s^\star s} = \frac{a}{s} \in S^{-1}I$, as needed. □

Recall that a ring R is called S-reduced if $N_S(R) = 0$, that is, there is no nonzero S-nilpotent element of R.

Proposition 3.10. *Given an S-pure ideal I of an S-reduced ring R. Then $S^{-1}I = S^{-1}\sqrt[S]{I}$.*

Proof. One can easily see that $S^{-1}I \subseteq S^{-1}\sqrt[S]{I}$. Now pick $\frac{a}{s} \in S^{-1}\sqrt[S]{I}$. Then $ua \in \sqrt[S]{I}$ for some $u \in S$. So we can find an $s' \in S$ providing $s'(ua)^n \in I$ for some $n \in \mathbb{Z}^+$. As I is S-pure, $ts'(ua)^n = s'(ua)^n b$ for some $b \in I$ and $t \in S$. This gives $s'(ua)^n(t-b) = 0$. As $s'(ua)^n(t-b)^n = 0$ and R is S-reduced, we obtain $uta = uab \in I$. So we get $uta \in I$ which gives $\frac{a}{s} = \frac{uta}{uts} \in S^{-1}I$. □

Proposition 3.11. *Let I, J be S-pure ideals of R satisfying $(I : s) = I$, $(J : s) = J$ for all $s \in S$ and $\sqrt[S]{I} = \sqrt[S]{J}$. Then $I = J$.*

Proof. Let $a \in I$. Then $sa = ab$ for some $s \in S$ and for some $b \in I$. Since $b \in I \subseteq \sqrt[S]{I} = \sqrt[S]{J}$, $s'b^n \in J$ for some $s' \in S$ and $n \in \mathbb{Z}^+$. Then $s'sab^{n-1} = s'ab^n$. It implies that $s's^n a = s'ab^n$. Put $s's^n = t$. Then $ta = as'b^n \in J$. Thus $a \in (J : t) = J$. Similar argument shows that $J \subseteq I$, as desired. □

Corollary 3.12. *Let I, J be S-pure ideals of R satisfying $\sqrt[S]{I} = \sqrt[S]{J}$. Then $S^{-1}I = S^{-1}J$.*

Lemma 3.13. *Let R be a ring, I be any ideal and P be an S-prime ideal of R. Then, $S^{-1}I \subseteq S^{-1}P$ if and only if $P \in V_S(I)$.*

Proof. Assume that $S^{-1}I \subseteq S^{-1}P$. Then for all $a \in I$, $\frac{a}{s} \in S^{-1}P$. This means that there exists $t \in S$ such that $ta \in P$ for all $a \in I$ by [12, Lemma 2.16]. Hence $tI \subseteq P$ which gives $P \in V_S(I)$, as desired. On the other hand, suppose $P \in V_S(I)$ and take $\frac{a}{s'} \in S^{-1}I$. Then $ua \in I$ for some $u \in S$ and there exists $s \in S$ such that $sI \subseteq P$. This gives $sua \in sI \subseteq P$. Hence $\frac{a}{s'} = \frac{sua}{sus'} \in S^{-1}P$, as needed. □

Definition 3.14. A subset E of $Spec_S(R)$ is called stable under generalization if $Q \in E$ and P is an S-prime ideal of R with $P \subset Q$ implies $P \in E$.

The following theorem is the S-version of [15, Theorem 3.12].

Theorem 3.15. *Given a m.c.s. S of R, the map $\phi : I \mapsto V_S(I)$ is an injection from the set of S-pure ideals with $(I : s) = I$ for all $s \in S$ to the set of closed subsets of the S-prime spectrum of R which are stable under generalization.*

Proof. Assume that I is an S-pure ideal of R with $(I : s) = I$ for all $s \in S$. Then we have to show $V_S(I)$ is stable under generalization. Let Q and P be two S-prime ideals such that $Q \subseteq P$. Suppose $P \in V_S(I)$. Then $sI \subseteq P$ for some $s \in S$. Now assume that $Q \notin V_S(I)$ implying $sI \nsubseteq Q$ for all $s \in S$. It indicates that there exists $a \in I$ such that $sa \notin Q$ for all $s \in S$. This gives $\frac{a}{s'} = \frac{sa}{ss'} \notin S^{-1}Q$ for all $s' \in S$. Choose $\frac{r}{t} \in Ann(\frac{a}{s'})$. Then $\frac{r}{t}\frac{a}{s'} = 0 \in S^{-1}Q$. Then we can find $u \in S$ providing $ura \in Q$. Since Q is an S-prime ideal and

$sa \notin Q$ for all $s \in S$, we get $s''r \in Q$ for some $s'' \in S$. It gives $\frac{s''r}{s''t} = \frac{r}{t} \in S^{-1}Q$ and so $Ann(\frac{a}{s'}) \subseteq S^{-1}Q$. By using this fact and Lemma 3.13, we have

$$Ann(\frac{a}{s'}) + S^{-1}I \subseteq S^{-1}Q + S^{-1}I \subseteq S^{-1}Q + S^{-1}P \subseteq S^{-1}P \tag{2}$$

It contradicts with $Ann(\frac{a}{s'}) + S^{-1}I = S^{-1}R$. Thus, $Q \in V_S(I)$, that is, $V_S(I)$ is stable under generalization and this map is well defined. For injectivity, assume that $\phi(I) = \phi(J)$. Let $I \nsubseteq J$. If we assume $S^{-1}I \subseteq S^{-1}J$, then for all $a \in I$ we have $\frac{a}{s} \in S^{-1}I \subseteq S^{-1}J$. So $ua \in J$ for some $u \in S$ and this implies $a \in (J : u) = J$, a contradiction. Thus $S^{-1}I \nsubseteq S^{-1}J$. Then there exists $\frac{a}{s} \in S^{-1}I$ such that $\frac{a}{s} \notin S^{-1}J$. This means $Ann(\frac{a}{s}) + S^{-1}J \neq S^{-1}R$. Thus we can find a prime ideal P of R with $P \cap S = \varnothing$ such that $Ann(\frac{a}{s}) + S^{-1}J \subseteq S^{-1}P$. Since $S^{-1}J \subseteq S^{-1}P$, we have $J \subseteq P$ and thus $P \in V_S(J) = V_S(I)$. Then we have $S^{-1}I \subseteq S^{-1}P$ by Lemma 3.13. Hence $Ann(\frac{a}{s}) + S^{-1}I \subseteq S^{-1}P$, a contradiction. Because I is S-pure ideal and so $Ann(\frac{a}{s}) + S^{-1}I = S^{-1}R$. Then we conclude that $I \subseteq J$. Similar argument shows $J \subseteq I$. Therefore, we obtain $I = J$, as desired. □

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