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TOPOLOGICAL SENSITIVITY AND ITS STRONGER FORMS ON SEMIFLOWS

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Abstract. In this paper we introduce and study the notions of topological sensitivity and its stronger forms on semiflows and on product semiflows. We give a relationship between multi-topological sensitivity and thick topological sensitivity on semiflows. We prove that for a Urysohn space X, a syndetically transitive semiflow (T, X, π) having a point of proper compact orbit is syndetic topologically sensitive. Moreover, it is proved that for a T_3 space X, a transitive, nonminimal semiflow (T, X, π) having a dense set of almost periodic points is syndetic topologically sensitive. Also, wherever necessary examples/counterexamples are given.

1. Introduction

Chaos is an important concept when the complexity of a dynamical system is studied. Sensitivity, first introduced by Ruelle and Takens, is a metric dependent property and is an important ingredient in the study of almost all types of chaos [18]. It tells us about the unpredictability of a dynamical system. Roughly speaking, it says that points in any neighborhood of a point can be significantly separated by a certain constant. A dynamical system may be sensitive for some period of time, therefore when we talk about chaotic phenomena it becomes important to measure the sensitivity of a system or the time at which a system remains chaotic or sensitive. Considering this fact, Moothathu introduced and studied stronger forms of sensitivity including syndetic sensitivity, cofinite sensitivity, multi-sensitivity [15]. Since then stronger forms of sensitivity have been studied by many authors in different settings of dynamical systems such as autonomous dynamical systems, iterated function systems, non-autonomous dynamical systems, group actions, etc., [3, 10, 11, 13, 19]. In 2018, Fedeli generalized the concept of sensitivity defined on metric spaces to topological spaces, known as topological sensitivity, by using open covers for

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topological spaces [4]. Recently, in 2020, Wang et al. have defined and studied the topological sensitivity with respect to Furstenberg families [24].

Devaney's definition of chaos contains three notions namely (i) transitivity (ii) dense set of periodic points and (iii) sensitivity. In [2] it is proved that transitivity and dense set of periodic points of a map on an infinite metric space imply its sensitivity. A nonminimal transitive map is sensitive is proved in [1,5]. Miller and Money have generalized this result and proved that a non-minimal transitive semiflow is sensitive [14]. Relations among various forms of sensitivity and various forms of transitivity, for example weakly mixing, thickly transitive, syndetically transitive, multi-transitive etc. have been obtained by many researchers for dynamical systems in different settings [6,10,14,15,20]. Relations between thick sensitivity and multi-sensitivity in case of semiflows for uniform spaces are studied in [25]. Different forms of sensitivity have been defined and studied by many authors for semiflows or semigroup actions, cf., [7,12,13,17,22]. Sensitivity and its stronger forms have been studied by many authors for product semiflows [8,9,23,26]. In [21], authors have proved that product semiflow is multi-sensitive if and only if at least one of the factors is multi-sensitive. Relations between sensitivity and topological sensitivity for a map have been studied by Fedeli in [4]. The author also proved that a transitive map with dense set of almost periodic points on an infinite Urysohn space, having an eventually periodic point, is topologically sensitive. Topological sensitivity, for a T_4 space, in case of semiflows is studied in [25]. In this paper, we will study about topological sensitivity, which is the most general case of sensitivity, its stronger forms and their relation with various types of transitivity in case of semiflows.

The paper is organized as follows. In Section 2, we give the preliminaries required for remaining sections of the paper. In Section 3, we obtain relations between stronger forms of sensitivity and their topological version. Then we prove that topological sensitivity and its stronger forms are preserved under conjugacy. We prove that for any discretely syndetic subsemigroup P of T, thick topological sensitivity of (T, X, π) implies that (P, X, π) is thick topologically sensitive and if (P, X, π) is syndetic topologically sensitive, then so is (T, X, π) and show that multi-topological sensitivity of (P, X, π) and (T, X, π) are equivalent. It is proved that if T is a discrete abelian semigroup and every $t \in T$ is almost open, then multi-topological sensitivity of (T, X, π) implies its thick topological sensitivity and transitivity with thick topological sensitivity of a semiflow imply multi-topological sensitivity of the semiflow. We also prove that if any one of (T, X, π) or (T, Y, ϕ) is multi-topologically sensitive, then their product semiflow $(T, X \times Y, \pi \times \phi)$ is multi-topologically sensitive and the converse is true if X and Y are both compact. In Section 4, we prove that for a Urysohn space X if there exists an $x \in X$ having a proper compact orbit, then syndetic transitivity of (T, X, π) implies its syndetic topological sensitivity. We also prove that for a T_3 space X, if (T, X, π) is nonminimal and syndetically transitive, then (T, X, π) is syndetic topologically sensitive and hence we get that for a T_3 space X, if (T, X, π) is nonminimal, transitive and set of almost periodic points is dense in X, then (T, X, π) is syndetic topologically sensitive.

2. Preliminaries

We shall denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$.

Let T be a topological semigroup with operation $+$. For any two subsets $K_1 \subset T$, $K_2 \subset T$, we denote $K_1 + K_2 = \{k_1 + k_2 : k_1 \in K_1, k_2 \in K_2\}$ and for any $t \in T$, $(t + K) = (\{t\} + K)$. A set $S \subset T$ is called *syndetic* if there exists a compact set $K \subset T$ such that for any $t \in T$, $(t + K) \cap S \neq \emptyset$. A subset $P \subset T$ is called *discretely syndetic* if there exists a finite set $F \subset T$ such that for any $t \in T$, $\{t + F\} \cap P \neq \emptyset$. Any set of form T/F for some finite set F is called *cofinite*. A set $S \subset T$ is called *thick* if for any compact subset $K \subset T$ there exists a $t \in T$ such that $(t + K) \subset S$. Let $\mathcal{P}$ be the collection of all subsets of T, a nonempty family $\mathcal{F}(\neq \mathcal{P})$ of subsets of T is called a *Furstenberg family* if it is hereditary upward, i.e., if $F_1 \in \mathcal{F}$ and $F_2 \supset F_1$, then $F_2 \in \mathcal{F}$.

Let X be a topological space. A continuous semigroup action $\pi : T \times X \to X$ of a topological semigroup T is called a *semiflow* and is denoted by (T, X, π) or simply (T, X). An element $\pi(t, x)$ for $t \in T$ and $x \in X$ will be denoted by $t.x$. By the definition of semigroup action we have $s.(t.x) = (s + t).x$. Any $t \in T$ is called *almost open* if for any nonempty open set $G \subset X$, $t.G$ contains an open set. Let $x \in X$ be any point. Then *orbit* of x is given by $T(x) = \{t.x : t \in T\}$. A semiflow is called *minimal* if orbit of every point of X is dense in X. A semiflow is called *nonminimal* if it is not minimal. A point $x \in X$ is called *periodic* if there exists a syndetic set $S \subset T$ such that $s.x = x$ for every $s \in S$ [14]. We say that $x \in X$ is an *almost periodic point* if for any nonempty open set $G \subset X$ with $x \in G$ there exists a compact set $K \subset T$ such that for any $t \in T$ there exists a $t' \in K$ such that $(t + t').x \in G$. We say that an $x \in X$ has a proper compact orbit if orbit of x is a proper compact subset of X. A topological space X is called a *Urysohn space* if for any elements $x, y \in X, x \neq y$ there exist open sets $U, V \subset X$ with $x \in U, y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. A topological space X is called *metrizable* if there exists a metric d on X generating topology of X. A topological space is called *nonmetrizable* if it is not metrizable.

For any pair of nonempty open subsets U and V of X, denote $N(U, V) = \{t \in T : t^{-1}.U \cap V \neq \emptyset\}$, where $t^{-1}.U = \{x \in X : t.x \in U\}$. If X is a metric space, then diameter of any set $A \subset X$, is denoted by $\mathrm{diam}(A)$.

The product semiflow of two semiflows (T, X, π) and (T, Y, ϕ) is denoted by $(T, X \times Y, \pi \times \phi)$, where $\pi \times \phi : T \times (X \times Y) \to X \times Y$ is defined by $(\pi \times \phi)(t, (x, y)) = (\pi(t, x), \phi(t, y))$ and $X \times Y$ has a product topology [21].

Definition ([12])**.** A semiflow (T, X, π) is said to be

(1) *topologically transitive* if for any pair of nonempty open sets $U, V \subset X$, we have that $N(U, V) \neq \emptyset$.

(2) $\mathcal{F}$*-transitive* if for any collection of nonempty open sets $U, V \subset X$, we have that $N(U, V) \in \mathcal{F}$.
(3) *weakly mixing* if for any nonempty open subsets U_1, U_2; V_1, V_2 of X, we have $N(U_1, V_1) \bigcap N(U_2, V_2) \neq \emptyset$.

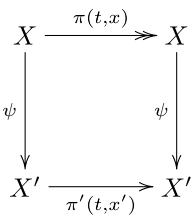
Let X be a metric space, (T, X, π) be any semiflow and $\delta > 0$ be a real number. For any nonempty open subset $G \subset X$ we denote

$$D(G, \delta) = \{t \in T : \text{diam}(t.G) > \delta\}.$$

Definition ([26])**.** For a metric space X, a semiflow (T, X, π) is called

(1) *sensitive* if there exists a $\delta > 0$ such that for any nonempty open subset $G \subset X$, $D(G, \delta)$ is nonempty.
(2) *syndetically sensitive* if there exists a $\delta > 0$ such that for any nonempty open subset $G \subset X$, $D(G, \delta)$ is syndetic.
(3) *multi sensitive* if there exists a $\delta > 0$ such that for any $n \in \mathbb{N}$ and for nonempty open subsets $G_1, G_2, \ldots, G_n \subset X$, $\bigcap_{i=1}^n D(G_i, \delta)$ is nonempty.
(4) $\mathcal{F}$*-sensitive* if there exists a $\delta > 0$ such that for any nonempty open set $G \subset X$, $D(G, \delta) \in \mathcal{F}$.

Definition ([16])**.** Let (T, X, π) and (T, X', π') be two semiflows. We say that (T, X, π) is *conjugate* to (T, X', π') if there exists a homeomorphism $\psi : X \to X'$ such that for every $t \in T$ the following diagram commutes:



i.e., $\pi'(t, \psi(x)) = \psi(\pi(t, x))$ for every $x \in X$.

3. Topological sensitivity and its stronger forms

In this section we introduce stronger forms of topological sensitivity. We prove that $\mathcal{F}$-topological sensitivity is preserved under conjugacy, for any Furstenberg family $\mathcal{F}$. We discuss relations between stronger forms of topological sensitivities. We also study about sensitivity on product semiflows.

Let (T, X, π) be a semiflow and $\mathscr{E}$ be an open cover for X. For any nonempty open subset U of X define

$$D(U, \mathscr{E}) = \{t \in T : \text{there exist } x, y \in U \text{ such that } (x, y) \notin \bigcup\{t^{-1}.E \times t^{-1}.E : E \in \mathscr{E}\}\}.$$

Definition. Let (T, X, π) be a semiflow. Then an open cover $\mathscr{E}$ of X is said to be

(1) a *sensitivity cover* (*s-cover*) if for any nonempty open subset U of X, we have $D(U,\mathscr{E}) \neq \emptyset$ and (T,X,π) is *topologically sensitive* if it has an s-cover.
(2) a *syndetic sensitivity cover* (*ss-cover*) (*respectively, thick sensitivity cover* (*ts-cover*)) if $D(U,\mathscr{E})$ is syndetic (respectively, thick) for any nonempty open subset U of X and (T,X,π) is *syndetic topologically sensitive* (*respectively, thick topologically sensitive*) if it has an ss-cover (respectively, ts-cover).
(3) a *cofinite sensitivity cover* (*cs-cover*) if $D(U,\mathscr{E})$ is cofinite for any nonempty open subset U of X and (T,X,π) is *cofinite topologically sensitive* if it has a cs-cover.
(4) a *multi-sensitivity cover* (*ms-cover*) if for any collection of nonempty open subsets $U_1,U_2,\ldots,U_n$ of X, we have $\bigcap_{i=1}^n D(U_i,\mathscr{E}) \neq \emptyset$ and (T,X,π) is *multi-topologically sensitive* if it has an ms-cover.
(5) an $\mathcal{F}$-*s-cover* if for any nonempty open subset U of X, we have $D(U,\mathscr{E}) \in \mathcal{F}$ and (T,X,π) is $\mathcal{F}$ *topologically sensitive* if it has an $\mathcal{F}$-s-cover.

Proposition 3.1. *Let (T,X,π) be a topologically sensitive semiflow with s-cover $\mathscr{E}$. Then for any nonempty open set $G \subset X$, $D(G,\mathscr{E})$ is infinite.*

Proof. Let (T,X,π) be a topologically sensitive semiflow with s-cover $\mathscr{E}$ and assume that for some nonempty open set $G \subset X$, $D(G,\mathscr{E})$ is finite, say, $D(G,\mathscr{E}) = \{t_1,\ldots,t_k\}$. Now, take a $U_1 \in \mathscr{E}$ such that $(t_1.G) \cap U_1 \neq \emptyset$ then $G_1 = G \cap t_1^{-1}.U_1$ is nonempty open and $t_1 \notin D(G_1,\mathscr{E})$. Similarly, taking a $U_2 \in \mathscr{E}$ such that $(t_2.G_1) \cap U_2 \neq \emptyset$, we get that $G_2 = G_1 \cap t_2^{-1}.U_2$ is a nonempty, open subset of G_1. Since $t_1 \notin D(G_1,\mathscr{E})$ and $t_2.G_2 \subset U_2$, therefore $\{t_1,t_2\} \cap D(G_2,\mathscr{E}) = \emptyset$. Proceeding as above, we get nonempty open sets $G_k \subset G_{k-1} \subset \cdots \subset G_2 \subset G_1 \subset G$ such that $\{t_1,t_2,\ldots,t_k\} \cap D(G_k,\mathscr{E}) = \emptyset$. By topological sensitivity of (T,X,π), we get that $D(G_k,\mathscr{E}) \neq \emptyset$ and hence there exists an $s \in T$ such that $s \in D(G_k,\mathscr{E})$ and as $\{t_1,t_2,t_3,\ldots,t_k\} \cap D(G_k,\mathscr{E}) = \emptyset$, we get that $s \notin \{t_1,t_2,t_3,\ldots,t_k\}$. Now, using the fact that $G_k \subset G$, we get that $D(G_k,\mathscr{E}) \subset D(G,\mathscr{E})$ which implies that $s \in D(G,\mathscr{E})$ which is a contradiction to the fact that $D(G,\mathscr{E}) = \{t_1,t_2,t_3,\ldots,t_k\}$. Thus, $D(G,\mathscr{E})$ is infinite for every nonempty open set $G \subset X$. ☐

Theorem 3.2. *Let (T,X,π) be a semiflow, where (X,d) is a metric space. Consider the following statements*:

(i) *(T,X,π) is syndetically sensitive*;
(ii) *there exists an $\epsilon > 0$ such that $\{B_d(x,\epsilon) : x \in X\}$ is an ss-cover for (T,X,π)*;
(iii) *(T,X,π) is syndetic topologically sensitive.*

Then (i) $\iff$ (ii) $\implies$ (iii) *and if X is taken to be compact, then* (iii) $\implies$ (ii) *also holds.*

Proof. (i) $\implies$ (ii) Let $\delta > 0$ be a syndetically sensitivity constant for (T,X,π). Then for any nonempty open set $G \subset X, D(G,\delta)$ is syndetic. Therefore, for

$\epsilon = \delta/2$ taking $\mathcal{E} = \{B_d(x, \epsilon) : x \in X\}$ we get that $D(G, \delta) \subset D(G, \mathcal{E})$. Hence, $D(G, \mathcal{E})$ is syndetic and thus $\mathcal{E}$ is an ss-cover for (T, X, π).

(ii) $\Longrightarrow$ (i) Let $\mathcal{E} = \{B_d(x, \epsilon) : x \in X\}$ be an ss-cover for (T, X, π). Then for any nonempty open set $G \subset X$, $D(G, \mathcal{E})$ is syndetic. By definition of $\mathcal{E}$, we have for some $\epsilon > 0$ and for any $s \in D(G, \mathcal{E})$, $\operatorname{diam}(s.G) > \epsilon$. So, by taking $\delta = \epsilon$, we get that $D(G, \delta)$ is syndetic which proves that (T, X, π) is syndetically sensitive.

(ii) $\Longrightarrow$ (iii) Clearly follows.

Now assume that X is compact. We prove that (iii) $\Longrightarrow$ (ii). Let $\mathcal{U}$ be an ss-cover for (T, X, π). Since X is compact, therefore there exists a Lebesgue number $\delta > 0$ such that any ball $B_d(x, \delta)$ is contained in U for some $U \in \mathcal{U}$. So, if $s \in D(G, \mathcal{U})$, then $s.G \not\subset B_d(x, \delta)$ for any $x \in X$ and hence $\{B_d(x, \delta) : x \in X\}$ is an ss-cover for (T, X, π). □

Remark 3.3. Note that by similar arguments, Theorem 3.2 is also true for $\mathcal{F}$ topological sensitivity and multi-topological sensitivity. Also, the compactness of the metric space is necessary for equivalence in Theorem 3.2 as justified by the following example (for another such example see [4]).

Example 3.4. Consider the space $X = (0, 1]$ with the usual metric and define $f : X \to X$ by $f(x) = x^2$. Let $T = \mathbb{N}$ and $\pi : T \times X \to X$ be defined by $\pi(t, x) = t.x = f^t(x)$. Then (T, X, π) is a semiflow which is not sensitive with respect to usual metric. Now, consider the metric $d(x, y) = |\ln(x) - \ln(y)|$ on X. Then for any nonempty open set $U \subset (0, 1]$ $x, y \in U$, $x \neq y$ implies $d(n.x, n.y) = |\ln(x^{2^n}) - \ln(y^{2^n})| = 2^n |\ln(x) - \ln(y)|$. So, we can find an $n_0 \in \mathbb{N}$ such that $d(n.x, n.y) \geq 1$ for all $n \geq n_0$ which implies that (T, X, π) is cofinitely sensitive and hence cofinite topologically sensitive. Note that the topology generated by both metrics is same. Therefore, the semiflow (T, X, π) is topologically sensitive but not sensitive with respect to usual metric.

In [4, Theorem 2.4] it is proved that topological sensitivity for maps is preserved under conjugacy. In the next theorem we prove that, for any Furstenberg family $\mathcal{F}$, $\mathcal{F}$ topological sensitivity is preserved under conjugacy in case of semiflows also.

Proposition 3.5. *If semiflow (T, X, π) is conjugate to (T, X', π') and (T, X', π') is $\mathcal{F}$ topologically sensitive, then (T, X, π) is also $\mathcal{F}$ topologically sensitive.*

Proof. Since (T, X, π) is conjugate to (T, X', π'), therefore there exists a homeomorphism $\psi : X \to X'$ such that for every $t \in T$,

$$\pi'(t, \psi(x)) = \psi(\pi(t, x)) \text{ for each } x \in X.$$

Let $\mathcal{U}'$ be an $\mathcal{F}$-s-cover for (T, X', π'). Then we show that the open cover $\mathcal{U} = \{\psi^{-1}(U') : U' \in \mathcal{U}'\}$ is an $\mathcal{F}$-s-cover for (T, X, π).

Let $G \subset X$ be any nonempty open set. Then $\psi(G) \subset X'$ is a nonempty open set. So, $D(\psi(G), \mathcal{U}') \in \mathcal{F}$. Choose $t \in D(\psi(G), \mathcal{U}')$ and $x', y' \in \psi(G)$ such that $(t.x', t.y') \notin \{U' \times U' : U' \in \mathcal{U}'\}$. Take $x, y \in G$ such that $\psi(x) = x'$, $\psi(y) = y'$

then $(t.\psi(x), t.\psi(y) \notin \{U' \times U' : U' \in \mathcal{U}'\})$, i.e., $(\psi(t.x), \psi(t.y)) \notin \{U' \times U' : U' \in \mathcal{U}'\}$. Therefore, $(t.x, t.y) \notin \{\psi^{-1}(U)' \times \psi^{-1}(U') : U' \in \mathcal{U}'\}$. Hence, $(t.x, t.y) \notin \{U \times U : U \in \mathcal{U}\}$, i.e., $t \in D(G,\mathcal{U})$ which implies $D(\psi(G),\mathcal{U}') \subset D(G,\mathcal{U})$. So, (T, X, π) is $\mathcal{F}$ topologically sensitive. □

Theorem 3.6. *Let (T, X, π) be a semiflow. Then we have following*:

1. *If (P, X, π) is syndetic topologically sensitive for some discretely syndetic subsemigroup P of T, then (T, X, π) is syndetic topologically sensitive.*
2. *If (T, X, π) is thick topologically sensitive, then (P, X, π) is thick topologically sensitive for any discretely syndetic subsemigroup P of T.*

Proof. (1) Since $P \subset T$ is discretely syndetic so there exists a finite set $F \subset T$ such that for every $t \in T$ there exists an $f \in F$ such that $f + t \in P$. Let $\mathcal{E}$ be an ss-cover for (P, X, π). Let $G \subset X$ be any nonempty open set. Then there exists a compact set $K \subset P$ such that for any $s \in P$ there exists a $k \in K$ such that $s + k \in D(G, \mathcal{E})$, i.e., $(s + k).G \not\subset U$ for any $U \in \mathcal{E}$. Now take any $t \in T$ then there exists an $f \in F$ such that $t + f \in P$ and there exists a $k \in K$ such that $t + f + k \in D(G, \mathcal{E})$, i.e., $(t + f + k).G \not\subset U$ for any $U \in \mathcal{E}$. Hence, for the compact set $F + K \subset T$ and for any $t \in T$ there exists an $f + k \in F + K$ such that $t + f + k \in D(G, \mathcal{E})$. Thus, (T, X, π) is syndetic topologically sensitive.

(2) Since P is a discretely syndetic subsemigroup of T, so let $F \subset T$ be a finite set such that for any $t \in T$, $\{t + F\} \cap P \neq \emptyset$. Let $K \subset P$ be any compact subset. Then $K + F \subset T$ is a compact set and let $G \subset X$ be any nonempty open set. Then there exists a $t \in T$ such that $t + K + F \subset D(G, \mathcal{E})$ for some cover $\mathcal{E}$ of X. Since P is discretely syndetic so there exists an $f \in F$ such that $t + f \in P$, then $t + f + K \subset D(G, \mathcal{E})$. Hence, (P, X, π) is thick topologocially sensitive. □

Theorem 3.7. *Let (T, X, π) be a semiflow and every $t \in T$ is almost open. Then the following are equivalent*:

1. *(T, X, π) is multi-topologically sensitive*;
2. *(P, X, π) is multi-topologically sensitive, for any discretely syndetic subsemigroup P of T.*

Proof. (2) $\implies$ (1) is obvious.

We prove (1) $\implies$ (2).

Let (T, X, π) be multi-topologically sensitive with ms-cover $\mathcal{E}$. Let $F = \{f_1, f_2, f_3, \ldots, f_n\}$ be a finite set such that for every $t \in T$ there exists an $f_i \in F$ such that $t + f_i \in P$. Now, consider a collection of nonempty open subsets $G_1, G_2, \ldots, G_k$ of X, then since each f_i is almost open so $f_i.G_j$ has nonempty interior for any $f_i \in F$ and any $j \in \{1, 2, \ldots, k\}$. As (T, X, π) is multi-topologically sensitive so there exists an $s \in T$ such that

$$s \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{k} D(f_i.G_j, \mathcal{E}).$$

Hence, for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k\}$ $s.f_i.G_j \not\subset U$ for all $U \in \mathcal{E}$ and for some $f_i \in F$, $s + f_i \in P$. Thus, we have an i such that $s + f_i \in P$ and $s + f_i \in \bigcap_{j=1}^{k} D(G_j, \mathcal{E})$ which proves that (P, X, π) is multi-topologically sensitive. □

Next, we provide an example justifying that every $t \in T$ is almost open in Theorem 3.7 is a necessary condition.

Example 3.8. Define $f : [0, 1] \to [0, 1]$ by $f(x) = 1 - |1 - 2x|$. Take $X = [0, 1]$ and $T = \mathbb{N} \times \{0, 1\}$ with discrete topology and operation $+ : T \to T$ by $(n, 0) + (m, 0) = (n + m, 0)$ and $(n, i) + (m, j) = (n + m, 1)$ if any one of i or j is nonzero. Define $\pi : T \times X \to X$ by $\pi((n, 0), x) = f^n(x)$, $\pi((n, 1), x) = 0$ then (T, X, π) is a semiflow. One can easily verify that (T, X, π) is multi-topologically sensitive.

Now, take subsemigroup $P = \mathbb{N} \times \{1\}$ of T. Since for any $n \in \mathbb{N}$, $(n, 1).x = 0$ for each $x \in X$, therefore (P, X, π) is not multi-topologically sensitive.

Proposition 3.9. *Let (T, X, π) be a semiflow, where T is a discrete abelian semigroup such that every $t \in T$ is almost open. If (T, X, π) is multi-topologically sensitive, then it is thick topologically sensitive.*

Proof. Let $K = \{t_1, t_2, \ldots, t_k\} \subset T$ be any compact set and $U \subset X$ be any nonempty open set. Take nonempty open sets $U_i \subset t_i.U$, $i \in \{1, \ldots, k\}$ then since (T, X, π) is multi-topologically sensitive therefore there exists an $s \in T$ such that $s \in \bigcap_{i=1}^{k} D(U_i, \mathcal{V})$, where $\mathcal{V}$ is an ms-cover for (T, X, π). Hence, $s.U_i \not\subset V$ for any $V \in \mathcal{V}$ which implies that $(s + t_i).U \not\subset V$ for any $V \in \mathcal{V}$. Thus, $s + K \subset D(U, \mathcal{V})$. Therefore, (T, X, π) is thick topologically sensitive. □

Next, we provide an example of a multi-topologically sensitive semiflow (T, X, π), where T is a discrete abelian semigroup and every $t \in T$ is almost open. So by above result it is thick topologically sensitive.

Example 3.10. Let $\mathbb{R}$ be the real line with usual topology. Consider the map $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$. Take $T = \{(n_1, n_2, n_3, \ldots) : n_i \in \mathbb{N}\}$ with coordinate-wise addition and discrete topology. Let $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ with box topology. Define $\pi : T \times X \to X$ by $(n_1, n_2, n_3, \ldots).(x_1, x_2, x_3, \ldots) = (f^{n_1}(x_1), f^{n_2}(x_2), f^{n_3}(x_3), \ldots)$. Now we will prove that (T, X, π) is a multi-topologically sensitive semiflow. Take open cover $\mathcal{U} = \{\prod_{i=1}^{\infty} B(x_i, 1) : x_i \in \mathbb{R}\}$, where $B(x_i, 1) = (x_i - 1, x_i + 1)$. Take nonempty open sets $U^j = {U_1}^j \times {U_2}^j \times {U_3}^j \times \cdots \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ for $j \in \{1, 2, 3, \ldots, k\}$. For any $i \in \mathbb{N}$, for open sets $\{{U_i}^j \subset \mathbb{R}, j \in \{1, 2, 3, \ldots, k\}\}$ there exists an $n_i \in \mathbb{N}$ such that $\operatorname{diam}(f^{n_i}({U_i}^j)) > 2$ for all $j \in \{1, 2, 3, \ldots, k\}$, i.e., $f^{n_i}({U_i}^j) \not\subset B(x_i, 1)$ for any $x_i \in \mathbb{R}$ and for any $j \in \{1, 2, 3, \ldots, k\}$. So, we have $(n_1, n_2, n_3, \ldots).({U_1}^j, {U_2}^j, {U_3}^j, \ldots) \not\subset \prod_{i=1}^{\infty} B(x_i, 1)$ for any $x_i \in \mathbb{R}$ and for any $j \in \{1, 2, 3, \ldots, k\}$, i.e., $(n_1, n_2, n_3, \ldots) \in \bigcap_{j=1}^{k} D(U^j, \mathcal{U})$. Hence, (T, X, π) is multi-topologically sensitive and clearly every $t \in T$ is almost open. Thus, by Proposition 3.9 (T, X, π) is thick topologically sensitive.

Remark 3.11. In Example 3.10 the semiflow (T, X, π) is thick topologically sensitive but not cofinite topologically sensitive. As for any open cover $\mathcal{U}$ of X and any nonempty open set $U \in \mathcal{U}$, for all $i \in \mathbb{N}$ there exist an $x_i \in \mathbb{R}$ and $\delta_i > 0$ such that $\prod_{i=1}^{\infty} B(x_i, \delta_i) \subset U$. Therefore, taking nonempty open subset $G = \prod_{i=1}^{\infty} B(x_i, \delta_i/i)$, we get that for any $t \in \{1\} \times \{1, 2\} \times \{1, 2, 3\} \times \cdots \subset T$, $t \notin D(G, \mathcal{U})$. Hence, (T, X, π) is not cofinite topologically sensitive.

Proposition 3.12. *Let (T, X, π) be a semiflow. If (T, X, π) is transitive and thick topologically sensitive, then it is multi-topologically sensitive.*

Proof. Let $U_1, U_2, \dots, U_n$ be a collection of nonempty open subsets of X. Then as (T, X, π) is transitive so there exist $t_1, t_2, \dots, t_n$ in T such that $U = t_1^{-1}U_1 \cap t_2^{-1}U_2 \cap \cdots \cap t_n^{-1}U_n$ is nonempty and open. Take compact set $K = \{t_1, t_2, \dots, t_n\}$ then there exists an $s \in T$ such that $s + K \subset D(U, \mathcal{U})$, where $\mathcal{U}$ is a thick sensitivity cover for (T, X, π). Since $D(U, \mathcal{U}) \subset D(t_i^{-1}U_i, \mathcal{U})$ and $s + t_i \in D(U, \mathcal{U})$ for all $i \in \{1, 2, 3, \dots, n\}$, therefore $s + t_i \in D(t_i^{-1}U_i, \mathcal{U})$ which implies that $s \in D(U_i, \mathcal{U})$ for all $i \in \{1, 2, 3, \dots, n\}$. Hence, $s \in \bigcap_{i=1}^{n} D(U_i, \mathcal{U})$ which proves that (T, X, π) is multi-topologically sensitive. □

Next, we prove some results for product semiflows. Similar results on sensitivity for product semiflows on metric spaces have been proved in [21].

Theorem 3.13. *Let (T, X, π) and (T, Y, ϕ) be two semiflows. If either (T, X, π) or (T, Y, ϕ) is multi-topologically sensitive, then product $(T, X \times Y, \pi \times \phi)$ is multi-topologically sensitive.*

Proof. Let (T, X, π) be multi-topologically sensitive with multi-sensitivity cover $\mathcal{U}$. We prove that $\mathcal{V} = \{U \times Y : U \in \mathcal{U}\}$ is a multi-sensitivity cover for $(T, X \times Y, \pi \times \phi)$. Let $G_i \times H_i \subset X \times Y$ be a nonempty open subset for every $i \in \{1, 2, 3, \dots, k\}$ and for every $k \in \mathbb{N}$. Since (T, X, π) is multi-topologically sensitive, so there exists a $t \in T$ such that $t \in \bigcap_{i=1}^{k} D(G_i, \mathcal{U})$, i.e., $t.G_i \not\subset U$ for any $U \in \mathcal{U}$ which implies that $t.(G_i \times H_i) \not\subset U \times Y$ for any $U \times Y \in \mathcal{V}$, i.e., $t \in \bigcap_{i=1}^{k} D(G_i \times H_i, \mathcal{V})$. Hence, $\mathcal{V}$ is a multi-sensitivity cover for $(T, X \times Y, \pi \times \phi)$ implying $(T, X \times Y, \pi \times \phi)$ is multi-topologically sensitive. □

Remark 3.14. Similarly one can also prove that if any one of (T, X, π) or (T, Y, ϕ) is $\mathcal{F}$ topologically sensitive for any Furstenberg family $\mathcal{F}$, then the product semiflow $(T, X \times Y, \pi \times \phi)$ is also $\mathcal{F}$ topologically sensitive.

The converse of the above theorem is also true for compact spaces. To prove the converse part, we first prove the following lemma.

Lemma 3.15. *Let X, Y be compact and $\mathcal{V}$ be any open cover of $X \times Y$. Then there exist open covers $\mathcal{U}_1$ and $\mathcal{U}_2$ of X and Y, respectively, such that for any $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ there exists a $V \in \mathcal{V}$ such that $U_1 \times U_2 \subset V$.*

Proof. Let $\mathcal{V}$ be any open cover of $X \times Y$. Then for any $(x, y) \in X \times Y$ there exists a $V \in \mathcal{V}$ such that $(x, y) \in V$ and as V is open in $X \times Y$ so there exist open subsets V^1_{xy}, V^2_{xy} of X and Y, respectively, such that $(x,y) \in V^1_{xy} \times V^2_{xy} \subset V \in \mathcal{V}$. Now since X, Y are compact, therefore $X \times Y$ is also compact and $\{V^1_{xy} \times V^2_{xy} : (x,y) \in X \times Y\}$ is an open cover of $X \times Y$, therefore it has a finite subcover, say,

$$\{V^1_{x_i y_j} \times V^2_{x_i y_j} : (i,j) \in \{1,2,3,\ldots,m\} \times \{1,2,3,\ldots,n\}\}.$$

Let $S_{xy} \subset \{1,2,3,\ldots,m\} \times \{1,2,3,\ldots,n\}$ be the set such that for any $(i,j) \in S_{xy}$, $(x,y) \in V^1_{x_i y_j} \times V^2_{x_i y_j}$. Now, for any $x \in X$ and $y \in Y$, take

$$U_x = \bigcap_{(i,j)\in S_{xy}, y \in Y} V^1_{x_i y_j} \quad \text{and} \quad U_y = \bigcap_{(i,j)\in S_{xy}, x \in X} V^2_{x_i y_j}.$$

Then for any $x \in X$ and $y \in Y$ note that the sets U_x and U_y are nonempty and open in X and Y, respectively. Hence, $\{U_x : x \in X\}$ and $\{U_y : y \in Y\}$ are open covers of X and Y, respectively, and for any $(x,y) \in X \times Y$, S_{xy} is nonempty, so for $(i,j) \in S_{xy}$ we have $(x,y) \in U_x \times U_y \subset V^1_{x_i y_j} \times V^2_{x_i y_j} \subset V$ for some $V \in \mathcal{V}$. ☐

Theorem 3.16. *Let (T,X,π) and (T,Y,ϕ) be two semiflows, where X, Y are compact. If $(T, X \times Y, \pi \times \phi)$ is multi-topologically sensitive, then at least one of (T,X,π) or (T,Y,ϕ) is multi-topologically sensitive.*

Proof. Let $(T, X \times Y, \pi \times \phi)$ be multi-topologically sensitive with multi-sensitivity cover $\mathcal{V}$. Assume that none of (T,X,π) or (T,Y,ϕ) is multi-topologically sensitive. Then by Lemma 3.15, we have open covers $\mathcal{U}_1$ and $\mathcal{U}_2$ of X and Y, respectively, such that for any $U_1 \times U_2 \in \mathcal{U}_1 \times \mathcal{U}_2$, there exists a $V \in \mathcal{V}$ such that $U_1 \times U_2 \subset V$. Now, since none of (T,X,π) or (T,Y,ϕ) is multi-topologically sensitive, therefore there exist collections of nonempty open subsets $G_1, G_2, \ldots, G_n$ of X and $H_1, H_2, \ldots, H_k$ of Y such that $\bigcap_{i=1}^n D(G_i, \mathcal{U}_1) = \emptyset$ and $\bigcap_{j=1}^k D(H_j, \mathcal{U}_2) = \emptyset$. Hence, for every $t \in T$, $t.(G_i \times H_j) \subset U_1 \times U_2$ for some $(i,j) \in \{1,2,3,\ldots,n\} \times \{1,2,3,\ldots,k\}$, $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$. Thus $t.(G_i \times H_j) \subset V$ for some $(i,j) \in \{1,2,3,\ldots,n\} \times \{1,2,3,\ldots,k\}$ for some $V \in \mathcal{V}$, i.e.,

$$\bigcap_{i \in M_n,\ j \in M_k} D(G_i \times H_j, \mathcal{V}) = \emptyset, \quad \text{where } M_i = \{1,2,\ldots,i\},$$

which is a contradiction to the fact that $(T, X \times Y, \pi \times \phi)$ is multi-topologically sensitive and hence at least one of (T,X,π) or (T,Y,ϕ) is multi-topologically sensitive. ☐

4. Some sufficient conditions for topological sensitivity

In this section, we discuss relations among stronger forms of transitivity and sensitivity. We prove that for a Urysohn space X if there exists an $x \in X$ having proper compact orbit and (T,X,π) is syndetically transitive, then (T,X,π) is syndetic topologically sensitive. We also prove that if there exists an $x \in X$ having proper compact orbit and (T,X,π) is transitive with set of almost periodic points dense in X, then (T, X, π) is syndetic topologically sensitive. Moreover, it is proved that for a T_3 space X if (T, X, π) is nonminimal, transitive and set of almost periodic points is dense in X, then (T, X, π) is syndetic topologically sensitive.

In [4, Proposition 2.7] Fedeli has proved that for a weakly mixing map $f : X \to X$, if X has two nonempty open sets U, V such that $\overline{U} \cap \overline{V} = \emptyset$, then the map f is topologically sensitive. Here, we prove this result on semiflows using next lemma.

Lemma 4.1 ([13]). *Let (T, X, π) be any weakly mixing semiflow with T being abelian. Then for any collection of nonempty open sets $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ there exists a $t \in T$ such that $t.U_i \cap V_i \neq \emptyset$.*

Theorem 4.2. *If (T, X, π) is weakly mixing with T being abelian and there exist nonempty open sets U and V such that $\overline{U} \cap \overline{V} = \emptyset$, then (T, X, π) is multi-topologically sensitive.*

Proof. We shall show that the open cover $\mathcal{U} = \{X/\overline{U}, X/\overline{V}\}$ is a multi-sensitivity cover for (T, X, π). Let $G_1, G_2, \ldots, G_n$ be a collection of nonempty open subsets of X. Then since (T, X, π) is weakly mixing, there exists an $s \in T$ such that $s.G_i \cap U \neq \emptyset$ and $s.G_i \cap V \neq \emptyset$ for all $i \in \{1, 2, 3, \ldots, n\}$ which implies that $s.G_i \not\subset X/\overline{U}$ and $s.G_i \not\subset X/\overline{V}$ for all $i \in \{1, 2, 3, \ldots, n\}$. Hence $s \in D(G_i, \mathcal{U})$ for all $i \in \{1, 2, 3, \ldots, n\}$. Thus, $\bigcap_{i=1}^{n} D(G_i, \mathcal{U}) \neq \emptyset$, i.e., (T, X, π) is multi-topologically sensitive. □

In [14, Theorem 4.4] authors have proved that in case of metric spaces a syndetic transitive, nonminimal semiflow is syndetically sensitive and in [25, Theorem 1.2] authors have proved that a topologically transitive and nonminimal action of a semigroup on an infinite T_4 space is topologically sensitive if the space has dense set of almost periodic points. We will now prove similar results on a T_3 and Urysohn space X. For a map $f : L \times Z \to Y$, for any $r \in L$ and $G \subset Y$ denote $r^{-1}.G = \{z \in Z : f(r, z) \in G\}$.

Lemma 4.3. *Let $f : L \times Z \to Y$ be a continuous map with $f(r, z) = r.z$. For any nonempty open set $G \subset Y$, if L is compact, then $\bigcap_{r \in L} r^{-1}.G$ is open in Z.*

Proof. Take $z \in \bigcap_{r \in L} r^{-1}.G$ then $r.z \in G$ for each $r \in L$. So, $f(r, z) \in G$ for each $r \in L$ and hence $(r, z) \in f^{-1}(G)$ for each $r \in L$, i.e., $L \times \{z\} \subset f^{-1}(G)$. Since G is nonempty open, $f^{-1}(G)$ is also nonempty open and as L is compact, therefore by Tube Lemma there exists a nonempty open set $U \subset Z$ with $z \in U$ such that $L \times U \subset f^{-1}(G)$. Thus, for each $z' \in U$, $f(r, z') \in G$ for each $r \in L$, i.e., $r.z' \in G$ for each $r \in L$. So, $z' \in r^{-1}.G$ for each $r \in L$, i.e., $z' \in \bigcap_{r \in L} r^{-1}.G$ for each $z' \in U$ and this implies that $U \subset \bigcap_{r \in L} r^{-1}.G$. Hence, $\bigcap_{r \in L} r^{-1}.G$ is open. □

Theorem 4.4. *Let (T, X, π) be a syndetically transitive semiflow, where X is a Urysohn space. If there exist an $x \in X$ such that $T(x)$ is a proper compact subset of X, then (T, X, π) is syndetic topologically sensitive.*

Proof. Let $y \in X$ be such that $y \notin T(x)$. Then there exist nonempty open sets U, V such that $y \in U$, $T(x) \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. Take open cover $\mathcal{U} = \{X/\overline{U}, X/\overline{V}\}$. We show that $\mathcal{U}$ is an *ss*-cover.

Let G be any nonempty open subset of X. Then $N(U, G)$ is syndetic. Let K_1 be a compact set such that for any $t \in T$ there exists a $t_1 \in K_1$ such that $t + t_1 \in N(U, G)$. Now, for any $r \in T$ as $T(x) \subset V$, $r.x \in V$. Therefore, $x \in \bigcap_{r \in K_1} r^{-1}.V$, i.e., $\bigcap_{r \in K_1} r^{-1}.V \neq \emptyset$ and by Lemma 4.3, $\bigcap_{r \in K_1} r^{-1}.V$ is open. Hence, $N(\bigcap_{r \in K_1} r^{-1}.V, G)$ is syndetic. Let K_2 be a compact set such that for any $t \in T$ there exists a $t_2 \in K_2$ such that $t + t_2 \in N(\bigcap_{r \in K_1} r^{-1}.V, G)$. Thus, for any $t \in T$ there exist a $t_2 \in K_2$ and $y \in G$ such that $(t + t_2 + r).y \in V$ for each $r \in K_1$. For $t + t_2 \in T$ there exists a $t_1 \in K_1$ such that $t + t_2 + t_1 \in N(U, G)$. So taking $r = t_1$ we have $(t + t_2 + t_1).y \in V$, $(t + t_2 + t_1).x \in U$. Therefore, $(t + t_2 + t_1).y \notin X/\overline{V}$, $(t + t_2 + t_1).x \notin X/\overline{U}$. Hence, for open cover $\mathcal{U} = \{X/\overline{U}, X/\overline{V}\}$ we have $t + t_2 + t_1 \in D(G, \mathcal{U})$. Thus, for any $t \in T$ there exists $t_2 + t_1 \in K_2 + K_1$ such that $t + t_2 + t_1 \in D(G, \mathcal{U})$. Also K_2, K_1 are compact, so $K_2 + K_1$ is also compact. Therefore, $D(G, \mathcal{U})$ is syndetic and hence (T, X, π) is syndetic topologically sensitive. □

Corollary 4.5. *Let (T, X, π) be a transitive semiflow with dense set of almost periodic points, where X is a Urysohn space with an $x \in X$ such that $T(x)$ is a proper compact subset of X. Then (T, X, π) is syndetic topologically sensitive.*

Proof. Let $U, V \subset X$ be nonempty open sets. Take an $s \in T$ such that $U \cap s^{-1}.V \neq \emptyset$ and open. Let $x \in U \cap s^{-1}.V$ be an almost periodic point and K be a compact set such that for any $t \in T$ there exists a $t_1 \in K$ such that $(t + t_1).x \in U \cap s^{-1}V$. We will prove that $N(V, U)$ is syndetic. Take any $t \in T$ then there exists a $t_1 \in K$ such that $(t + t_1).x \in U \cap s^{-1}.V$. So, $x \in U$ and $(t + t_1 + s).x \in V$, i.e., $(t + t_1 + s).U \cap V \neq \emptyset$. Take $K_1 = K + \{s\}$ then for any $t \in T$ we have $t_1 + s \in K_1$ such that $t + t_1 + s \in N(V, U)$. Hence, $N(V, U)$ is syndetic which proves that (T, X, π) is syndetically transitive. Thus, by Theorem 4.4 (T, X, π) is syndetic topologically sensitive. □

Now, we give an example of a transitive semiflow (T, X, π), where X is a Urysohn space having a dense set of almost periodic points and an $x \in X$ with proper compact orbit, so (T, X, π) is syndetic topologically sensitive.

Example 4.6. Let $T = [1, \infty)$ with usual multiplication operation and usual topology and $X = S^1 = [0, 1)$ with metric $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$. Define $\pi : T \times X \to X$ by $\pi(t, x) = t.x = (t \times x) \mod 1$. Then (T, X, π) is a semiflow. Now, we will prove that (T, X, π) is a transitive semiflow with a dense set of almost periodic points.

Let U, V be nonempty open subsets of X and $x (\neq 0) \in U$ and $y (\neq 0) \in V$. Take $r = (1 + y)/x$ then $r.x = (\frac{1+y}{x} \times x) \mod 1 = y$. So, $r.U \cap V \neq \emptyset$ and hence (T, X, π) is topologically transitive.

Next, we will prove that every $x \in X$ is a periodic point. Clearly 0 is a periodic point and for any $x(\neq 0) \in X$ there exists a syndetic set

$$S = \left\{1 + \frac{k}{x} : k \in \mathbb{N}\right\} = \left\{1 + \frac{1}{x}, 1 + \frac{2}{x}, 1 + \frac{3}{x}, \dots\right\}$$

such that for any $s \in S$ we have $s.x = x$ and hence every $x \in X$ is a periodic point. So, almost periodic points are dense in X.

As $T(0) = \{0\}$, so $T(0)$ is a proper compact subset of X and obviously X with usual topology is a Urysohn space. Hence, (T, X, π) is a transitive semiflow with a dense set of almost periodic points, where X is a Urysohn space and there exists an $x \in X$ such that $T(x)$ is a proper compact subset of X. Thus, by Corollary 4.5 we get that (T, X, π) is syndetic topologically sensitive.

Remark 4.7. Let (T, X, π) be a syndetically transitive, nonminimal semiflow, where T is compact and X is a Urysohn space. Then (T, X, π) is syndetic topologically sensitive.

Remark 4.8. Every transitive and periodically dense semiflow (T, X, π), where X is a Urysohn space such that $T(x) \neq X$ for any $x \in X$, is syndetic topologically sensitive.

Theorem 4.9. *Let (T, X, π) be a syndetically transitive, nonminimal semiflow, where X is a T_3 space. Then (T, X, π) is syndetic topologically sensitive.*

Proof. Let x be a point such that $T(x)$ is not dense in X and U be a nonempty open set with $T(x) \cap U = \emptyset$. Now, since X is a T_3 space there exist nonempty open sets V, V_1 such that $V \subset \overline{V} \subset V_1 \subset \overline{V_1} \subset U$. Take open cover $\mathcal{U} = \{V_1, X/\overline{V}\}$ and any nonempty open set $G \subset X$ then $N(V, G)$ is syndetic. Let K_1 be the compact set such that for all $t \in T$ there exists a $t_1 \in K_1$ such that $t + t_1 \in N(V, G)$.

Now, for any $r \in T$, $r.x \in U \subset X/\overline{V}_1$. So, $r^{-1}.(X/\overline{V}) \neq \emptyset$ and by Lemma 4.3, $\bigcap_{r \in K_1} r^{-1}.(X/\overline{V}_1)$ is open and hence $N(\bigcap_{r \in K_1} r^{-1}.(X/\overline{V}_1), G)$ is syndetic. Let K_2 be a compact set such that for all $t \in T$ there exists a $t_2 \in K_2$ such that $t + t_2 \in N(\bigcap_{r \in K_1} r^{-1}.(X/\overline{V}_1), G)$. Since K_1, K_2 are compact sets therefore, $K_2 + K_1$ is also compact. Take any $t \in T$ then there exists a $t_2 \in K_2$ such that $t + t_2 \in N(\bigcap_{r \in K_1} r^{-1}.(X/\overline{V}_1), G)$. Consequently, there exists an $x \in G$ such that $(t + t_2 + r).x \in (X/\overline{V}_1)$ for all $r \in K_1$. Now, for $t + t_2 \in T$ there exists a $t_1 \in K_1$ such that $t + t_2 + t_1 \in N(V, G)$ then there will exist a $y \in G$ such that $(t + t_2 + t_1).y \in V$ and for $r = t_1$ we have $(t + t_2 + t_1).x \in (X/\overline{V}_1)$. Thus, for any $t \in T$ there exist $t_2 + t_1 \in K_2 + K_1$ such that $(t + t_2 + t_1).x \notin V_1$, $(t + t_2 + t_1).y \notin (X/\overline{V})$, i.e., $G \times G \not\subset \{(t + t_2 + t_1)^{-1}B \times (t + t_2 + t_1)^{-1}B$ for all $B \in \mathcal{U}\}$. Therefore, $D(G, \mathcal{U})$ is syndetic and hence (T, X, π) is syndetic topologically sensitive. □

Corollary 4.10. *Let (T, X, π) be a transitive, nonminimal semiflow with dense set of almost periodic points, where X is a T_3 space. Then (T, X, π) is syndetic topologically sensitive.*

Proof. The proof is similar to the proof of Corollary 4.5. □

Next we give an example of a transitive, nonminimal semiflow (T, X, π), where X is a nonmetrizable T_3 space having a dense set of almost periodic points so by above result (T, X, π) is syndetic topologically sensitive.

Example 4.11. Consider the set $X = S^1 \times I$, where $I = [0, 1]$ and $S^1 = \mathbb{R}/\mathbb{Z}$. For $a \in X,\ r > 0$ we write $B(a, r) = \{b : d(b, a) < r\}$, where d is the usual metric on X.

Consider the topology τ on X generated by the basic open sets $B(a, r)$ if $B(a, r)$ is not tangent to $\{(x, y) : x \in S^1,\ y = 0\}$ and $B(a, r) \cup \{c\}$ if $B(a, r)$ is tangent to $\{(x, y) : x \in S^1,\ y = 0\}$ with c being the point at which $B(a, r)$ touches $\{(x, y) : x \in S^1,\ y = 0\}$. Then topological space X is T_3 but not T_4. Next, consider the map $g : S^1 \to S^1$ by $g(x) = x + \alpha$, where α is an irrational number and $h : I \to I$ by $h(y) = \{2y : 0 \leq y < \frac{1}{2}, 2 - 2y : \frac{1}{2} \leq y \leq 1\}$. Define $f : X \to X$ by $f(x, y) = (g(x), h(y))$ then the map $f : X \to X$ is continuous. Take discrete semigroup $T = \{1, 2, \dots\} = \mathbb{N}$ with usual operation of addition. Define map $\pi : T \times X \to X$ by $\pi(t, (x, y)) = f^t(x, y)$. Then (T, X, π) is a semiflow.

First, we show that (T, X, π) is transitive. Let $U, V \subset X$ be any open sets. Then there exist U_1, V_1 nonempty open in S^1 and U_2, V_2 nonempty open in I such that $U_1 \times U_2 \subset U$ and $V_1 \times V_2 \subset V$. Now, there exists an $n \in \mathbb{N}$ such that $h^n(U_2) = I$ and $g^n(U_1) \cap V_1 \neq \emptyset$, so $g^n(U_1) \cap V_1 \neq \emptyset$ and $h^n(U_2) \cap V_2 \neq \emptyset$ and hence, $f^n(U) \cap (V) \neq \emptyset$. Thus, (T, X, π) is transitive.

Next, we show that almost periodic points of (T, X, π) are dense in X. Let V be any nonempty open subset of X. Then there exist nonempty open set $U'_1 \subset S^1$ and nonempty open set $V'_1 \subset I$, such that $U'_1 \times V'_1 \subset V$. Now there exist a $y \in V'_1$ and an $n_0 \in \mathbb{N}$ such that $h^{n_0}(y) = y$. We show that for any $x \in U'_1$ $(x, y) \in V$ is an almost periodic point. Since every point of S^1 is an almost periodic point for map $e(x) = x + n_0\alpha$ on S^1. So, for any $U_1 \times V_1 \subset X$ containing (x, y) there exists a $t \in \mathbb{N}$ such that $e^t(x) \in U_1$, i.e., $g^{n_0 t}(x) \in U_1$. So, $h^{n_0 t}(x, y) \in U_1 \times V_1$ implying that (x, y) is an almost periodic point. So, almost periodic points are dense in X.

Thus (T, X, π) has a dense set of almost periodic points. Since, $T.(S^1 \times \{0\}) \subset S^1 \times \{0\}$, therefore (T, X, π) is nonminimal. So, (T, X, π) is transitive, nonminimal with a dense set of almost periodic points on a T_3 space implying that (T, X, π) is syndetic topologically sensitive.

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