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ON DELAY DIFFERENTIAL EQUATIONS WITH MEROMORPHIC SOLUTIONS OF HYPER-ORDER LESS THAN ONE

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Abstract. We consider the delay differential equations

$$b(z)w(z+1)+c(z)w(z-1)+a(z)\frac{w'(z)}{w^k(z)}=\frac{P(z,w(z))}{Q(z,w(z))},$$

where $k\in\{1,2\}$, $a(z)$, $b(z)\not\equiv 0$, $c(z)\not\equiv 0$ are rational functions, and $P(z,w(z))$ and $Q(z,w(z))$ are polynomials in $w(z)$ with rational coefficients satisfying certain natural conditions regarding their roots. It is shown that if this equation has a non-rational meromorphic solution w with hyper-order $\rho_2(w)<1$, then either $\deg_w(P)=\deg_w(Q)+1\leq 3$ or $\max\{\deg_w(P),\deg_w(Q)\}\leq 1$. In addition, it is shown that in the case $\max\{\deg_w(P),\deg_w(Q)\}=0$ the equations above can have such a solution, with an additional zero density requirement, only if the coefficients of the equation satisfy certain strict conditions.

1. Introduction

Ablowitz, Halburd and Herbst [1] have suggested that the existence of sufficiently many finite-order meromorphic solutions of a difference equation is a good indication that the equation in question is of Painlevé type. Further work in this direction have supported their hypothesis, see, e.g., [9,18] as well as the review papers [6,10] and the references therein. Halburd and one of us [11] have found necessary conditions for certain types of rational delay differential equations to admit a non-rational meromorphic solution of hyper-order less than one. The equations singled out by this method include a delay equation of Painlevé type and equations that can be explicitly solved by elliptic functions. For more recent studies applying Nevanlinna theory to delay differential equations, see, e.g., [3,4,15,19].

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Grammaticos, Ramani and Moreira [7] have examined Painlevé-type delay differential equations from the point of view of a version of singularity confinement. Viallet [5] has introduced a notion of algebraic entropy for such equations. Recently, Berntson [2] has considered elliptic and soliton-type solutions of examples of delay differential Painlevé equations, while Stokes [20] has conducted research on the geometric interpretation of singularity confinement phenomena in such equations.

We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [13, 14]. We recall the definitions of the order and the hyper-order for a meromorphic function w as follows:

$$\rho(w) = \limsup_{r\to\infty} \frac{\log T(r,w)}{\log r}, \quad \rho_2(w) = \limsup_{r\to\infty} \frac{\log\log T(r,w)}{\log r}.$$

Recently, Halburd and one of us [11] applied Nevanlinna theory to study delay differential equations and obtained the following theorem:

Theorem 1.1 ([11]). *Let $w(z)$ be a non-rational meromorphic solution of*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z,w(z)) = \frac{P(z,w(z))}{Q(z,w(z))}, \tag{1.1}$$

where $a(z)$ is rational, $P(z,w(z))$ is a polynomial in $w(z)$ having rational coefficients in z, and $Q(z,0)\not\equiv 0$ is a monic polynomial in $w(z)$ with roots that are rational in z and not roots of $P(z,w(z))$. If $\rho_2(w)<1$, then

$$\deg_w(P) = \deg_w(Q) + 1 \le 3$$

or the degree of $R(z,w(z))$ as a rational function in $w(z)$ is either 0 *or* 1.

The coefficients on the left hand side of equation (1.1) are selected to be of a specific form so that the equation contains the equation

$$w(z+1) - w(z-1) + a\frac{w'(z)}{w(z)} = b, \quad a,b\in\mathbb{C}, \tag{1.2}$$

obtained by Quispel, Capel and Sahadevan [17] as a symmetry reduction of the Kac-van Moerbeke equation. Note that if $a\neq 0$, then (1.2) can be mapped, using the transformation $w(z)=af(z)$, into

$$f(z+1) - f(z-1) + \frac{f'(z)}{f(z)} = C, \tag{1.3}$$

where $C=b/a$. Equation (1.2) is one of the few delay differential equations with a known continuum limit to a Painlevé equation. It is natural to ask how restrictive is the choice made in (1.1), and what happens if we consider a more general equation, for example

$$b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z,w(z)) = \frac{P(z,w(z))}{Q(z,w(z))}, \tag{1.4}$$

where, as in (1.1), we assume that $a(z), b(z), c(z)$ is rational, $P(z,w(z))$ is a polynomial in $w(z)$ having rational coefficients in z, and $Q(z,0)\not\equiv 0$ is a monic polynomial in $w(z)$ with roots that are rational in z and not roots of $P(z,w(z))$. In the special case, where one of $b(z)$, $c(z)$ vanishes identically, the equation (1.4) has been considered in [19]. In this paper, we consider the case $b(z)\not\equiv 0$, $c(z)\not\equiv 0$ and obtain the following theorem.

Theorem 1.2. *Let $w(z)$ be a non-rational meromorphic solution of equation (1.4). If $\rho_2(w)<1$, then*

$$\deg_w(P)=\deg_w(Q)+1\le 3 \quad \text{or} \quad \deg_w(R)\le 1. \tag{1.5}$$

If, in addition, $\deg_w(R)=3$, then $T(r,w)=\overline{N}(r,w)+S(r,w)$.

The proof of Theorem 1.2 in Section 3 below is a simplified version of the proof of Theorem 1.1 in [11]. Halburd and one of us [11] considered more carefully the special case, where $\deg_w(R(z,w))=0$ in (1.1):

Theorem 1.3 ([11])**.** *Let $w(z)$ be a non-rational meromorphic solution of*

$$w(z+1)-w(z-1)+a(z)\frac{w'(z)}{w(z)}=b(z), \tag{1.6}$$

where $a(z)\not\equiv 0$ and $b(z)$ are rational. If $\rho_2(w)<1$, and for any $\epsilon>0$

$$\overline{N}(r,\frac{1}{w})\ge(\frac{3}{4}+\epsilon)T(r,w)+S(r,w),$$

then the coefficients $a(z)$ and $b(z)$ are both constants.

Under the assumptions of Theorem 1.3 the equation (1.6) reduces exactly into equation (1.2) discovered by Quispel, Capel and Sahadevan. Similarly, we consider the case, where $R(z,w(z))=d(z)$ does not depend on $w(z)$, and the equation (1.4) becomes

$$a(z)w(z+1)+b(z)w(z-1)+c(z)\frac{w'(z)}{w(z)}=d(z), \tag{1.7}$$

where $a(z)\not\equiv 0$, $b(z)\not\equiv 0$, $c(z)$, $d(z)$ are rational. We obtain the following generalization of Theorem 1.3.

Theorem 1.4. *Let $w(z)$ be a non-rational meromorphic solution of equation (1.7), where $a(z)\not\equiv 0$, $b(z)\not\equiv 0$, $c(z)$, $d(z)$ are rational. If $\rho_2(w)<1$, and for any $\varepsilon>0$*

$$\overline{N}(r,\frac{1}{w})\ge(\frac{3}{4}+\varepsilon)T(r,w)+S(r,w), \tag{1.8}$$

then (1.7) reduces by a linear change in $w(z)$ into (1.3), where $f(z)=\frac{a(z-1)}{c(z-1)}w(z)$ and $C\in\mathbb{C}$.

Finally, we consider an equation outside the class (1.4).

Theorem 1.5. *Let $w(z)$ be a non-rational meromorphic solution of*

$$\alpha(z)w(z+1)+\beta(z)w(z-1)=\frac{a(z)w'(z)+b(z)w(z)}{w^2(z)}+c(z), \tag{1.9}$$

where $\alpha(z) \not\equiv 0$, $\beta(z) \not\equiv 0$, $a(z) \not\equiv 0$, $b(z)$, $c(z)$ *are rational. If* $\rho_2(w) < 1$, *and for any* $\varepsilon > 0$

$$\overline{N}(r, \frac{1}{w}) \geq (\frac{3}{4} + \varepsilon)T(r, w) + S(r, w), \tag{1.10}$$

then $c(z) \equiv 0$ *and*

$$\frac{-\beta(z+2)}{\alpha(z)} = \frac{\alpha(z+1)a(z+2) + \beta(z+2)a(z+1)}{\alpha(z)a(z+1) + \beta(z+1)a(z)}$$

and

$$\frac{b(z+2)}{a(z+2)} - \frac{b(z)}{a(z)} = \frac{a'(z)}{a(z)} - \frac{a'(z+2)}{a(z+2)} + \gamma(z),$$

where $\gamma(z) = \frac{\beta'(z+2)}{\beta(z+2)} - \frac{\alpha'(z)}{\alpha(z)}$.

The theorem above is a generalization of [11, Theorem 1.3], which is a special case of Theorem 1.5 corresponding to the choices $\alpha(z) = 1$ and $\beta(z) = -1$. In the final result we consider a version of the equation (1.9), where the right hand side is a rational function of $w(z)$ with rational coefficients.

Theorem 1.6. *Let* $w(z)$ *be a non-rational meromorphic solution of*

$$b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w^2(z)} = R(z, w(z)) = \frac{P(z, w(z))}{Q(z, w(z))}, \tag{1.11}$$

where $a(z)$ *is rational,* $P(z, w(z))$ *is a polynomial in* $w(z)$ *having rational coefficients in* z, *and* $Q(z, 0) \not\equiv 0$ *is a monic polynomial in* $w(z)$ *with roots that are rational in* z *and not roots of* $P(z, w(z))$. *If* $\rho_2(w) < 1$, *then*

$$\deg_w(P) = \deg_w(Q) + 1 \leq 3 \quad \textit{or} \quad \deg_w(R) \leq 1. \tag{1.12}$$

If, in addition, $\deg_w(R) = 3$, *then* $N\left(r, \frac{1}{w}\right) = S(r, w)$.

The remainder of the paper is organized as follows. Section 2 contains two auxiliary lemmas needed in the proofs of Theorems 1.2 and 1.4–1.6 in Sections 3–6. Section 3 contains the proof of Theorem 1.2, while Sections 4–6 present the proofs of Theorems 1.4–1.6.

2. Lemmas

The Valiron-Mohon'ko identity [16, 21] is a useful tool to estimate the characteristic function of rational functions. Its proof can be found, for example, in [14, Theorem 2.2.5].

Lemma 2.1 ([14], Theorem 2.2.5)**.** *Let* w *be a meromorphic function and* $R(z, w)$ *be a rational function in* w *and meromorphic in* z. *If the coefficients of* $R(z, w)$ *are small compared to* w, *then*

$$T(r, R(z, w)) = \deg_w(R)T(r, w) + S(r, w).$$

The following lemma, related to the value distribution of meromorphic solutions of a large class of differential-difference equations, is an important tool in this article. A differential-difference polynomial in $w(z)$ is defined by

$$P(z,w)=\sum_{l\in L} b_l(z)w(z)^{l_{0,0}}w(z+c_1)^{l_{1,0}}\cdots w(z+c_\nu)^{l_{\nu,0}}w'(z)^{l_{0,1}}\cdots w^{(\mu)}(z+c_\nu)^{l_{\nu,\mu}},$$

where $c_1,\dots,c_\nu$ are distinct complex constants, L is a finite index set consisting of elements of the form $l=(l_{0,0},\dots,l_{\nu,\mu})$ and the coefficients $b_l(z)$ are rational functions of z for all $l\in L$.

Lemma 2.2 ([11], Lemma 2.1)**.** *Let w be a non-rational meromorphic solution of*

$$P(z,w)=0, \tag{2.1}$$

where $P(z,w)$ is a differential-difference polynomial in $w(z)$ with rational coefficients, and let $a_1,\dots,a_k$ be rational functions. If the following two conditions

(1) *$P(z,a_j)\not\equiv 0$ for all $j\in\{i,\dots,k\}$;*
(2) *there exist $s>0$ and $\tau\in(0,1)$ such that*

$$\sum_{j=1}^{k} n\left(r,\frac{1}{w-a_j}\right)\le k\tau n(r+s,w)+O(1) \tag{2.2}$$

are satisfied, then $\rho_2(w)\ge 1$.

3. Proof of Theorem 1.2

Liu and Song [15, Remark 1.1] found a clever way to simplify the first part of the proof of [11, Theorem 1.1]. In the current proof of Theorem 1.2, we have introduced new ideas to further simplify the proof method of [11, Theorem 1.1].

The first step we prove is that $\deg_w(R)\le 3$. In the second step we discuss four cases which depend on the numbers of the roots of $Q(z,w)$. Suppose that (1.5) has a non-rational meromorphic solution $w(z)$ with $\rho_2(w)<1$.

First step: Since $w=0$ is not a pole of $R(z,w(z))$, we see that either $w(z)$ has finitely many zeros which are the zeros of $a(z)$ or $w(z)$ has infinite many zeros which are poles of $w(z+1)$ or $w(z-1)$ or both. Thus using [15, Remark 1.1] and [12, Lemma 8.3] we obtain

$$\begin{aligned}
&N\left(r,b(z)w(z+1)+c(z)w(z-1)+a(z)\frac{w'(z)}{w(z)}\right)\\
&\le N(r,w(z+1))+N(r,w(z-1))+\overline{N}(r,w(z))+S(r,w)\\
&\le 2N(r,w(z))+\overline{N}(r,w(z))+S(r,w).
\end{aligned}$$

From using Lemma 2.1, the logarithmic derivative lemma and its difference analogue, it follows that

$$\deg_w(R(z,w(z)))T(r,w(z))\le T\left(r,b(z)w(z+1)+c(z)w(z-1)+a(z)\frac{w'(z)}{w(z)}\right)$$

$$\leq N\left(r, b(z)w(z+1)+c(z)w(z-1)+a(z)\frac{w'(z)}{w(z)}\right)+m(r,w(z))+S(r,w)$$
$$\leq 2N(r,w(z))+\overline{N}(r,w(z))+m(r,w(z))+S(r,w)$$
$$\leq 2T(r,w(z))+\overline{N}(r,w(z))+S(r,w).$$

Therefore,

$$(\deg_w(R(z,w(z)))-2)T(r,w(z))\leq \overline{N}(r,w(z))+S(r,w),$$

which implies that $\deg_w R(z)\leq 3$, i.e., $\deg_w(P)\leq 3$, and $\deg_w(Q)\leq 3$. Also, if $\deg_w R(z)=3$, it follows that $T(r,w)=\overline{N}(r,w)+S(r,w)$.

Second step: Case 1. If $Q(z,w(z))$ in (1.4) has at least two distinct non-zero rational roots for w, say $d_1(z)\not\equiv 0$ and $d_2(z)\not\equiv 0$, then (1.4) can be written as

$$\begin{aligned} &b(z)w(z+1)+c(z)w(z-1)+a(z)\frac{w'(z)}{w(z)} \\ &=\frac{P(z,w(z))}{(w(z)-d_1(z))(w(z)-d_2(z))\tilde{Q}(z,w(z))}, \end{aligned} \tag{3.1}$$

where $\deg_w(P)\leq 3$ and $\deg_w(\tilde{Q})\leq 1$. Here, there exists the possibility that $\tilde{Q}(z,d_1(z))\equiv 0$ or $\tilde{Q}(z,d_2(z))\equiv 0$. We also assume that $P(z,w(z))$ and $\tilde{Q}(z,w(z))$ do not have common roots. Since $P(z,d_j)\not\equiv 0$ for $j=1,2$, neither $d_1(z)$ nor $d_2(z)$ is a solution of (3.1), and thus the first condition of Lemma 2.2 is satisfied.

Assume that $\hat{z}\in\mathbb{C}$ is any point satisfying

$$w(\hat{z})=d_1(\hat{z}), \tag{3.2}$$

and such that none of the rational coefficients of (3.1) and their shifts have a zero or a pole at $\hat{z}$ and $P(\hat{z},w(\hat{z}))\neq 0$. Let p denote the order of the zero of $w-d_1$ at $z=\hat{z}$. Here, $\hat{z}$ is called a generic root of $w-d_1$ of order p.

We will only consider generic roots from now on. Since the coefficients are rational, the contributions from the non-generic roots can always be included in an error term of the type $O(\log r)$. Next we discuss whether $z=\hat{z}$ is a zero or a pole of $w(z+n)$ $(n=1,2,3)$ or not.

Now, by (3.1), it follows that $w(\hat{z}+1)=\infty$ or $w(\hat{z}-1)=\infty$ and the order is at least p. Without loss of generality we may assume that $w(\hat{z}+1)=\infty$. Then, by shifting the equation (3.1), we have

$$\begin{aligned} &b(z+1)w(z+2)+c(z+1)w(z)+a(z+1)\frac{w'(z+1)}{w(z+1)} \\ &=\frac{P(z+1,w(z+1))}{(w(z+1)-d_1(z+1))(w(z+1)-d_2(z+1))\tilde{Q}(z+1,w(z+1))}. \end{aligned} \tag{3.3}$$

Subcase 1.1. Let

$$\deg_w(P)\leq \deg_w(\tilde{Q})+2.$$

Now, by (3.3), $\hat{z}+2$ is a pole of $w(z)$ with order one. Therefore, for any $p \geq 1$ there is a pole of order at least p at $z = \hat{z}+1$, which can be paired up with the root of $w - d_1$ at $z = \hat{z}$.

Using the same discussions for the roots of $w - d_2$ without the possible overlap in the pairing of poles with the zeros of $w - d_1$ and $w - d_2$, by adding up all points $\hat{z}$ such that (3.2) is valid, and similarly for $w(\hat{z}) = d_2(\hat{z})$, it follows that

$$n\left(r, \frac{1}{w-d_1}\right) + n\left(r, \frac{1}{w-d_2}\right) \leq n(r+1, w) + O(1).$$

Therefore the second condition (2.2) of Lemma 2.2 is satisfied, and so $\rho_2(w) \geq 1$, which is a contradiction with $\rho_2(w) < 1$.

Subcase 1.2. Let

$$\deg_w(P) > \deg_w(\tilde{Q}) + 2.$$

Since $\deg_w(P) \leq 3$, then $\deg_w(P) = 3$, and it immediately follows that $\deg_w(Q) = 2$. Thus the assertion (1.5) holds in this case.

Case 2. Suppose that $Q(z, w(z))$ in (1.4) has at least one non-zero rational root, say $d_1(z) \not\equiv 0$. Then (1.4) can be written as

$$b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z, w(z))}{(w(z) - d_1(z))^n \check{Q}(z, w(z))}, \tag{3.4}$$

where $\deg_w(P) \leq 3$ and $n + l \leq 3$, $\deg_w(\check{Q}) = l$. Then $d_1(z)$ is not a solution of (3.4), and thus the first condition of Lemma 2.2 is satisfied for d_1. We assume that $n \in \{2, 3\}$ and consider the case $n = 1$ as a part of Case 3 below. Suppose that $\hat{z}$ is a generic root of $w(z) - d_1(z)$ of order p. Next without loss of generality suppose that $\hat{z}+1$ is a pole of $w(z)$ with order np at least.

Subcase 2.1. Let

$$\deg_w(P) \leq n + l.$$

Then $\hat{z}+2$ is a pole of $w(z)$ with order one, and $\hat{z}+3$ is a pole of $w(z)$ with order np, at least. By continuing the iteration, it yields three possible cases as follows:

(a) $w(\hat{z}+4) = \infty$;
(b) $w(\hat{z}+4) \neq \infty$ and $w(\hat{z}+4) \neq d_1(\hat{z}+4)$;
(c) $w(\hat{z}+4) = d_1(\hat{z}+4)$.

If the case (a) or (b) is valid, then $\hat{z}+5$ is a pole of $w(z)$ with order np, and we have even more poles for every root of $w - d_1$. For the case (c), it is at least in principle possible that $w(\hat{z}+5)$ is a finite value. By adding up the contribution from all points $\hat{z}$ to corresponding counting functions, it follows that

$$n\left(r, \frac{1}{w-d_1}\right) \leq \frac{1}{n} n(r+4, w) + O(1).$$

Thus both conditions of Lemma 2.2 are satisfied, and so $\rho_2(w) \geq 1$.

Subcase 2.2. Let

$$\deg_w(P) \geq n + l + 1.$$

Suppose again that $\hat{z}$ is a generic root of $w(z) - d_1(z)$ of order p. Similarly as before, say $\hat{z}+1$ is a pole of $w(z)$ with order np at least. This implies that $\hat{z}+2$ is a pole of $w(z)$ with order np at least, and so, the only way that $w(\hat{z}+4)$ can be finite is that $w(\hat{z}+3) = d_1(\hat{z}+3)$, or $w(\hat{z}+3)$ is a root of $\check{Q}(z, w(z))$, with multiplicity p. In this case, we have

$$n\left(r, \frac{1}{w - d_1}\right) \leq \frac{1}{n} n(r + 3, w) + O(1)$$

by going through all roots of $w - d_1$ in this way. Lemma 2.2 thus implies that $\rho_2(w) \geq 1$.

Case 3. Suppose now that $Q(z, w)$ in the equation (1.4) has only one simple root, say $d_1(z) \not\equiv 0$. Then (1.4) can be written as

$$b(z)w(z + 1) + c(z)w(z - 1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z, w(z))}{w(z) - d_1(z)}.$$

Subcase 3.1. Assume first that

$$\deg_w(P) = 3.$$

Let $\hat{z}$ be a generic root of $w(z) - d_1(z)$ of order p. Similarly as before, say $\hat{z}+1$ is a pole of $w(z)$ with order p. Then $\hat{z} + 2$ is a pole of $w(z)$ with order $2p$ at least, and $\hat{z} + 3$ is a pole of $w(z)$ with order $4p$, and so on. We have

$$n\left(r, \frac{1}{w - d_1}\right) \leq \frac{1}{3} n(r + 2, w) + O(1).$$

Lemma 2.2 thus implies that $\rho_2(w) \geq 1$.

Subcase 3.2. Assume that

$$\deg_w(P) \leq 2.$$

If $\deg_w(P) = 2$, then $\deg_w(P) = \deg_w(Q) + 1$ and thus the assertion (1.5) holds. If $\deg_w(P) \leq 1$, then $\deg_w(R) = 1$.

Case 4. $R(z, w(z))$ is a polynomial in $w(z)$. Then (1.4) takes the form

$$b(z)w(z + 1) + c(z)w(z - 1) + a(z)\frac{w'(z)}{w(z)} = P(z, w(z)), \tag{3.5}$$

where $\deg_w(P) \leq 3$. If $\deg_w(P) \leq 1$, then $\deg_w(R) \leq 1$.

Assume therefore that

$$\deg_w(P) \geq 2,$$

and suppose first that $w(z)$ has infinitely many poles. Next by applying the reasoning in the proof of [11, Lemma 3.2], we get $\rho_2(w) \geq \lambda_2(\frac{1}{w}) \geq 1$.

Suppose now that $w(z)$ has finitely many poles, and that $\rho_2(w) < 1$. In this case, from (3.5), we get

$$b(z)w(z)w(z + 1) + c(z)w(z)w(z - 1) + a(z)w'(z) = P(z, w(z))w(z). \tag{3.6}$$

Since $\deg_w(P) \geq 2$, using the difference analogue of Clunie Lemma [8] with [12, Remark 5.3], $m(r,w) = S(r,w)$, so $T(r,w) = S(r,w)$, which is a contradiction. The proof of Theorem 1.2 is completed.

4. Proof of Theorem 1.4

First let's rewrite equation (1.7) as

$$\alpha(z)w(z+1) + \beta(z)w(z-1) + \frac{w'(z)}{w(z)} = \gamma(z), \tag{4.1}$$

where $\alpha(z) = \frac{a(z)}{c(z)} \not\equiv 0$, $\beta(z) = \frac{b(z)}{c(z)} \not\equiv 0$, $\gamma(z) = \frac{d(z)}{c(z)}$ are rational.

By (1.8) and by the assumption that $w(z)$ is non-rational, it follows that $w(z)$ has infinitely many zeros. Since $\gamma(z)$ is rational, next we only consider the case that $\hat{z}$ is a generic zero of $w(z)$. We need to consider two cases.

Case 1. Suppose first that $w(\hat{z}+1) = \infty$ and $w(\hat{z}-1) = \infty$. Then from (4.1) it follows that $w(\hat{z}+2) = \infty$ and $w(\hat{z}-2) = \infty$. Now, at least in principle we may have $w(\hat{z}-3) = 0 = w(\hat{z}+3)$. Hence, in this case we can find at least four poles of $w(z)$ (ignoring multiplicity) which correspond to three zeros (also ignoring multiplicity) of $w(z)$ and to no other zeros.

Case 2. Assume now that $w(\hat{z}+1) = \infty$ or $w(\hat{z}-1) = \infty$. Without loss of generality we can then suppose that $w(\hat{z}+1) = \infty$ (the case $w(\hat{z}-1) = \infty$ is completely analogous). We will begin by showing that we need only consider simple generic zeros of $w(z)$. Let $N_1\left(r, \frac{1}{w}\right)$ denote the integrated counting function for the simple zeros of w and let $N_{[p}\left(r, \frac{1}{w}\right)$ be the counting function for the zeros of w, which are of order p or higher. Then $N(r, \frac{1}{w}) = N_1\left(r, \frac{1}{w}\right) + N_{[2}\left(r, \frac{1}{w}\right)$ and

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{w}\right) &= N_1\left(r, \frac{1}{w}\right) + \overline{N}_{[2}\left(r, \frac{1}{w}\right) \\
&\leq N_1\left(r, \frac{1}{w}\right) + \frac{1}{2}N_{[2}\left(r, \frac{1}{w}\right) \\
&\leq \frac{1}{2}N_1\left(r, \frac{1}{w}\right) + \frac{1}{2}N\left(r, \frac{1}{w}\right).
\end{aligned}$$

Hence, using the assumption (1.8),

$$\begin{aligned}
N_1\left(r, \frac{1}{w}\right) &\geq 2\overline{N}\left(r, \frac{1}{w}\right) - N\left(r, \frac{1}{w}\right) \\
&\geq \left(\frac{3}{2} + \varepsilon\right) T(r,w) - N\left(r, \frac{1}{w}\right) \\
&\geq \left(\frac{1}{2} + \varepsilon\right) T(r,w) + S(r,w).
\end{aligned}$$

Thus there are at least "$\left(\frac{1}{2}+\varepsilon\right)T(r,w)$" worth of simple zeros of w. So we consider the case in which the zeros of w at $\hat{z}$ are simple, and we have

$$
\begin{aligned}
w(z-1) &= K + O(z-\hat{z}), \quad K\in\mathbb{C},\\
w(z) &= A(z-\hat{z}) + O((z-\hat{z})^2), \quad A\in\mathbb{C}\backslash\{0\},\\
\alpha(z)w(z+1) &= -\frac{1}{z-\hat{z}} + O(1),\\
\alpha(z+1)w(z+2) &= \frac{1}{z-\hat{z}} + O(1),\\
\alpha(z+2)w(z+3) &= \frac{\alpha(z)+\beta(z+2)}{\alpha(z)}\cdot\frac{1}{z-\hat{z}} + O(z-\hat{z})
\end{aligned}
\tag{4.2}
$$

in a neighborhood of $\hat{z}$.

If $\alpha(\hat{z})+\beta(\hat{z}+2)\neq 0$, then

$$\alpha(z+3)w(z+4) = \frac{\alpha(z+1)-\beta(z+3)}{\alpha(z+1)}\cdot\frac{1}{z-\hat{z}} + O(z-\hat{z}).$$

Therefore either we have infinitely many points such that $\alpha(z)=-\beta(z+2)$ or we can find at least four poles of $w(z)$ for every two simple zeros of $w(z)$, if $w(z+4)=\infty$. Even if $w(z+4)=0$, there are three poles of $w(z)$ for every two simple zeros of $w(z)$, and then, either way,

$$\overline{n}\left(r,\frac{1}{w}\right)\leq\frac{2}{3}n(r+1,w)+O(1).$$

Hence, for any $\varepsilon>0$,

$$\overline{N}\left(r,\frac{1}{w}\right)\leq\left(\frac{2}{3}+\frac{\varepsilon}{2}\right)N(r+1,w)+O(\log r),$$

and so by using [12, Lemma 8.3] to deduce that $N(r+1,w)=N(r,w)+S(r,w)$, we have

$$\overline{N}\left(r,\frac{1}{w}\right)\leq\left(\frac{2}{3}+\frac{\varepsilon}{2}\right)T(r,w)+S(r,w).$$

This is a contradiction with the assumption (1.8).

So

$$\beta(z+2) = -\alpha(z). \tag{4.3}$$

By substituting (4.3) into (1.7), it follows that

$$\alpha(z)w(z+1)-\alpha(z-2)w(z-1)+\frac{w'(z)}{w(z)}=\gamma(z). \tag{4.4}$$

Letting $f(z)=\alpha(z-1)w(z)$, then (4.4) can be written

$$f(z+1)-f(z-1)+\frac{f'(z)}{f(z)}=\gamma(z)+\frac{\alpha'(z-1)}{\alpha(z-1)}.$$

By using Theorem 1.3, we get $\gamma(z)+\frac{\alpha'(z-1)}{\alpha(z-1)}=C$ $(C\in\mathbb{C})$.

5. Proof of Theorem 1.5

By (1.10) and by the assumption that $w(z)$ is non-rational, it follows that $w(z)$ has infinitely many zeros. Since $a(\hat{z})$, $b(\hat{z})$ and $c(\hat{z})$ are rational, it is sufficient to just think about the case, where $z=\hat{z}$ is a generic zero of $w(z)$ of order p. Then by (1.9) there is a pole of $w(z)$ of order $p+1$, at least, at $z=\hat{z}+1$ or at $z=\hat{z}-1$ (or at both points). We need to consider two cases.

Case 1. Assume now that $w(\hat{z}+1)=\infty$ or $w(\hat{z}-1)=\infty$. Without loss of generality we can then suppose that $w(\hat{z}+1)=\infty$.

Subcase 1.1. The zero is simple, and suppose that $c(z)\not\equiv 0$. Then, in a neighborhood of $\hat{z}$,

$$\begin{aligned} & w(z-1)=K+O(z-\hat{z}), \quad K\in\mathbb{C}, \\ & w(z)=\delta(z-\hat{z})+O((z-\hat{z})^2), \quad \delta\in\mathbb{C}\backslash\{0\}, \\ & \alpha(z)w(z+1)=\frac{a(z)}{\delta(z-\hat{z})^2}+\frac{b(z)}{\delta(z-\hat{z})}+c(z)-K\beta(z)+O(z-\hat{z}), \\ & \alpha(z+1)w(z+2)=c(z+1)+O(z-\hat{z}), \\ & \alpha(z+2)w(z+3)=\frac{-\beta(z+2)}{\alpha(z)}\left(\frac{a(z)}{\delta(z-\hat{z})^2}+\frac{b(z)}{\delta(z-\hat{z})}\right)+O(1), \end{aligned} \tag{5.1}$$

where there can be at most finitely many $\hat{z}$ such that $c(\hat{z}+1)=0$. Hence there are two poles of $w(z)$ (counting multiplicity) corresponding to one zero (counting multiplicity) in this case.

Assume now that $c(z)\equiv 0$, $w(z)$ has a pole at $z=\hat{z}+1$, and that $w(\hat{z}-1)$ is finite. Then, in a neighborhood of $\hat{z}$,

$$\begin{aligned} & w(z-1)=K+O(z-\hat{z}), \quad K\in\mathbb{C}, \\ & w(z)=\delta(z-\hat{z})+O((z-\hat{z})^2), \quad \delta\in\mathbb{C}\backslash\{0\}, \\ & \alpha(z)w(z+1)=\frac{a(z)}{\delta(z-\hat{z})^2}+\frac{b(z)}{\delta(z-\hat{z})}+O(1), \\ & \alpha(z+1)w(z+2)=\left(-\beta(z+1)-\frac{2\alpha(z)a(z+1)}{a(z)}\right)\delta(z-\hat{z}) \\ & \qquad\qquad +O((z-\hat{z})^2), \\ & \alpha(z+2)w(z+3)=\frac{A(z)}{\delta(z-\hat{z})^2}+\frac{B(z)}{\delta(z-\hat{z})}+O(1), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} A(z)= & \frac{-\beta(z+2)a(z)}{\alpha(z)}+\frac{\alpha(z+1)a(z)a(z+2)}{-\beta(z+1)a(z)-2\alpha(z)a(z+1)}, \\ B(z)= & \frac{-\beta(z+2)D(z)b(z)+\alpha(z)\alpha(z+1)a(z)b(z+2)}{\alpha(z)D(z)} \\ & +\frac{a(z+2)(\alpha(z+1)D'(z)a(z)-D(z)(\alpha'(z+1)a(z)+\alpha(z+1)a'(z)))}{D^2(z)}, \end{aligned}$$

$D(z) = -\beta(z+1)a(z) - 2\alpha(z)a(z+1).$

From (5.2), we find if $A(z) \not\equiv 0$, there are at least four poles (counting multiplicity) with the two zeros of $w(z)$ (counting multiplicity). If

$$A(z) = \frac{-\beta(z+2)a(z)}{\alpha(z)} + \frac{\alpha(z+1)a(z)a(z+2)}{-\beta(z+1)a(z) - 2\alpha(z)a(z+1)} = 0 \tag{5.3}$$

and $B(z) \not\equiv 0$, from equation (1.9) it follows that

$$\alpha(z+3)w(z+4) = -\frac{\alpha(z+2)\delta a(z+3)}{B(z)} + O(z-\hat{z})$$

for all z in a neighborhood of $\hat{z}$, and so $w(\hat{z}+4)$ is finite and non-zero with at most finitely many exceptions. Thus we can group together three poles of $w(z)$ (counting multiplicity) and two zeros of $w(z)$ (ignoring multiplicity). If $A(z) \equiv 0$ and $B(z) \equiv 0$, then $w(\hat{z}+3)$ can be finite.

Subcase 1.2. If the order of the zero of $w(z)$ at $z = \hat{z}$ is $p \geq 2$, then there are always at least three poles of $w(z)$ (counting multiplicity) for each two zeros of $w(z)$ (ignoring multiplicity) in sequences (5.1) and (5.2).

If there are only finitely many zeros $\hat{z}$ of $w(z)$ such that $A(z) \equiv 0$ and $B(z) \equiv 0$ both hold, then

$$\overline{n}\left(r, \frac{1}{w}\right) \leq \frac{2}{3} n(r+1, w) + O(1).$$

Hence, for any $\varepsilon > 0$,

$$\overline{N}\left(r, \frac{1}{w}\right) \leq \left(\frac{2}{3} + \frac{\varepsilon}{2}\right) N(r+1, w) + O(\log r),$$

and so by using [12, Lemma 8.3] to deduce that $N(r+1, w) = N(r, w) + S(r, w)$, we have

$$\overline{N}\left(r, \frac{1}{w}\right) \leq \left(\frac{2}{3} + \frac{\varepsilon}{2}\right) T(r, w) + S(r, w).$$

This is in contradiction with (1.10), and so there must be infinitely many points $\hat{z}$ such that $A(z) \equiv 0$ and $B(z) \equiv 0$ are both satisfied.

By $A(z) \equiv 0$, we get

$$\frac{-\beta(z+2)}{\alpha(z)} = \frac{\alpha(z+1)a(z+2) + \beta(z+2)a(z+1)}{\alpha(z)a(z+1) + \beta(z+1)a(z)}$$

and by $B(z) \equiv 0$, it follows that

$$\frac{b(z+2)}{a(z+2)} - \frac{b(z)}{a(z)} = \frac{a'(z)}{a(z)} - \frac{a'(z+2)}{a(z+2)} + \gamma(z),$$

where $\gamma(z) = \frac{\beta'(z+2)}{\beta(z+2)} - \frac{\alpha'(z)}{\alpha(z)}$.

Case 2. Suppose that $w(\hat{z}+1) = \infty$ and $w(\hat{z}-1) = \infty$. Then, even if $w(\hat{z}+2) = 0$ and $w(\hat{z}-2) = 0$, we can group together three zeros of $w(z)$ (ignoring multiplicity) with at least four poles of $w(z)$ (counting multiplicity).

6. Proof of Theorem 1.6

In the proof, as the first step we prove that $\deg_w(R) \leq 3$. In the second step we discuss four cases depending on the numbers of the roots of $Q(z,w)$. Suppose that (1.11) has a non-rational meromorphic solution $w(z)$ with $\rho_2(w) < 1$.

First step: Since $w = 0$ is not a pole of $R(z, w(z))$, we see that either $w(z)$ has finitely many zeros which are the zeros of $a(z)$ or $w(z)$ has infinite many zeros which are poles of $w(z+1)$ or $w(z-1)$ or both. Next similarly to Case 1 in the proof of Theorem 1.2, by using [12, Lemma 8.3], Lemma 2.1, the logarithmic derivative lemma and its difference analogue, it follows that

$$\begin{aligned}
&\deg_w(R(z,w(z)))T(r,w(z)) \\
&\leq T\left(r, b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w^2(z)}\right) \\
&\leq 2N(r,w(z)) + m(r,w) + m\left(r, \frac{1}{w(z)}\right) + S(r,w) \\
&\leq 2T(r,w(z)) + m\left(r, \frac{1}{w(z)}\right) + S(r,w).
\end{aligned}$$

Therefore,

$$(\deg_w(R(z,w(z))) - 2)T(r,w(z)) \leq m\left(r, \frac{1}{w(z)}\right) + S(r,w),$$

which implies that $\deg_w R(z) \leq 3$, i.e., $\deg_w(P) \leq 3$, and $\deg_w(Q) \leq 3$. Furthermore, if $\deg_w(R) = 3$, we have $N\left(r, \frac{1}{w}\right) = S(r,w)$.

Second step: Case 1. If $Q(z, w(z))$ in (1.11) has at least two distinct non-zero rational roots for w, say $d_1(z) \not\equiv 0$ and $d_2(z) \not\equiv 0$, then (1.11) can be written as

$$\begin{aligned}
&b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w^2(z)} \\
&= \frac{P(z,w(z))}{(w(z)-d_1(z))(w(z)-d_2(z))\tilde{Q}(z,w(z))},
\end{aligned} \tag{6.1}$$

where $\deg_w(P) \leq 3$ and $\deg_w(\tilde{Q}) \leq 1$. Here, there exists the possibility that $\tilde{Q}(z, d_1(z)) \equiv 0$ or $\tilde{Q}(z, d_2(z)) \equiv 0$. We also assume that $P(z,w(z))$ and $\tilde{Q}(z,w(z))$ do not have common roots. Then neither $d_1(z)$ nor $d_2(z)$ is a solution of (6.1), and so they satisfy the first condition of Lemma 2.2.

Assume that $\hat{z} \in \mathbb{C}$ is a generic root of $w - d_1$ of order p, where the generic root has been defined in the proof of Theorem 1.2. Similarly to Case 1 in the proof of Theorem 1.2, next we discuss whether $z = \hat{z}$ is a zero or a pole of $w(z+n)$ $(n = 1,2,3)$ or not.

Now, by (6.1), it follows that $w(z+1)$ or $w(z-1)$ has a pole at $z = \hat{z}$ of order at least p. Without loss of generality we may assume that $w(z+1)$ has such a pole at $\hat{z}$.

Subcase 1.1. Let

$$\deg_w(P) \le \deg_w(\tilde{Q}) + 2. \tag{6.2}$$

If $p > 1$, we obtain

$$\begin{aligned} w(z) &= d_1(z) + \alpha_1(z-\hat{z})^p + O((z-\hat{z})^{p+1}),\\ w(z+1) &= \frac{\alpha_2}{(z-\hat{z})^p} + O((z-\hat{z})^{1-p}),\\ w(z+2) &= -\frac{c(z+1)d_1(z)}{b(z+1)} + \frac{pa(z+1)}{\alpha_2 b(z+1)}\cdot(z-\hat{z})^{p-1}\\ &\quad - \frac{\alpha_1 c(z+1)}{b(z+1)}(z-\hat{z})^p + O((z-\hat{z})^{p+1}),\\ w(z+3) &= -\frac{\alpha_2 c(z+2)}{b(z+2)}\cdot\frac{1}{(z-\hat{z})^p} + O((z-\hat{z})^{1-p}),\\ w(z+4) &= \frac{c(z+3)c(z+1)}{b(z+3)b(z+1)}\cdot d_1(z) + O((z-\hat{z})^{p-1}), \end{aligned} \tag{6.3}$$

where α_j $(j=1,2)$ are non-zero constants. From (6.3), it may be that $w(\hat{z}+2) = d_j(\hat{z}+2)$ and $w(\hat{z}+4) = d_j(\hat{z}+4)$ $(j=1,2)$, both with the order $p-1$. In addition, we have $w(\hat{z}+5)=\infty$, with the order p. This is the scenario, where there are the least number of poles for the biggest number of roots of $w-d_j$ $(j=1,2)$. Namely, if $w(\hat{z}+2)\neq d_j(\hat{z}+2)$ and $w(\hat{z}+4)\neq d_j(\hat{z}+4)$ $(j=1,2)$, then we have even more poles for every root of $w-d_1$. Identical reasoning holds also for the roots of $w-d_2$. Hence in this case, we have

$$n\left(r, \frac{1}{w-d_1}\right) + n\left(r, \frac{1}{w-d_2}\right) \le n(r+3, w) + O(1). \tag{6.4}$$

If $p=1$, we have

$$\begin{aligned} w(z) &= d_1(z) + \alpha_1(z-\hat{z}) + O((z-\hat{z})^2),\\ w(z+1) &= \frac{\alpha_2}{(z-\hat{z})} + O(1),\\ w(z+2) &= -\frac{c(z+1)}{b(z+1)}\cdot d_1(z) + \frac{a(z+1)}{\alpha_2 b(z+1)} - \frac{\alpha_1 c(z+1)}{b(z+1)}(z-\hat{z}) + O((z-\hat{z})^2),\\ w(z+3) &= -\frac{\alpha_2 c(z+2)}{b(z+2)}\cdot\frac{1}{(z-\hat{z})} + O(1),\\ w(z+4) &= \frac{c(z+3)c(z+1)}{b(z+3)b(z+1)}\cdot d_1(z) + \frac{a(z+3)b(z+2)}{\alpha_1 c(z+2)b(z+3)} - \frac{a(z+1)c(z+3)}{\alpha_2 b(z+1)b(z+3)}\\ &\quad + O(z-\hat{z}), \end{aligned}$$

where α_j $(j=1,2)$ are non-zero constants. Similarly as in the case $p>1$, we can still get

$$n\left(r, \frac{1}{w-d_1}\right) + n\left(r, \frac{1}{w-d_2}\right) \le n(r+3, w) + O(1). \tag{6.5}$$

Hence, the second condition of Lemma 2.2 is satisfied again, which yields $\rho_2(w) \geq 1$.

Subcase 1.2. Let

$$\deg_w(P) > \deg_w(\tilde{Q}) + 2.$$

If $\deg_w(P) = 3$, it immediately follows that $\deg_w(Q) = 2$, and so the assertion (1.12) holds in this case.

Case 2. Suppose that $Q(z, w(z))$ in (1.11) has at least one multiple non-zero rational root, say $d_1(z) \not\equiv 0$. Then (1.11) can be written as

$$b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w^2(z)} = \frac{P(z, w(z))}{(w(z) - d_1(z))^n \check{Q}(z, w(z))}, \tag{6.6}$$

where $\deg_w(P) \leq 3$ and $n + l \leq 3$, $\deg_w(\check{Q}) = l$. Then $d_1(z)$ is not a solution of (6.6), and thus the first condition of Lemma 2.2 is satisfied for d_1. Now we have that $n \in \{2, 3\}$, and we moreover suppose that $\hat{z}$ is a generic root of $w(z) - d_1(z)$ of order p. Then either $w(z+1)$ or $w(z-1)$ has a pole order np at least, at $z = \hat{z}$, and we suppose without loss of generality that $w(\hat{z}+1) = \infty$ is such a pole.

Subcase 2.1. Let

$$\deg_w(P) \leq n + l.$$

Since $n > 1$, we always have $np > 1$, and so

$$\begin{aligned}
w(z) &= d_1(z) + \alpha_1(z - \hat{z})^p + O((z - \hat{z})^{p+1}), \\
w(z+1) &= \frac{\alpha_2}{(z - \hat{z})^{np}} + O(1), \\
w(z+2) &= -\frac{c(z+1)d_1(z)}{b(z+1)} + \frac{npa(z+1)}{\alpha_2 b(z+1)} \cdot (z - \hat{z})^{np-1} \\
&\quad - \frac{\alpha_1 c(z+1)}{b(z+1)}(z - \hat{z})^p + O((z - \hat{z})^{p+1}), \\
w(z+3) &= -\frac{\alpha_2 c(z+2)}{b(z+2)} \cdot \frac{1}{(z - \hat{z})^{np}} + O(1), \\
w(z+4) &= \frac{c(z+3)c(z+1)}{b(z+3)b(z+1)} \cdot d_1(z) + O((z - \hat{z})^p),
\end{aligned} \tag{6.7}$$

where α_j $(j = 1, 2)$ are non-zero constants. From (6.7), it follows that we cannot (at least immediately) rule out the possibility that $w(\hat{z}+2) = d_1(\hat{z}+2)$ and $w(\hat{z}+4) = d_1(\hat{z}+4)$, both with order at most p. It also follows that $w(\hat{z}+5) = \infty$, with order np. This is the case, where the amount of roots of $w - d_1$ is maximal compared to the number of poles of w. Hence in this case, we have

$$n\left(r, \frac{1}{w - d_1}\right) \leq \frac{1}{n} n(r+3, w) + O(1). \tag{6.8}$$

Hence, the second condition of Lemma 2.2 is satisfied again, which yields $\rho_2(w) \geq 1$.

Subcase 2.2. Let

$$\deg_w(P) \geq n + l + 1.$$

This case is exactly the same as Subcase 2.2 in the proof of Theorem 1.2, so we get $\rho_2(w) \geq 1$.

Case 3. Suppose now that $Q(z,w)$ in the equation (1.11) has only one simple root, say $d_1(z) \not\equiv 0$. Then (1.11) can be written as

$$b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w^2(z)} = \frac{P(z,w(z))}{w(z) - d_1(z)}.$$

Subcase 3.1. Assume first that

$$\deg_w(P) = 3.$$

This case is the same as Subcase 3.1 in the proof of Theorem 1.2, so $\rho_2(w) \geq 1$.

Subcase 3.2. Assume that

$$\deg_w(P) \leq 2.$$

If $\deg_w(P) = 2$, then $\deg_w(P) = \deg_w(Q) + 1$ and thus the assertion (1.12) holds. If $\deg_w(P) \leq 1$, then $\deg_w(R) = 1$.

Case 4. The final remaining case is the one, where $R(z,w(z))$ is a polynomial in $w(z)$. Then (1.11) takes the form

$$b(z)w(z+1) + c(z)w(z-1) + a(z)\frac{w'(z)}{w^2(z)} = P(z,w(z)), \tag{6.9}$$

where $\deg_w(P) \leq 3$. If $\deg_w(P) \leq 1$, then $\deg_w(R) \leq 1$.

Assume therefore that

$$\deg_w(P) \geq 2,$$

and suppose first that $w(z)$ has infinitely many poles. By applying the reasoning in the proof of [11, Lemma 3.2], we get $\rho_2(w) \geq 1$.

Suppose now that $w(z)$ has finitely many poles, and that $\rho_2(w) < 1$. In this case, from (6.9), we get

$$b(z)w^2(z)w(z+1) + c(z)w^2(z)w(z-1) + a(z)w'(z) = P(z,w(z))w^2(z). \tag{6.10}$$

Since $\deg_w(P) \geq 2$, using the difference analogue of Clunie Lemma [8] with [12, Remark 5.3] implies $m(r,w) = S(r,w)$, so $T(r,w) = S(r,w)$, which is a contradiction. The proof of Theorem 1.6 is completed.

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