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LIST INJECTIVE COLORING OF PLANAR GRAPHS WITH GIRTH AT LEAST FIVE

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Abstract. A vertex coloring of a graph G is called injective if any two vertices with a common neighbor receive distinct colors. A graph G is injectively k-choosable if any list L of admissible colors on $V(G)$ of size k allows an injective coloring φ such that $\varphi(v) \in L(v)$ whenever $v \in V(G)$. The least k for which G is injectively k-choosable is denoted by $\chi_i^l(G)$. For a planar graph G, Bu et al. proved that $\chi_i^l(G) \leq \Delta + 6$ if girth $g \geq 5$ and maximum degree $\Delta(G) \geq 8$. In this paper, we improve this result by showing that $\chi_i^l(G) \leq \Delta + 6$ for $g \geq 5$ and arbitrary $\Delta(G)$.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, $\delta(G)$ and $g(G)$ be the vertex set, edge set, face set, maximum degree, minimum degree and girth of G, respectively, and let $N_G(v) = \{u \mid uv \in E(G)\}$.

An *injective* k-coloring of a graph G is a mapping c: $V(G) \to \{1, 2, \ldots, k\}$ such that for any two vertices $u, v \in V(G)$, $c(u) \neq c(v)$ if $N(u) \cap N(v) \neq \emptyset$. The *injective chromatic number* of G, denoted by $\chi_i(G)$, is the least integer k such that G has an injective k-coloring.

A *list assignment* of a graph G is a mapping L which assigns a color list $L(v)$ to each vertex $v \in V(G)$. Given a list assignment L of G, an injective coloring φ of G is called an *injective L-coloring* if $\varphi(v) \in L(v)$ for each $v \in V(G)$. A graph G is injectively *k-choosable* if G has an injective L-coloring for any list assignment L with $|L(v)| \geq k$ for each $v \in V(G)$. The *injective choosability number* of G, denoted by $\chi_i^l(G)$, is the least integer k such that G is injectively k-choosable. Note that $\chi_i(G) \leq \chi_i^l(G)$ for every graph G. Borodin et al. [1] proved that for a planar graph, $\chi_i^l(G) = \chi_i(G) = \Delta$ if $\Delta \geq 16$ and $g = 7$, or $\Delta \geq 10$ and $8 \leq g \leq 9$, $\Delta \geq 6$ and $10 \leq g \leq 11$, or $\Delta = 5$ and $g \geq 12$.

A 2-*distance* k-coloring of a graph G is a mapping c: $V(G) \to \{1, 2, \ldots, k\}$ such that for any two vertices $u, v \in V(G)$, $c(u) \neq c(v)$ if $1 \leq d(v_1, v_2) \leq 2$.

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The 2-*distance chromatic number* of G, denoted by $\chi_2(G)$, is the least integer k such that G has a 2-distance k-coloring.

The concept of injective coloring was introduced by Hahn et al. [15] in 2002. They showed the injective chromatic number of complete graphs, paths, cycles, stars and proved that $\chi(G) \leq \chi_i(G) \leq \Delta^2(G) - \Delta(G) + 1$ if G is connected and $G \neq K_2$.

Obviously, an injective coloring is not necessarily proper, and this is the only difference between an injective coloring and a 2-distance coloring. But if every edge of a graph is incident with a triangle, they are the same. For the 2-distance coloring of a planar graph, Wegner [19] posed the following conjecture in 1977.

Conjecture A. Let G be a planar graph with maximum degree Δ.
(1) $\chi_2(G) \leq 7$ if $\Delta = 3$;
(2) $\chi_2(G) \leq \Delta + 5$ if $4 \leq \Delta \leq 7$;
(3) $\chi_2(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$ if $\Delta \geq 8$.

On the trivial fact that $\chi_i(G) \leq \chi_2(G)$, in 2010, Lužar posed the following conjecture about planar graphs in [17]. The upper bounds are tight if Conjecture B is true.

Conjecture B. Let G be a planar graph with maximum degree Δ.
(1) $\chi_i(G) \leq 5$ if $\Delta = 3$;
(2) $\chi_i(G) \leq \Delta + 5$ if $4 \leq \Delta \leq 7$;
(3) $\chi_i(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$ if $\Delta \geq 8$.

Clearly, $\Delta(G) \leq \chi_i(G) \leq |V(G)|$, so it seems natural to describe graphs of $\chi_i(G) = \Delta(G)$. For a planar graph, the following sufficient conditions (in terms of g and Δ) are known: $\Delta \geq 71$ and $g \geq 7$ [2], $\Delta \geq 4$ and $g \geq 13$ [9], and $\Delta \geq 3$ and $g \geq 19$ [18].

Many researches about the injective chromatic number have been studied under the limitation of maximum degree Δ and maximum average degree $mad(G)$, where $\mathrm{mad}(G) = \max_{\emptyset \neq H \subseteq G}\{\frac{2|E(H)|}{|V(H)|}\}$, there are the following results.

Theorem 1. *Let G be a graph with maximum degree Δ.*

(1) $\chi_i(G) \leq \Delta + 3$ *if* $mad(G) < \frac{14}{5}$; $\chi_i(G) \leq \Delta + 4$ *if* $mad(G) < 3$; $\chi_i(G) \leq \Delta + 8$ *if* $mad(G) < \frac{10}{3}$ [14].

(2) $\chi_i^l(G) \leq \Delta + 2$ *if* $mad(G) < \frac{14}{5}$ *and* $\Delta \geq 4$; $\chi_i^l(G) \leq 5$ *if* $mad(G) < \frac{36}{13}$ *and* $\Delta = 3$ [10].

(3) $\chi_i^l(G) \leq \Delta + 2$ *if* $mad(G) < 3$ *and* $\Delta \geq 12$; $\chi_i^l(G) \leq \Delta + 4$ *if* $mad(G) < \frac{10}{3}$ *and* $\Delta \geq 30$; $\chi_i^l(G) \leq \Delta + 5$ *if* $mad(G) < \frac{10}{3}$ *and* $\Delta \geq 18$; $\chi_i^l(G) \leq \Delta + 6$ *if* $mad(G) < \frac{10}{3}$ *and* $\Delta \geq 14$ [16].

(4) $\chi_i(G) \leq \Delta + 1$ *if* $mad(G) \leq \frac{5}{2}$; $\chi_i(G) = \Delta$ *if* $mad(G) < \frac{42}{19}$ [9].

For a planar graph G with girth at least g, $\mathrm{mad}(G) < \frac{2g}{g-2}$. The issue of the injective chromatic number is discussed under the limitation of girth and maximum degree in [1, 4, 5, 7, 8, 11, 12], which can be described as follows.

Theorem 2. *Let G be a planar graph with $g(G) \geq g'$ and $\Delta(G) \geq D$.*

(1) *If $(g', D) \in \{(9,4),(7,7),(6,17)\}$, then $\chi_i(G) \leq \Delta + 1$.*

(2) *If $(g', D) \in \{(7,1),(6,9)\}$, then $\chi_i(G) \leq \Delta + 2$.*

(3) *If $(g', D) = \{(8,5)\}$, then $\chi_i^l(G) \leq \Delta + 1$. If $(g', D) = \{(6,8)\}$, then $\chi_i^l(G) \leq \Delta + 2$. If $(g', D) = \{(6,24)\}$, then $\chi_i^l(G) \leq \Delta + 1$. If $g' = 6$, then $\chi_i^l(G) \leq \Delta + 3$.*

For a planar graph G with girth $g \geq 5$, Bu et al. [7] proved that if $\Delta \geq 8$, then $\chi_i^l(G) \leq \Delta + 6$. In [5], they proved that if $\Delta \geq 13$, then $\chi_i^l(G) \leq \Delta + 4$, and for any Δ, $\chi_i^l(G) \leq \Delta + 7$, and in [3], they improved the result and showed that if $\Delta \geq 11$, then $\chi_i^l(G) \leq \Delta + 4$. In [6], Bu et al. proved that if $\Delta \geq 10$, then $\chi_i^l(G) \leq \Delta + 5$. So far, for a planar graph G with girth $g \geq 5$ and for any Δ, the best result of injective chromatic number is $\chi_i(G) \leq \Delta + 6$ [13]. In this paper, we improve these results by proving the following theorem, which is closer to Conjecture B.

Theorem 3. *If G is a planar graph with girth $g(G) \geq 5$, then $\chi_i^l(G) \leq \Delta + 6$.*

2. Structural properties of critical graphs

A graph G is called *k-critical* if G does not admit any injective L-coloring with $|L(v)| \geq k$ for each $v \in V(G)$, but any subgraph G does. In this section, we will investigate some structural properties of critical graphs.

For convenience, we introduce some notations. A k-, k^+- or k^-- vertex is a vertex of degree k, at least k, or at most k, respectively. Similarly, we can define the k-, k^+- or k^--face. A k-, k^+- or k^--neighbor of v is a k-, k^+- or k^--vertex adjacent to v. For each $v \in V(G)$, let $v_1, v_2, \ldots, v_{d(v)}$ be the neighbors of v with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_{d(v)})$. Let $n_k(v)$ be the number of k-neighbors of v, $n_{k^+}(v)$ be the number of k^+-neighbors of v, and $S_G(v) = \sum_{u \in N(v)}(d(u) - 1) = \sum_{u \in N(v)} d(u) - d(v)$. Obviously, the number of vertices that have a common neighbor with v in G is at most $S_G(v)$. So, every vertex v has at most $S_G(v)$ forbidden colors if the other vertices are injectively colored. Let G be a $(\Delta + 6)$-critical graph, a 3-vertex v of G is called *bad* if $S_G(v) \leq \Delta + 5$. For integers k and d, a $k(d)$-vertex is a k-vertex adjacent to d 2-vertices.

At the end of this section, we present the following properties of $(\Delta + 6)$-critical graphs which have been proved in [4].

Lemma 4. $\delta(G) \geq 2$.

Lemma 5. *For any edge $uv \in E(G)$, $\max\{S_G(u), S_G(v)\} \geq \Delta + 6$.*

Lemma 6. *G has no adjacent 2-vertices.*

Lemma 7. *Suppose that $3 \leq d(v) \leq 7$. If v_1 is a 2-neighbor of v, $r = n_{3^+}(v)$ and u_i $(i = 1, \ldots, r)$ is the 3^+-neighbor of v, then*

(1) *$r \geq 2$,*

(2) *$\sum_{i=1}^{r} d(u_i) \geq \Delta + 6 + 2r - d(v)$.*

3. Proof of Theorem 3

In this section, we always assume that a planar graph G has been embedded in the plane. The theorem is proved by contradiction. Suppose that the theorem is false. Let G be a $(\Delta+6)$-critical graph. It is easy to see that G is connected and $\delta(G) \geq 2$.

We apply a discharging procedure to complete the proof by showing that G does not exist. We assign to each vertex v a charge $\omega(v)$ such that $\omega(v) = 3d(v) - 10$ and to each face f a charge $\omega(f) = 2d(f) - 10$. Applying Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and the Handshaking Lemmas for vertices and faces for a plane graph, we have

$$\sum_{x \in V \cup F} \omega(x) = -20.$$

If we obtain a new weight $\omega^*(x)$ for all $x \in V(G) \cup F(G)$ by transferring weights from one element to another, then we also have $\sum \omega^*(x) = -20$. If these transfers result in $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and the theorem is proved.

In [4], Bu et al. have proved that for a planar graph with girth $g \geq 5$ and $\Delta \geq 8$, $\chi_i^l(G) \leq \Delta + 6$. In the following, we only need to consider the case when $\Delta \leq 7$.

Claim 1. The following configurations are forbidden.
(1) A 5(3)-vertex adjacent to a 5(1)-vertex.
(2) A 6(4)-vertex adjacent to a 4(1)-vertex.
(3) A 6(4)-vertex adjacent to a 5(3)-vertex.
(4) A 6(3)-vertex adjacent to two 4(2)-vertices.
(5) A 6(3)-vertex adjacent to a 5(3)-vertex, a 3-vertex and a 4-vertex.
(6) A 7(5)-vertex adjacent to a 3(1)-vertex.

Proof. For (1), suppose that v is a 5(3)-vertex with $d(v_i) = 2$ for $1 \leq i \leq 3$ and $d(v_4) = 5$, where v_4 is a 5(1)-vertex. Let u be the adjacent 2-vertex of v_4. For convenience, we assume that $d(v_5) = \Delta$. Let L be an arbitrary list assignment of G with $|L(x)| \geq \Delta + 6$ for each $x \in V(G)$. By the choice of G, $G - vv_1$ has an injective L-coloring c. Now we erase the colors on u, v and v_1. Our aim is to recolor u, v and v_1 to extend c from $G - vv_1$ to the whole graph G to obtain a contradiction. Let $L'_c(v)$ be the set of available colors of v. Obviously,

$$\begin{aligned} L'_c(v_1) &\geq \Delta + 6 - (\Delta + 5 - d(v_1)) \geq 3, \\ L'_c(v) &\geq \Delta + 6 - (2 \times 3 + 5 + \Delta - d(v) - 1) \geq 1, \\ L'_c(u) &\geq \Delta + 6 - (\Delta + 5 - d(u) - 1) \geq 4. \end{aligned}$$

So we can recolor v, v_1, u in turn. The obtained coloring is an injective L-coloring of G.

For (2), the proof is quite similar to that of (1), and we omit it.

For (3), suppose that v is a 6(4)-vertex with $d(v_i) = 2$ for $1 \le i \le 4$ and $d(v_5) = 5$, where v_5 is a 5(3)-vertex. Let u, w, z be the adjacent 2-vertices of v_5. For convenience, we consider the worst case and assume that $d(v_6) = \Delta$. Let L be an arbitrary list assignment of G with $|L(x)| \ge \Delta+6$ for each $x \in V(G)$. By the choice of G, $G - vv_1$ has an injective L-coloring c. Now we erase the colors on v_1, v, u, w, z. Our aim is to recolor v_1, v, u, w, z to extend c from $G - vv_1$ to the whole graph G to obtain a contradiction. Let $L'_c(v)$ be the set of available colors of v. Obviously,

$$\begin{aligned} L'_c(v_1) &\ge \Delta + 6 - (\Delta + 6 - d(v_1)) \ge 2, \\ L'_c(v) &\ge \Delta + 6 - (2 \times 4 + 5 + \Delta - d(v) - 3) \ge 2, \\ L'_c(u) &\ge \Delta + 6 - (\Delta + 5 - d(u) - 3) \ge 6, \\ L'_c(w) &\ge \Delta + 6 - (\Delta + 5 - d(w) - 3) \ge 6, \\ L'_c(z) &\ge \Delta + 6 - (\Delta + 5 - d(z) - 3) \ge 6. \end{aligned}$$

So we can recolor v, v_1, u, w, z in turn to obtain an injective L-coloring of G.

For (4), suppose that v is a 6(3)-vertex with $d(v_i) = 2$ for $1 \le i \le 3$ and $d(v_4) = d(v_5) = 4$. Let x_i, y_i, z_i be the other neighbor of v_i and $d(x_i) = d(y_i) = 2$ for $i = 4, 5$, respectively. For convenience, we consider the worst case and assume that $d(v_6) = \Delta$. Let L be an arbitrary list assignment of G with $|L(x)| \ge \Delta+6$ for each $x \in V(G)$. By the choice of G, $G - vv_1$ has an injective L-coloring c. Now we erase the colors on $v_1, v, x_4, y_4, x_5, y_5$. Our aim is to recolor $v_1, v, x_4, y_4, x_5, y_5$ to extend c from $G - vv_1$ to the whole graph G to obtain a contradiction. Let $L'_c(v)$ be the set of available colors of v. Obviously,

$$\begin{aligned} L'_c(v_1) &\ge \Delta + 6 - (\Delta + 6 - d(v_1)) \ge 2, \\ L'_c(v) &\ge \Delta + 6 - (2 \times 3 + 2 \times 4 + \Delta - d(v) - 4) \ge 2, \\ L'_c(x_4) &\ge \Delta + 6 - (\Delta + 4 - d(x_4) - 2) \ge 6, \\ L'_c(y_4) &\ge 6, \ L'_c(x_5) \ge 6, \ L'_c(y_5) \ge 6. \end{aligned}$$

So we can recolor $v, v_1, x_4, y_4, x_5, y_5$ in turn. The obtained coloring is an injective L-coloring of G.

For (5), suppose that v is a 6(3)-vertex with $d(v_i) = 2$ for $1 \le i \le 3$, $d(v_4) = 3$, $d(v_5) = 4$ and $d(v_6) = 5$, where v_6 is a 5(3)-vertex. Let u, w, z be the adjacent 2-vertices of v_6. Let L be an arbitrary list assignment of G with $|L(x)| \ge \Delta + 6$ for each $x \in V(G)$. By the choice of G, $G - vv_1$ has an injective L-coloring c. Now we erase the colors on v_1, v, u, w, z. Our aim is to recolor v_1, v, u, w, z to extend c from $G - vv_1$ to the whole graph G to obtain a contradiction. Let $L'_c(v)$ be the set of available colors of v. Obviously,

$$\begin{aligned} L'_c(v_1) &\ge \Delta + 6 - (\Delta + 6 - d(v_1)) \ge 2, \\ L'_c(v) &\ge \Delta + 6 - (2 \times 3 + 3 + 4 + 5 - d(v) - 3) \ge 3, \\ L'_c(u) &\ge \Delta + 6 - (\Delta + 5 - d(u) - 3) \ge 6, \ L'_c(w) \ge 6, \ L'_c(z) \ge 6. \end{aligned}$$

So we can recolor v, v_1, u, w, z in turn to obtain an injective L-coloring of G.

For (6), suppose that v is a 7(5)-vertex with $d(v_i) = 2$ for $1 \le i \le 5$, $d(v_6) = 3$, where v_6 is a 3(1)-vertex. Let u be the adjacent 2-vertices of v_6. For convenience, we assume that $d(v_7) = \Delta$. Let L be an arbitrary list assignment of G with $|L(x)| \ge \Delta+6$ for each $x \in V(G)$. By the choice of G, $G-vv_1$ has an injective L-coloring c. Now we erase the colors on v_1, v, u. Our aim is to recolor v_1, v, u to extend c from $G-vv_1$ to the whole graph G to obtain a contradiction. Let $L'_c(v)$ be the set of available colors of v. Obviously, $L'_c(v_1) \ge 1$, $L'_c(v) \ge 1$, $L'_c(u) \ge 6$. So we can recolor v, v_1, u in turn to obtain an injective L-coloring of G. □

We list the following discharging rules.

R1. Each 2-vertex receives 2 from each adjacent 3^+-vertex.

R2. Suppose $d(v) = 3$ and $uv \in E$.
- (1) A 3(1)-vertex receives $\frac{3}{2}$ from each adjacent 7-vertex.
- (2) Suppose $d(u) = 3$ and $S_G(u) \ge \Delta + 6$. If $S_G(v) \le \Delta + 5$, then v receives $\frac{1}{3}$ from u. Otherwise, v receives nothing from u.
- (3) If $4 \le d(u) \le 5$, then v receives $\frac{1}{3}$ from u.
- (4) If $6 \le d(u) \le 7$ and v is adjacent to a bad 3-vertex, then v receives $\frac{2}{3}$ from u. Otherwise, v receives $\frac{1}{2}$ from u except v is a 3(1)-vertex.

R3. Each 4(2)-vertex receives 1 from each adjacent 6^+-vertex. Each 4(1)-vertex receives $\frac{1}{6}$ from each adjacent 5^+-vertex.

R4. Each 5(3)-vertex receives $\frac{1}{2}$ from each adjacent 5(0)-vertex, $\frac{1}{2}$ from each adjacent 6^+-vertex.

Let f be a k-face of G, $k \ge 5$. Obviously, $\omega^*(f) = 2k - 10 \ge 0$.

Let v be a k-vertex of G, $k \ge 2$. We will check that each vertex has a non-negative charge after the discharging process.

If $k = 2$, then $\omega(v) = -4$. By R1, $\omega^*(v) = -4 + 2 \times 2 = 0$.

If $k = 3$, then $\omega(v) = -1$. By Lemma 7(1), $n_2(v) \le 1$. If $n_2(v) = 1$, then $d(v_2) + d(v_3) \ge \Delta + 7$ by Lemma 7(2), that is, $\Delta = 7$ and $d(v_2) = d(v_3) = 7$. So $\omega^*(v) = -1 - 2 + 2 \times \frac{3}{2} = 0$ by R1 and R2(1).

Suppose $n_2(v) = 0$. If $2 \le n_3(v) \le 3$, it is obviously $S_G(v) \le \Delta + 5$. So $S_G(v_i) \ge \Delta + 6$ for $i = 1, 2, 3$. By R2, v receives at least $\frac{1}{3}$ from v_i. So $\omega^*(v) \ge -1 + 3 \times \frac{1}{3} = 0$.

Suppose $n_3(v) = 1$. If $S_G(v) \le \Delta+5$, then $S_G(v_1) \ge \Delta+6$ by Lemma 5. By R2, v receives at least $\frac{1}{3}$ from v_i, $i = 1, 2, 3$. So $\omega^*(v) \ge -1+3\times\frac{1}{3} = 0$. Suppose $S_G(v) \ge \Delta + 6$. Then $d(v_3) \ge d(v_2) \ge 6$. If $S_G(v_1) \le \Delta + 5$, then v sends $\frac{1}{3}$ to v_1 and receives $\frac{2}{3}$ from each of v_2 and v_3. So $\omega^*(v) \ge -1 - \frac{1}{3} + 2 \times \frac{2}{3} = 0$. If $S_G(v_1) \ge \Delta + 6$, then v receives $\frac{1}{2}$ from each of v_2 and v_3. So $\omega^*(v) \ge -1 + 2 \times \frac{1}{2} = 0$.

If $n_3(v) = 0$, then v receives at least $\frac{1}{3}$ from each adjacent vertex by R2. So $\omega^*(v) = -1 + 3 \times \frac{1}{3} = 0$.

If $k = 4$, then $\omega(v) = 2$. By Lemma 7(1), $n_2(v) \le 2$.

If $n_2(v) = 2$, then the other two adjacent vertices must be 6^+-vertices. By R3, $\omega^*(v) \geq 2 - 2 \times 2 + 2 \times 1 = 0$.

If $n_2(v) = 1$, then $n_3(v) \leq 1$. (If $n_3(v) \geq 2$, then $S_G(v) < \Delta + 6$, $S_G(v_1) < \Delta + 6$, a contradiction to Lemma 5.) If $n_3(v) = 1$, then v_3, v_4 are 5^+-vertices by Lemma 5. So $\omega^*(v) \geq 2 - 2 - \frac{1}{3} + 2 \times \frac{1}{6} = 0$ by R3. If $n_3(v) = 0$, then $\omega^*(v) \geq 2 - 2 = 0$.

If $n_2(v) = 0$, then $\omega^*(v) \geq 2 - 4 \times \frac{1}{3} > 0$.

If $k = 5$, then $\omega(v) = 5$. By Lemma 7(1), $n_2(v) \leq 3$. Suppose $n_2(v) = 3$. By Lemma 7(2), the other two adjacent vertices must be 5^+-vertices, and the 5-vertex adjacent to v must be a 5(0)-vertex by Claim 1(1). It follows that $\omega^*(v) \geq 5 - 3 \times 2 + 2 \times \frac{1}{2} = 0$ by R4. If $n_2(v) = 2$, then $n_3(v) \leq 1$ by Lemma 5. Hence, $\omega^*(v) \geq 5 - 2 \times 2 - 1 \times \frac{1}{3} - 2 \times \frac{1}{6} > 0$ by R1, R2, R3 and R4. If $n_2(v) = 1$, then $\omega^*(v) \geq 5 - 1 \times 2 - 4 \times \frac{1}{3} > 0$. If $n_2(v) = 0$, then v sends at most $\frac{1}{2}$ to its neighbors by R2, R3 and R4. So $\omega^*(v) \geq 5 - 5 \times \frac{1}{2} > 0$.

If $k = 6$, then $\Delta \geq 6$ and $\omega(v) = 8$. By Lemma 7(1), $n_2(v) \leq 4$. If $n_2(v) = 4$, then $d(v_5) + d(v_6) \geq \Delta + 4$ by Lemma 7(2). So $\min\{d(v_5), d(v_6)\} \geq 4$. By Claim 1(2) and Claim 1(3), v_5, v_6 are not 4(1)-vertices, 4(2)-vertices, 5(3)-vertices. By R3, R4, v sends nothing to v_5, v_6. So $\omega^*(v) \geq 8 - 2 \times 4 = 0$. If $n_2(v) = 3$, by Lemma 7(2), $d(v_4) + d(v_5) + d(v_6) \geq \Delta + 6 \geq 12$. So the degree sequence of v_4, v_5, v_6 is $(3, 3, 6^+)$, or $(3, 4, 5^+)$ (the 5-vertex is not a 5(3)-vertex by Claim 1(5)), or $(3, 5^+, 5^+)$, or $(4, 4^+, 4^+)$ (there exists at most one 4(2)-vertex by Claim 1(4)), or $(5^+, 5^+, 5^+)$. By R2, R3 and R4, v sends at most $\max\{2 \times \frac{2}{3}, \frac{2}{3} + 1, \frac{2}{3} + 2 \times \frac{1}{2}, 1 + 2 \times \frac{1}{2}, 3 \times \frac{1}{2}\} = 2$ to v_4, v_5 and v_6 in total. So $\omega * (v) \geq 8 - 3 \times 2 - 2 = 0$. If $0 \leq n_2(v) \leq 2$, then $\omega^*(v) \geq 8 - 2n_2(v) - 1 \times (k - n_2(v)) \geq 0$.

If $k = 7$, then $\Delta = 7$ and $\omega(v) = 11$. By Lemma 7(1), $n_2(v) \leq 5$.

Suppose $n_2(v) = 5$. By Lemma 7(2), $d(v_6) + d(v_7) \geq \Delta + 3 = 10$. So the degree sequence of (v_6, v_7) is (3,7), or $(4, 6^+)$ or $(5^+, 5^+)$. By Claim 1(6), a 7(5)-vertex is not adjacent to a 3(1)-vertex, and v sends at most $\max\{\frac{2}{3}, 1, 2 \times \frac{1}{2}\} = 1$ to v_6 and v_7 in total. So $\omega^*(v) \geq 11 - 5 \times 2 - 1 = 0$.

Suppose $n_2(v) = 4$. By Lemma 7(2), $d(v_5) + d(v_6) + d(v_7) \geq \Delta + 5 = 12$. So the degree sequence of (v_5, v_6, v_7) is $(3, 3, 6^+)$, or $(3, 4^+, 5^+)$ or $(4^+, 4^+, 4^+)$. Thus v sends at most $\max\{\frac{3}{2} \times 2, \frac{3}{2} + 1 + \frac{1}{2}, 3 \times 1\} = 3$ to v_5, v_6 and v_7 in total. So $\omega^*(v) \geq 11 - 4 \times 2 - 3 = 0$.

Suppose $n_2(v) = 3$. Then $d(v_4) + d(v_5) + d(v_6) + d(v_7) \geq \Delta + 7 = 14$ by Lemma 7(2). So $n_3(v) \leq 3$. If $n_3(v) = 3$, then $d(v_7) \geq 5$. So $\omega^*(v) \geq 11 - 3 \times 2 - 3 \times \frac{3}{2} - \frac{1}{2} = 0$. If $n_3(v) \leq 2$, then $\omega^*(v) \geq 11 - 3 \times 2 - 2 \times \frac{3}{2} - 2 \times 1 = 0$.

Suppose $n_2(v) = 2$. It is easy to check $n_3(v) \leq 4$. So $\omega^*(v) \geq 11 - 2 \times 2 - 4 \times \frac{3}{2} - 1 = 0$ by the discharging rules.

Suppose $n_2(v) \leq 1$. Then v sends 2 to each 2-neighbor and at most $\frac{3}{2}$ to each other neighbor. So $\omega^*(v) \geq 11 - 2 \times 1 - 6 \times \frac{3}{2} = 0$.

We have checked $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, a contradiction occurs because $0 \leq \sum_{x \in V \cup F} \omega^*(x) = \sum_{x \in V \cup F} \omega(x) = -20$.

This completes the proof when $\Delta \leq 7$ and hence that of the whole Theorem 3.

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References

[1] O. V. Borodin and A. O. Ivanova, *List injective colorings of planar graphs*, Discrete Math. **311** (2011), no. 2-3, 154–165. https://doi.org/10.1016/j.disc.2010.10.008

[2] Y. H. Bu, D. Chen, A. Raspaud, and W. Wang, *Injective coloring of planar graphs*, Discrete Appl. Math. **157** (2009), no. 4, 663–672. https://doi.org/10.1016/j.dam.2008.08.016

[3] Y. H. Bu and C. Huang, *List injective coloring of a class of planar graphs without short cycles*, Discrete Math. Algorithms Appl. **10** (2018), no. 5, 1850068, 12 pp. https://doi.org/10.1142/S1793830918500684

[4] Y. H. Bu and K. Lu, *Injective coloring of planar graphs with girth* 7, Discrete Math. Algorithms Appl. **4** (2012), no. 2, 1250034, 8 pp. https://doi.org/10.1142/S1793830912500346

[5] Y. H. Bu and K. Lu, *List injective coloring of planar graphs with girth* 5*,* 6*,* 8, Discrete Appl. Math. **161** (2013), no. 10-11, 1367–1377. https://doi.org/10.1016/j.dam.2012.12.017

[6] Y. H. Bu, C. Wang, and S. Yang, *List injective coloring of planar graphs*, Ars Combin. **141** (2018), 191–211.

[7] Y. H. Bu and S. Yang, *List injective coloring of planar graphs with girth* $g \geq 5$, Discrete Math. Algorithms Appl. **6** (2014), no. 1, 1450006, 12 pp. https://doi.org/10.1142/S1793830914500062

[8] H. Y. Chen and J. L. Wu, *List injective coloring of planar graphs with girth* $g \geq 6$, Discrete Math. **339** (2016), no. 12, 3043–3051. https://doi.org/10.1016/j.disc.2016.06.017

[9] D. W. Cranston, S.-J. Kim, and G. Yu, *Injective colorings of sparse graphs*, Discrete Math. **310** (2010), no. 21, 2965–2973. https://doi.org/10.1016/j.disc.2010.07.003

[10] D. W. Cranston, S.-J. Kim, and G. Yu, *Injective colorings of graphs with low average degree*, Algorithmica **60** (2011), no. 3, 553–568. https://doi.org/10.1007/s00453-010-9425-x

[11] D. Dimitrov, B. Lužar, and R.Škrekovski, *Injective coloring of planar graphs with* $\Delta + 2$ *colors*, preprint.

[12] W. Dong and W. Lin, *Injective coloring of planar graphs with girth* 6, Discrete Math. **313** (2013), no. 12, 1302–1311. https://doi.org/10.1016/j.disc.2013.02.014

[13] W. Dong and W. Lin, *Injective coloring of plane graphs with girth* 5, Discrete Math. **315** (2014), 120–127. https://doi.org/10.1016/j.disc.2013.10.017

[14] A. Doyon, G. Hahn, and A. Raspaud, *Some bounds on the injective chromatic number of graphs*, Discrete Math. **310** (2010), no. 3, 585–590. https://doi.org/10.1016/j.disc.2009.04.020

[15] G. Hahn, J. Kratochvíl, J. Širáň, and D. Sotteau, *On the injective chromatic number of graphs*, Discrete Math. **256** (2002), no. 1-2, 179–192. https://doi.org/10.1016/S0012-365X(01)00466-6

[16] R. Li and B. G. Xu, *Injective choosability of planar graphs of girth five and six*, Discrete Math. **312** (2012), no. 6, 1260–1265. https://doi.org/10.1016/j.disc.2011.10.029

[17] B. Lužar, *Planar graphs with largest injective chromatic numbers*, in: IMFM Preprint Series, 48, 2010, p. 1110.

[18] B. Lužar, R. Škrekovski, and M. Tancer, *Injective colorings of planar graphs with few colors*, Discrete Math. **309** (2009), no. 18, 5636–5649. https://doi.org/10.1016/j.disc.2008.04.005

[19] G. Wegner, *Graphs with Given Diameter and a Coloring Problem*, Technical Report, University of Dortmund, Germany, 1977.

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