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PERIODIC SHADOWABLE POINTS

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Abstract. In this paper, we consider the set of periodic shadowable points for homeomorphisms of a compact metric space, and we prove that this set satisfies some properties such as invariance and being a G_δ set. Then we investigate implication relations related to sets consisting of shadowable points, periodic shadowable points and uniformly expansive points, respectively. Assume that the set of periodic points and the set of periodic shadowable points of a homeomorphism on a compact metric space are dense in X. Then we show that a homeomorphism has the periodic shadowing property if and only if so is the restricted map to the set of periodic shadowable points. We also give some examples related to our results.

1. Introduction and preliminaries

The various notions of shadowing play an important role in the qualitative theory of hyperbolic dynamical systems (see [12]). Morales introduced the notion of shadowable points closely related to that of absolutely non-shadowable points and studied the relations between several invariant sets through pointwise dynamics (see [11]). In [6], Kawaguchi further extended this notion by introducing the concept of quantitative shadowable points for continuous maps and studied shadowable points with a given shadowing accuracy. Koo *et al.* [8] introduced the notions of topological stable points and finitely shadowable points for homeomorphisms on a compact metric space. They investigated the connection between pointwise topological stability in the sense of Walters and various shadowing properties. Some of the results concerning above works have been extended to a flow case by Aponte [2, 3].

Pilyugin and Plamenevskaya proved that the shadowing is a generic property in the set of homeomorphisms $H(M)$ on a compact smooth manifold M without boundary (see [13]). Also, Kościelniak and Mazur proved that if the space M is a closed smooth manifold, then periodic shadowing is a generic property in $H(M)$ (see [10]).

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Let us recall some basic notions of dynamical systems which are used in the sequel. Let X be a compact metric space with metric d and $f : X \to X$ be a homeomorphism.

For any $x \in X$, the set $\{f^n(x)\}_{n\in\mathbb{Z}}$ is called the *orbit* of x under f and is denoted by $O_f(x)$. A point $x \in X$ is said to be a *periodic point* if $f^n(x) = x$ for some $n \in \mathbb{N}$. The period of x is the smallest natural number of $\{n \in \mathbb{N} : f^n(x) = x\}$. Periodic points of period 1 are *fixed points.* We denote by $\mathrm{Per}(f)$ and $\mathrm{Fix}(f)$ the set of periodic points and the set of fixed points of f, respectively. A point $x \in X$ is said to be a *nonwandering point* if for any neighborhood U of x there is $n \in \mathbb{N}$ such that $U \cap f^n(U) \neq \emptyset$. The set $\Omega(f)$ of all nonwandering points is called the *nonwandering set.* For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in X is said to be a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. If for some positive integer $N \in \mathbb{N}$, we have $x_{i+N} = x_i$ (for all $i \in \mathbb{Z}$), then this δ-pseudo orbit is called *periodic.* A sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is called to be ϵ-*traced* (or *shadowed*) by $z \in X$ if $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. A point $x \in X$ is said to be *chain recurrent* if for any $\delta > 0$ there is a periodic δ-pseudo orbit which passes through x. The set $\mathrm{CR}(f)$ of all chain recurrent points is called the *chain recurrent set.* A subset B of X is said to be f-*invariant* for a homeomorphism f if $f(B) = B$. Obviously, $\mathrm{Per}(f)$, $\Omega(f)$ and $\mathrm{CR}(f)$ are f-invariant subsets of X.

We say that a finite δ-pseudo orbit $\{x_i\}_{i=0}^{l}$ of a homeomorphism f is a δ-*chain* from x_0 to x_l with length l. A non-empty subset A of X is said to be *chain transitive* whenever for any $x, y \in A$ and any $\delta > 0$ there exists a δ-chain from x to y. A homeomorphism $f : X \to X$ is said to be *chain transitive* if X is chain transitive.

We say that a homeomorphism $f : X \to X$ has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that any δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is ϵ-shadowed by a point $z \in X$.

We say that a homeomorphism $f : X \to X$ has the *periodic shadowing property* if for any $\epsilon > 0$ there is $\delta > 0$ such that if $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is a periodic δ-pseudo orbit, then there is $z \in \mathrm{Per}(f)$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

We recall that a point $x \in X$ is *shadowable* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ with $x_0 = x$ can be ϵ-shadowed (see [11, Definition 1.1]). We denote by $\mathrm{Sh}(f)$ the set of shadowable points of f. We say that a point $x \in X$ is *periodic shadowable* if for every $\epsilon > 0$ there is $\delta > 0$ such that every periodic δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}}$ with $x_0 = x$ can be ϵ-shadowed by some periodic point for f. We denote by $\mathrm{Sh}_{per}(f)$ the set of periodic shadowable points of f.

For the completion of the our results, we recall that a homeomorphism $f : X \to X$ is *expansive* if there is a positive constant e such that if $d(f^i(x), f^i(y)) \leq e$ for all $i \in \mathbb{Z}$, then $x = y$. We say that a point $x \in X$ is a *uniformly expansive point* if there exist a neighborhood U of x and an expansive constant $e > 0$ such that if $d(f^i(z), f^i(y)) \leq e$ for all $i \in \mathbb{Z}$, whenever $y, z \in U$, then $y = z$ (see [4]). Denote by $\mathrm{Exp}_u(f)$ the set of uniformly expansive points.

Clearly, a homeomorphism f is expansive if and only if every point of X is uniformly expansive (i.e., $\mathrm{Exp}_u(f) = X$).

In this paper, we consider the set of periodic shadowable points for homeomorphisms of a compact metric space, and we study some invariance and recurrence properties concerning $\mathrm{Sh}_{per}(f)$. Additionally, we show that $\mathrm{Sh}_{per}(f)$ is a G_δ subset of X. Then we investigate the relations between the sets of shadowable points, periodic shadowable points and uniformly expansive points, respectively. Also, we show that under the density of $\mathrm{Per}(f)$ and $\mathrm{Sh}_{per}(f)$, a homeomorphism has the periodic shadowing property if and only if so is the restricted map to the set of periodic shadowable points. Furthermore, we give some examples related to our results. More precisely, we state our main results.

Theorem 1.1. *Every uniformly expansive shadowable point of a homeomorphism on a compact metric space is a periodic shadowable point.*

Theorem 1.2. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Assume that $\mathrm{Per}(f)$ and $\mathrm{Sh}_{per}(f)$ are dense in X. Then a homeomorphism f has the periodic shadowing property if and only if $f|_{\mathrm{Sh}_{per}(f)}$ has the periodic shadowing property.*

2. Proof of main results

In this section, we introduce some results that are used to show our main results. Then we give proofs of main results of this paper. Also, we give examples to illustrate our main results.

Let us introduce the concept of the periodic shadowing property through $K \subset X$ for a homeomorphism $f : X \to X$ of a compact metric space X. We say that a homeomorphism $f : X \to X$ has the periodic shadowing property *through* K if for every $\epsilon > 0$ there is $\delta > 0$ such that every periodic δ-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ of f through K (i.e., $x_0 \in K$) can be ϵ-shadowed by a periodic point for f.

Lemma 2.1. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then f has the periodic shadowing property through a compact subset K of X if and only if every point in K is periodic shadowable.*

Proof. Obviously, we only have to prove the sufficiency. Suppose by contradiction that there is a compact subset K of X such that every point in K is periodic shadowable but f does not have the periodic shadowing property through K. Then there exist $\epsilon > 0$ and a sequence $\{\xi^k\}_{k\in\mathbb{N}}$ of periodic $\frac{1}{k}$-pseudo orbits with period $N_k \in \mathbb{N}$ through K that cannot be 2ϵ-shadowed. For every $k \in \mathbb{N}$, we write $\xi^k = \{\xi^k_i\}_{i\in\mathbb{Z}}$. By compactness of $K \subset X$, there is a point $y \in K$ such that ξ^k_0 converges to y as $k \to \infty$. Note that this $y \in K$ is periodic shadowable. Consequently, we can choose $\delta > 0$ from the periodic shadowableness of y for the above $\epsilon > 0$. Then we define a periodic sequence

$\hat{\xi^k} = \{\hat{\xi_i^k}\}_{i\in\mathbb{Z}}$ with period $N_k \in \mathbb{N}$ by

$$\hat{\xi_i^k} = \begin{cases} \xi_i^k, & \text{if } \ i \neq s \cdot N_k, \\ y, & \text{if } \ i = s \cdot N_k, \quad s \in \mathbb{Z}. \end{cases}$$

We see that all such periodic sequences are through the point $y \in K$. Since we have

$$d(f(\hat{\xi}_{i-1}^k), \hat{\xi}_i^k) = \begin{cases} d(f(\xi_{i-1}^k), \xi_i^k), & \text{if } \ i \neq s \cdot N_k, \ s \cdot N_k + 1, \\ d(f(y), \xi_i^k), & \text{if } \ i = s \cdot N_k + 1, \\ d(f(\xi_{i-1}^k), y), & \text{if } \ i = s \cdot N_k, \quad s \in \mathbb{Z}, \end{cases}$$

then

$$d(f(\hat{\xi}_{i-1}^k), \hat{\xi}_i^k) \leq \begin{cases} \frac{1}{k}, & \text{if } \ i \neq s \cdot N_k, \ s \cdot N_k + 1, \\ d(f(y), f(\xi_{i-1}^k)) + \frac{1}{k}, & \text{if } \ i = s \cdot N_k + 1, \\ d(\xi_i^k, y) + \frac{1}{k}, & \text{if } \ i = s \cdot N_k, \quad s \in \mathbb{Z}. \end{cases}$$

By continuity of f and $\xi_0^k \to y$, we get the periodic δ-pseudo orbit $\{\hat{\xi}_i^k\}_{i\in\mathbb{Z}}$ for sufficiently large k. Then for such a k it follows that there is $x_k \in \mathrm{Per}(f)$ such that $d(f^i(x_k), \hat{\xi}_i^k) \leq \epsilon$ for every $i \in \mathbb{Z}$. Then we see that $d(f^i(x_k), \xi_i^k) \leq \epsilon$ for $i \neq s \cdot N_k$. Since

$$d(x_k, \xi_{s\cdot N_k}^k) \leq d(x_k, y) + d(y, \xi_{s\cdot N_k}^k) \leq \epsilon + d(y, \xi_{s\cdot N_k}^k), \ \ k \in \mathbb{N},$$

we can conclude that $d(f^i(x_k), \xi_i^k) \leq 2\epsilon$ for $i = s \cdot N_k$ when k is sufficiently large. This implies that $\xi^k = \{\xi_i^k\}_{i\in\mathbb{Z}}$ can be 2ϵ-shadowed by a periodic point for k sufficiently large. This contradiction proves the result. □

We can state our results related to the periodic shadowableness and invariant sets.

Proposition 2.2. *The following properties hold for every homeomorphism $f : X \to X$ of a compact metric space X:*

(1) $\mathrm{Sh}_{per}(f)$ *is an f-invariant set.*
(2) *f has the periodic shadowing property if and only if* $\mathrm{Sh}_{per}(f) = X$.
(3) *If* $\mathrm{CR}(f) \subset \mathrm{Sh}_{per}(f)$, *then* $\overline{\mathrm{Per}(f)} = \Omega(f) = \mathrm{CR}(f)$. *Here $\overline{A}$ is denoted by the closure of A.*
(4) *If $h : X \to Y$ is a homeomorphism of metric spaces, then* $\mathrm{Sh}_{per}(h \circ f \circ h^{-1}) = h(\mathrm{Sh}_{per}(f))$.

Proof. First, we prove item (1). The proof is similar to that of Lemma 2.4 in [11]. It suffices to prove that if $x \in X$ is a periodic shadowable point of a homeomorphism $f : X \to X$ of a compact metric space X, then so are $f(x)$ and $f^{-1}(x)$. We only prove that $f(x)$ is a periodic shadowable point as the same proof works for $f^{-1}(x)$.

Fix $\epsilon > 0$. Since X is compact, f is uniformly continuous, so there is $\epsilon' > 0$ such that $d(f(y), f(z)) \leq \epsilon$ whenever $y, z \in X$ satisfy $d(y, z) \leq \epsilon'$. For this

ϵ', let $\delta' > 0$ be given by the periodic shadowableness of x. Again, since X is compact, so f^{-1} is uniformly continuous, thus there is $\delta > 0$ such that $d(f^{-1}(y), f^{-1}(z)) \leq \delta'$ whenever $y, z \in X$ satisfy $d(y, z) \leq \delta$.

Take a periodic δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ through $f(x)$ with period N. It follows from the choice of δ that $\{f^{-1}(x_i)\}_{i\in\mathbb{Z}}$ is a periodic δ'-pseudo orbit which is obviously through x with period N'. Then, the choice of δ' implies that $\{f^{-1}(x_i)\}_{i\in\mathbb{Z}}$ can be ϵ'-shadowed. So, the choice of ϵ' implies that $\{x_i\}_{i\in\mathbb{Z}}$ can be ϵ-shadowed.

Next we prove item (2). By taking $K = X$ in Lemma 2.1, then we have that f has the periodic shadowing property if and only if $\mathrm{Sh}_{per}(f) = X$.

Next we prove item (3). To prove $\mathrm{CR}(f) \subset \Omega(f)$, first we claim that $\mathrm{Sh}_{per}(f) \subset \Omega(f)$. Take a point $x \in \mathrm{Sh}_{per}(f)$ and $\epsilon > 0$. For this ϵ, let δ be given by the periodic shadowableness of x. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be a periodic δ-pseudo orbit with $x_0 = x$ and period N ($x_{i+N} = x_i$ for all $i \in \mathbb{Z}$) of f. Then there is a periodic point $z \in X$ such that

$$d(f^i(z), x_i) < \epsilon \quad \text{for all } i \in \mathbb{Z}.$$

In particular, $d(z, x) < \epsilon$ and $d(f^N(z), x) < \epsilon$. So $f^N(B(x, \epsilon)) \cap B(x, \epsilon) \neq \emptyset$, where $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$. Since ϵ is arbitrary, we get $x \in \Omega(f)$. So we see $\mathrm{CR}(f) \subset \Omega(f)$. Therefore, $\mathrm{CR}(f) = \Omega(f)$.

Now we will prove that $\overline{\mathrm{Per}(f)} = \mathrm{CR}(f)$. Clearly, $\overline{\mathrm{Per}(f)} \subset \mathrm{CR}(f)$. Thus, we need to show that $\mathrm{CR}(f) \subset \overline{\mathrm{Per}(f)}$. Let $x \in \mathrm{CR}(f)$. For any ϵ-neighborhood $B(x, \epsilon)$ of x, there is a periodic point $z \in \mathrm{Per}(f)$ such that $z \in B(x, \epsilon) \cap \mathrm{Per}(f)$.

Suppose $x \in \mathrm{CR}(f)$. For $\epsilon > 0$, let $0 < \delta < \epsilon$ be the number of the periodic shadowableness of x. Then there are $\delta > 0$ and $\xi = \{x_i\}_{i\in\mathbb{Z}}$ such that $x \in \xi$, where ξ is a periodic δ-pseudo orbit with period N. If $x = x_i$ for some $i \in \mathbb{Z}$, then $x = x_{i+N}$, i.e., there is a δ-chain such that $x = x_i, x_{i+1}, \ldots, x_{i+N} = x$.

By the periodic shadowableness of x, there is $z \in \mathrm{Per}(f)$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Thus we have a periodic point $z \in B(x, \epsilon) \cap \mathrm{Per}(f)$.

Lastly, we prove item (4). Let X and Y be compact metric spaces with metrics d and d', respectively and $h : X \to Y$ be a homeomorphism. Since h is uniformly continuous, it is easy to show that $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is a periodic δ-pseudo orbit of f with period N if and only if $h(\xi)$ also is a periodic δ'-pseudo orbit of g with period N. By applying similar argument in the proof of Lemma 2.3 in [7] for the periodic shadowableness, we can prove item (4). Hence the proof is complete. □

Proposition 2.3. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then $\mathrm{Sh}_{per}(f) = \mathrm{Sh}_{per}(f^n)$ for every $n \in \mathbb{N}$.*

Proof. Suppose $x \in \mathrm{Sh}_{per}(f)$. Let $\epsilon > 0$ and $\delta > 0$ be given by the periodic shadowableness of x. Given $n \in \mathbb{N}$, let $\{x_i\}_{i\in\mathbb{Z}}$ be a periodic δ-pseudo orbit of f^n with $x_0 = x$ and period N. Define a sequence $\eta = \{y_i\}_{i\in\mathbb{Z}}$ by

$$y_k = \begin{cases} x_q, & \text{if} \quad k = nq, \\ f^{k-nq}(x_q), & \text{if} \quad nq < k < n(q+1). \end{cases} \tag{2.1}$$

Indeed,

$$\eta = \{\dots, x_0, f(x_0), f^2(x_0), \dots, f^{n-1}(x_0), x_1, f(x_1), f^2(x_1), \dots, f^{n-1}(x_1), \dots\}$$

is also a periodic sequence. It is easy to see that $\eta = \{y_i\}_{i\in\mathbb{Z}}$ is a periodic δ-pseudo orbit of f with $y_0 = x_0 = x$ and period $N \cdot n$. By the periodic shadowableness of x, we find a periodic point $z \in X$ such that $d(f^i(z), y_i) < \epsilon$ for all $i \in \mathbb{Z}$. Putting $i = nq$ in the above inequality (2.1), we get $y_i = x_q$ and $d((f^n)^q(z), x_q) < \epsilon$ for all $i \in \mathbb{Z}$. Thus $x \in \mathrm{Sh}_{per}(f^n)$ for each $n \in \mathbb{N}$. Hence we have $\mathrm{Sh}_{per}(f) \subset \mathrm{Sh}_{per}(f^n)$. Similarly, we can also show the converse inclusion. This completes the proof. □

Proposition 2.4. *Every periodic shadowable point of a chain transitive homeomorphism on a compact metric space is shadowable.*

Proof. Let $f : X \to X$ be a chain transitive homeomorphism of a compact metric space X, and $x \in X$ be a periodic shadowable point of f. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be a non-periodic δ-pseudo orbit of f with $x_0 = x$. Since f is chain transitive, we can take a δ-chain from x_{-n} to x_n for each $n \in \mathbb{N}$;

$$\{x_n = y_0, y_1, \dots, y_{k_n} = x_{-n}\}$$

of length k_n such that $d(f(y_i), y_{i+1}) < \delta$ for $0 \le i \le k_n - 1$. Then we get a periodic δ-pseudo orbit

$$\{p_0^n = x_n = y_0, p_1^n = y_1, \dots, p_{k_n-1}^n = y_{k_n-1}, p_{k_n}^n = y_{k_n} = x_{-n}, \dots,$$
$$x_{-1}, x_0, x_1, \dots, p_{k_n+2n-1}^n = x_n = y_0\}$$

with $(k_n + 2n)$-period for each $n \in \mathbb{N}$. Since $x_0 = x$ is a periodic shadowable point of f, there is $z_n \in \mathrm{Per}(f)$ such that $d(f^i(z_n), p_i^n) < \epsilon$ for each $n \in \mathbb{N}$. Since X is compact, then there exists $\{z_n\}_{n\in\mathbb{N}}$ such that z_n converges to z. Note that if $i \le n$, then $p_i^n = x_i$. Then we see that

$$\lim_{n\to\infty} d(f^i(z_n), x_i) = \lim_{n\to\infty} d(f^i(z_n), p_i^n) = d(f^i(z), x_i) < \epsilon$$

for each $i \le n$. This completes the proof. □

Remark 2.5. Kawaguchi proved that if a homeomorphism $f : X \to X$ of a compact metric space is chain transitive, then either $\mathrm{Sh}(f) = \emptyset$ or $\mathrm{Sh}(f) = X$ (see [6, Theorem 1.1]). So Proposition 2.4 is also proved by using [6, Theorem 1.1] in the case when $\mathrm{Sh}(f) \neq \emptyset$.

Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $f : X \to X$ be a homeomorphism of a compact metric space X. Suppose that $x \in X$ is a uniformly expansive shadowable point. Then there exist an expansive constant $e > 0$ and a neighborhood U of x such that if $d(f^i(z), f^i(y)) < e$ for all $i \in \mathbb{Z}$, whenever $y, z \in U$, then $y = z$. Without loss of generality, it is sufficient to prove it for $\epsilon \le \frac{e}{2}$ satisfying $B(x, \epsilon) \subset U$. Let $0 < \epsilon \le \frac{e}{2}$ and $\delta > 0$ be given by the shadowableness of x. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be a periodic δ-pseudo orbit of f with $x_0 = x$ and period N

($x_{i+N} = x_i$ for all $i \in \mathbb{Z}$). By the shadowableness of x, there is a point $z \in X$ such that

$$d(f^i(z), x_i) < \epsilon \text{ for all } i \in \mathbb{Z}. \tag{2.2}$$

Since $x_{i+N} = x_i$ for all $i \in \mathbb{Z}$, we have

$$d(f^{i+N}(z), x_i) < \epsilon \text{ for all } i \in \mathbb{Z}.$$

Put $p = f^N(z)$. Then, we have

$$d(f^i(p), x_i) < \epsilon \text{ for all } i \in \mathbb{Z}. \tag{2.3}$$

Using inequalities (2.2) and (2.3), we get $d(f^i(z), f^i(p)) < 2\epsilon \leq e$ for all $i \in \mathbb{Z}$. The uniform expansivity of x implies $p = z = f^N(z)$. Thus $x \in \mathrm{Sh}_{per}(f)$ and $\mathrm{Exp}_u(f) \cap \mathrm{Sh}(f) \subset \mathrm{Sh}_{per}(f)$. This completes the proof. □

From Theorem 1.1 and Proposition 2.2, we immediately obtain the following result.

Corollary 2.6. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. Then the following properties hold*:

(1) *If* $\mathrm{CR}(f) \subset \mathrm{Exp}_u(f) \cap \mathrm{Sh}(f)$, *then* $\overline{\mathrm{Per}(f)} = \Omega(f) = \mathrm{CR}(f)$.
(2) *If f is expansive, then* $\mathrm{Sh}(f) \subset \mathrm{Sh}_{per}(f)$.

From Theorem 1.1 and Proposition 2.4, we obtain the following result as in Corollary 2.8 in [5].

Corollary 2.7. *For every expansive and chain transitive homeomorphism $f : X \to X$ of a compact metric space X, we have* $\mathrm{Sh}(f) = \mathrm{Sh}_{per}(f) = X$.

Given $\epsilon > 0$ and a subset K of X. We say that a homeomorphism f has *ϵ-periodic shadowing property* through a subset K of X if there exists $\delta > 0$ such that every periodic δ-pseudo orbit passing through K can be ϵ-shadowed by some periodic point of f. We denote by $B[\cdot, \delta]$ the closed δ-ball operation on X.

Lemma 2.8. *Let $f : X \to X$ be a homeomorphism of a compact metric space X and $\epsilon > 0$ be given. If f has the ϵ-periodic shadowing property through a compact subset K, then there is $\delta > 0$ such that f has the 2ϵ-periodic shadowing property through $B[K, \delta]$.*

Proof. Suppose by contradiction that a homeomorphism f has the ϵ-periodic shadowing property through a compact subset K but for every $\delta > 0$, we can find a periodic δ-pseudo orbit through $B[K, \delta]$ that cannot be 2ϵ-shadowed by some periodic point of f.

Take $\delta > 0$ from the ϵ-periodic shadowing property through K with $\delta < \epsilon$, and let $\{\xi_0^k\}_{k \in \mathbb{N}}$ be a sequence of periodic $\frac{1}{k}$-pseudo orbits with period N_k for f passing through $B[K, \frac{1}{k}]$ that cannot be 2ϵ-shadowed by some periodic point of f. For every $k \in \mathbb{N}$, we write $\xi^k = \{\xi_i^k\}_{i \in \mathbb{Z}}$. It follows from $B[K, \frac{1}{k}]$ that there is a sequence $x_k \in K$ such that $d(\xi_0^k, x^k) \leq \frac{1}{k}$ for all $k \in \mathbb{N}$. Since X

is compact, f is uniformly continuous, so we can choose k sufficiently large such that $\max\{d(f(\xi_0^k), f(x^k)), \frac{1}{k}\} \leq \frac{\delta}{2}$. Fix k and define a periodic sequence $\xi = \{\xi_i\}_{i\in\mathbb{Z}}$ with period N_k ($\xi_i = \xi_{i+N_k}$ for all $i \in \mathbb{Z}$) by

$$\xi_i = \begin{cases} \xi_i^k, & \text{if} \quad i \neq s \cdot N_k, \\ x^k, & \text{if} \quad i = s \cdot N_k, \quad s \in \mathbb{Z}. \end{cases}$$

Clearly, $d(f(\xi_i), \xi_{i+1}) \leq \frac{1}{k}$ for $n \neq s \cdot N_k - 1, s \cdot N_k, s \cdot N_k + 1$. Since

$$\begin{aligned} d(f(\xi_{s\cdot N_k - 1}), \xi_{s\cdot N_k}) &= d(f(\xi_{-1}), x^k) \leq d(f(\xi_{-1}), \xi_0^k) + d(\xi_0^k, x^k) \\ &\leq \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \leq \delta \end{aligned}$$

and

$$\begin{aligned} d(f(\xi_{s\cdot N_k}), \xi_{s\cdot N_k + 1}) &= d(f(x^k), \xi_1^k) \leq d(f(x^k), f(\xi_0^k)) + d(f(\xi_0^k), \xi_1^k) \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned}$$

we see that ξ is a periodic δ-pseudo orbit with period N_k. Since $\xi_0 = x^k \in K$ by definition, we see that ξ can be ϵ-shadowed by a periodic point z of f such that $d(f^i(z), \xi_i) \leq \epsilon$ for every $i \in \mathbb{Z}$. Clearly, $d(f^i(z), \xi_i^k) = d(f^i(z), \xi_i) \leq \epsilon \leq 2\epsilon$ for all $i \neq 0$. For $n = 0$, we have that

$$\begin{aligned} d(f^i(z), \xi_i^k) &= d(z, \xi_0^k) \leq d(z, x^k) + d(x^k, \xi_0^k) \\ &= d(z, \xi_0) + \frac{1}{k} \leq \epsilon + \frac{\delta}{2} \leq 2\epsilon \ \text{ for all } i \in \mathbb{Z}. \end{aligned}$$

Thus $d(f^i(z), \xi_i^k) \leq 2\epsilon$ for all $i \in \mathbb{Z}$. It follows that ξ_i^k is 2ϵ-shadowed by a periodic point z of f, which is a contradiction. This proves the result. □

A subset of a space X is a G_δ set if it is a countable intersection of open subsets of X.

We can obtain the following result related to the periodic shadowableness, which is adapted from Theorem 2.2.11 in [2].

Proposition 2.9. *The set of periodic shadowable points of a homeomorphism of a compact metric space is a G_δ set.*

Proof. Let $f : X \to X$ be a homeomorphism of a compact metric space X. Given $\epsilon > 0$ we denote by $\mathrm{Sh}_{per}(f, \epsilon)$ the set of points $p \in X$ such that f has the ϵ-periodic shadowing property through a subset $\{p\}$. We note that

$$\mathrm{Sh}_{per}(f) = \bigcap_{\epsilon>0} \mathrm{Sh}_{per}(f, \epsilon). \tag{2.4}$$

Let $\epsilon_0 > 0$ be given. We can suppose $\mathrm{Sh}_{per}(f) \neq \emptyset$. Given $x \in \mathrm{Sh}_{per}(f)$, since $x \in \mathrm{Sh}_{per}(f, \frac{\epsilon_0}{2})$ by Lemma 2.8, there is $\delta_{x,\epsilon_0} > 0$ such that every periodic δ_{x,ϵ_0}-pseudo orbit through $B[K, \delta_{x,\epsilon_0}]$ can be ϵ_0-shadowed by a periodic point of f. It follows that $B(x, \delta_{x,\epsilon_0}) \subset \mathrm{Sh}_{per}(f, \epsilon_0)$. So, for every $\epsilon > 0$,

$$\mathrm{Sh}_{per}(f, \epsilon) = A(\epsilon_0) \cup B(\epsilon_0),$$

where $A(\epsilon_0) = \bigcup_{x \in \mathrm{Sh}_{per}(f)} B(x, \delta_{x,\epsilon_0})$ and $B(\epsilon_0) = \mathrm{Sh}_{per}(f, \epsilon_0) \setminus A(\epsilon_0)$. Moreover, we note that $A(\epsilon_0)$ is open and $B(\epsilon_0) \subset \mathrm{Sh}_{per}(f, \epsilon_0) \setminus \mathrm{Sh}_{per}(f)$. By equality of (2.4), we have that $\bigcap_{\epsilon_0 > 0} B(\epsilon_0) = \emptyset$ and

$$\mathrm{Sh}_{per}(f) = \bigcap_{n \in \mathbb{N}} \mathrm{Sh}_{per}\Big(f, \frac{1}{n}\Big) = \bigcap_{n \in \mathbb{N}} A\Big(\frac{1}{n}\Big) \cup B\Big(\frac{1}{n}\Big) = \bigcap_{n \in \mathbb{N}} A\Big(\frac{1}{n}\Big).$$

Thus $\mathrm{Sh}_{per}(f)$ is a G_δ set of X. This completes the proof. □

Next, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that f has the periodic shadowing property. Let $\epsilon > 0$ and choose $\delta > 0$ such that every periodic δ-pseudo orbit of f in X is $\frac{\epsilon}{2}$-shadowed. Let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a periodic δ-pseudo orbit with period N ($x_i = x_{i+N}$ all $i \in \mathbb{Z}$) in $\mathrm{Sh}_{per}(f)$. Then ξ is a periodic δ-pseudo orbit in X. Since f has the periodic shadowing property, there exists a periodic point $z \in X$ such that $d(f^i(z), x_i) < \frac{\epsilon}{2}$ for all $i \in \mathbb{Z}$. Since f is continuous, there exist $r_{N-1} > 0$ with $r_{N-1} < \frac{\epsilon}{2}$ and $f(B(f^{N-1}(z), r_{N-1})) \subseteq B(f^N(z), \frac{\epsilon}{2})$. Also, there exist $r_{N-2} > 0$ with $r_{N-2} < r_{N-1}$ and $f(B(f^{N-2}(z), r_{N-2})) \subseteq B(f^{N-1}(z), r_{N-1})$. Continuing this process, we arrive at $r_1 > 0$ with $r_1 < r_2$ and $f(B(f(z), r_1)) \subseteq B(f^2(z), r_2)$. Finally, there exist $r_0 > 0$ with $r_0 < r_1$ and $f(B(z, r_0)) \subseteq B(f(z), r_1)$. Since $\mathrm{Per}(f)$ is dense in X and by construction, every $z_1 \in B(z, r_0) \cap \mathrm{Per}(f)$ is ϵ-shadows ξ, i.e.,

$$\begin{aligned} d(f^i(z_1), y_i) &\leq d(f^i(z_1), f^i(z)) + d(f^i(z), y_i) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } i \in \mathbb{Z}. \end{aligned}$$

Thus $f|_{\mathrm{Sh}_{per}(f)}$ has the periodic shadowing property.

The converse is proved by Theorem 3.1 in [9]. Hence the proof is complete. □

Corollary 2.10. *Let $f : X \to X$ be a homeomorphism of a compact metric space X. If $f|_{\overline{\mathrm{Sh}_{per}(f)}}$ has the periodic shadowing property, then the closure of the set of periodic shadowable points of $f|_{\overline{\mathrm{Sh}_{per}(f)}}$ coincides with $\overline{\mathrm{Sh}_{per}(f)}$. A similar statement also holds for the shadowableness of homeomorphisms.*

Proof. Suppose that the restricted map $f|_{\overline{\mathrm{Sh}_{per}(f)}} : \overline{\mathrm{Sh}_{per}(f)} \to \overline{\mathrm{Sh}_{per}(f)}$ has the periodic shadowing property. By Proposition 2.2(2), we have

$$\mathrm{Sh}_{per}(f|_{\overline{\mathrm{Sh}_{per}(f)}}) = \overline{\mathrm{Sh}_{per}(f)}.$$

Therefore $\overline{\mathrm{Sh}_{per}(f|_{\overline{\mathrm{Sh}_{per}(f)}})} = \overline{\mathrm{Sh}_{per}(f)}$. □

3. Examples

In this section, we give some examples related our main results and their key notions.

First, we give an example that there are a compact metric space X and a homeomorphism $f : X \to X$ such that an f-invariant set $\mathrm{Sh}_{per}(f)$ is a nonempty, open subset of X (see [11, Example 2.1]).

Example 3.1. Setting $X = C \cup [1, 2]$ with the subspace topology of $\mathbb{R}$, where C is the ternary Cantor set of $[0, 1]$, take a homeomorphism $f : X \to X$ as the identity of X. Since the identity map is an equicontinuous homeomorphism, we have that $\mathrm{Sh}_{per}(f)$ equals the set of totally disconnected points, so $\mathrm{Sh}_{per}(f) = C \setminus \{1\}$. Hence $\mathrm{Sh}_{per}(f)$ is an f-invariant, open subset of X.

Next, we give an example that there is a homeomorphism $f : X \to X$ of a compact metric space X such that $\mathrm{Sh}_{per}(f) = \emptyset$ but $\mathrm{Sh}(f) \neq \emptyset$.

Example 3.2. Let f be a minimal homeomorphism with a shadowable point of an infinite compact metric space X. From the minimality of an infinity space X, we see that $\mathrm{Per}(f) = \emptyset$, so $\mathrm{Sh}_{per}(f) = \emptyset$. Also we have $\mathrm{Sh}(f) \neq \emptyset$ by the assumption of f.

Also we give an example to illustrate our main results.

Example 3.3 ([1, Remark 3.1.9])**.** Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift homeomorphism. Note that the shift map σ is expansive with the shadowing property. Hence we have that $\mathrm{Sh}(\sigma) = \mathrm{Sh}_{per}(\sigma) = \Sigma_2$ by Theorem 1.1. Also, let $S = \{(x_i) \in \Sigma_2 : (x_i, x_{i+1}) \in C, i \in \mathbb{Z}\}$, where $C = \{(0, 0), (0, 1), (1, 1)\}$. Then $\sigma|_S : S \to S$ is a Markov subshift. Since $\sigma|_S$ is expansive with the shadowing property, $\mathrm{Sh}(\sigma|_S) = \mathrm{Sh}_{per}(\sigma|_S) = S$. So $\sigma|_S$ has the periodic shadowing property. Hence the periodic shadowing property for σ is equivalent to the periodic shadowing property for $\sigma|_S$.

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