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AN ABELIAN CATEGORY OF WEAKLY COFINITE MODULES

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Abstract. Let I be an ideal of a commutative Noetherian semi-local ring R and M be an R-module. It is shown that if $\dim M \leq 2$ and $\operatorname{Supp}_R M \subseteq V(I)$, then M is I-weakly cofinite if (and only if) the R-modules $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}^1_R(R/I, M)$ are weakly Laskerian. As a consequence of this result, it is shown that the category of all I-weakly cofinite modules X with $\dim X \leq 2$, forms an Abelian subcategory of the category of all R-modules. Finally, it is shown that if $\dim R/I \leq 2$, then for each pair of finitely generated R-modules M and N and each pair of the integers $i, j \geq 0$, the R-modules $\operatorname{Tor}^R_i(N, H^j_I(M))$ and $\operatorname{Ext}^i_R(N, H^j_I(M))$ are I-weakly cofinite.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring, and I be an ideal of R. For an R-module M, the ith local cohomology module of M with support in $V(I)$ is defined as:

$$H^i_I(M) = \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [8] or [13] for more details about local cohomology.

Hartshorne in [14] defined an R-module X to be I-*cofinite* if $\operatorname{Supp} X \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, X)$ is finitely generated for all $i \in \mathbb{N}_0$. Then he asked the following questions:

Question 1. For which Noetherian rings R and ideals J of R, the modules $H^i_J(M)$ are J-cofinite for all finitely generated R-modules M and all $i \in \mathbb{N}_0$?

Question 2. Whether the category of I-cofinite modules is an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-cofinite modules, are $\ker f$ and $\operatorname{coker} f$ I-cofinite?

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Concerning Question 1, there are several results in the literature (see [3–5, 9–11, 16, 17, 20, 26]).

Recall that an R-module M is said to be *weakly Laskerian* or *skinny* if the set of associated primes of any quotient module of M is finite. Bahmanpour in [2] proved that a given module M over a Noetherian ring R is weakly Laskerian if and only if it is FSF; see [2, Theorem 3.3]. Recall that by Quy's definition [25, Definition 2.1], an R-module M is said to be FSF if it possesses a finitely generated submodule N such that $\operatorname{Supp} M/N$ is a finite set. We recall that, if I is an ideal of R, then an R-module X is said to be I-*weakly cofinite* if $\operatorname{Supp} X \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, X)$ is weakly Laskerian for all $i \geq 0$.

In the sequel, we denote by $\mathscr{C}(R,I)_{cof}$ and $\mathscr{C}(R,I)_{wcof}$ the category of all I-cofinite modules and the category of all I-weakly cofinite modules, respectively. Also, in this paper for each integer $n \geq 0$, the symbols $\mathscr{C}^n(R,I)_{cof}$ and $\mathscr{C}^n(R,I)_{wcof}$ denote the category of all I-cofinite modules X with $\dim X \leq n$ and the category of all I-weakly cofinite modules Y with $\dim Y \leq n$, respectively.

With respect to Question 2, there are several papers devoted to this question; for example see [6, 11, 12, 14, 15, 18, 19, 22–24].

In [6] it was shown that for each ideal I of a Noetherian ring R, $\mathscr{C}^1(R,I)_{cof}$ is an Abelian category. A similar result was obtained for $\mathscr{C}^1(R,I)_{wcof}$ in [7].

In this paper we prove that $\mathscr{C}^2(R,I)_{wcof}$ is also an Abelian category provided that R has only finitely many maximal ideals. In order to prove this assertion, first we prove that for an ideal I of a semi-local ring R and a given R-module M, with $\dim M \leq 2$ and $\operatorname{Supp}_R M \subseteq V(I)$, M is I-weakly cofinite if (and only if) the R-modules $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}^1_R(R/I, M)$ are weakly Laskerian. Finally, we prove that if $\dim R/I \leq 2$, then for each pair of finitely generated R-modules M and N and each pair of the integers $i, j \geq 0$, the R-modules $\operatorname{Tor}^R_i(N, H^j_I(M))$ and $\operatorname{Ext}^i_R(N, H^j_I(M))$ are I-weakly cofinite. These results generalize the main results of E. Hatami and M. Aghapournahr in [15].

For each ideal I of a Noetherian ring R and each R-module M, we denote the submodule $\bigcup_{n=1}^{\infty}(0 :_M I^n)$ of M by $\Gamma_I(M)$. We denote $\operatorname{Supp}_R R/I = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$. For any unexplained notation and terminology we refer the reader to [8, 21].

2. The results

We start this section with some auxiliary lemmas.

Lemma 2.1 (see [15, Lemma 2.5]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be an R-module. Then the R-module M is weakly Laskerian if and only if the $\widehat{R}$-module $M \otimes_R \widehat{R}$ is weakly Laskerian.*

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I be an ideal of R and let M be an R-module. Then $M \in \mathscr{C}(R,I)_{wcof}$ if and only if $M \otimes_R \widehat{R} \in \mathscr{C}(\widehat{R}, I\widehat{R})_{wcof}$.*

Proof. The assertion follows immediately from Lemma 2.1. □

Lemma 2.3 (see [6, Proposition 2.6])**.** *Let I be an ideal of a Noetherian ring R and M be an R-module such that* $\dim M \leq 1$ *and* $\operatorname{Supp}_R M \subseteq V(I)$*. Then the following statements are equivalent*:

(i) *M is I-cofinite.*
(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

Lemma 2.4 (see [2, Theorem 3.3])**.** *Let R be a Noetherian ring and let M be an R-module. Then M is a weakly Laskerian R-module if and only if M possesses a finitely generated submodule N such that* $\operatorname{Supp}_R M/N$ *is a finite set.*

The following lemma plays a key role in the proof of Theorem 2.6.

Lemma 2.5. *Let I be an ideal of a Noetherian ring local ring* $(R, \mathfrak{m})$ *and let M be an R-module with* $\dim M \leq 2$ *and* $\operatorname{Supp}_R M \subseteq V(I)$*. Then the following statements are equivalent*:

(i) *M is I-weakly cofinite.*
(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are weakly Laskerian.*

Proof. (i)$\Longrightarrow$(ii) The assertion is clear.

(ii)$\Longrightarrow$(i) In contrary assume that M is not I-weakly cofinite. By using Lemma 2.1 and Corollary 2.2, without loss of generality we may assume that $(R, \mathfrak{m})$ is a complete Noetherian local ring. Now by the definition there exists an integer $j \geq 2$ such that the R-module $\operatorname{Ext}^j_R(R/I, M)$ is not weakly Laskerian. By Lemma 2.4 there are finitely generated submodules $U \subseteq \operatorname{Hom}_R(R/I, M)$ and $V \subseteq \operatorname{Ext}^1_R(R/I, M)$ such that the set

$$\Lambda := \{\mathfrak{m}\} \bigcup \big(\operatorname{Supp}_R \operatorname{Hom}_R(R/I, M)/U\big) \bigcup \big(\operatorname{Supp}_R \operatorname{Ext}^1_R(R/I, M)/V\big)$$

is finite. In this situation it is straightforward to see that $\dim R/\mathfrak{p} \leq 1$ for each $\mathfrak{p} \in \Lambda$. Since the R-module $\operatorname{Ext}^j_R(R/I, M)$ is not weakly Laskerian it follows that the R-module $\operatorname{Ext}^j_R(R/I, M)$ has a submodule W such that the set

$$T = \operatorname{Ass}_R \operatorname{Ext}^j_R(R/I, M)/W$$

is infinite. Therefore, there exists a countably infinite subset $\Omega = \{\mathfrak{q}_k\}_{k=1}^{\infty}$ of T such that $\Omega \cap \Lambda = \emptyset$. Then, we claim that

$$\left(\bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}\right) \not\subseteq \left(\bigcup_{k=1}^{\infty} \mathfrak{q}_k\right).$$

Assume the opposite. Then, by [20, Lemma 3.2], there exists an integer $n \geq 1$ such that $\cap_{\mathfrak{p}\in\Lambda} \mathfrak{p} \subseteq \mathfrak{q}_n$. Since $\mathfrak{q}_n \neq \mathfrak{m}$ we see that there is an element $\mathfrak{p} \in \Lambda$

such that $\mathfrak{p} \neq \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{q}_n$. Now from the relation $\dim R/\mathfrak{p} \leq 1$ one can deduce that $\mathfrak{q}_n = \mathfrak{p} \in \Lambda$, which is a contradiction. So, we have

$$\left(\bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}\right) \nsubseteq \left(\bigcup_{k=1}^{\infty} \mathfrak{q}_k\right).$$

Therefore, one can find an element $x \in \cap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$ such that $x \notin \cup_{k=1}^{\infty} \mathfrak{q}_k$.

Let S denote the multiplicatively closed subset $\{1_R, x, x^2, x^3, \ldots\}$ of R. Then it is easy to see that

$$\operatorname{Supp}_{S^{-1}R} S^{-1}M \subseteq V(S^{-1}I),$$

and the $S^{-1}R$-module $S^{-1}M$ is of dimension not exceeding one. Also, one sees that the $S^{-1}R$-modules

$$\operatorname{Hom}_{S^{-1}R}(S^{-1}R/S^{-1}I, S^{-1}M) \simeq S^{-1}(\operatorname{Hom}_R(R/I, M)) \simeq S^{-1}U$$

and

$$\operatorname{Ext}^1_{S^{-1}R}(S^{-1}R/S^{-1}I, S^{-1}M) \simeq S^{-1}(\operatorname{Ext}^1_R(R/I, M)) \simeq S^{-1}V$$

are finitely generated. Hence, by Lemma 2.3 the $S^{-1}R$-module $S^{-1}M$ is $S^{-1}I$-cofinite. Consequently, the $S^{-1}R$-module

$$S^{-1}(\operatorname{Ext}^j_R(R/I, M)/W) \simeq \operatorname{Ext}^j_{S^{-1}R}(S^{-1}R/S^{-1}I, S^{-1}M)/S^{-1}W,$$

is finitely generated and hence the set

$$\operatorname{Ass}_{S^{-1}R} S^{-1}(\operatorname{Ext}^j_R(R/I, M)/W)$$

is finite. But

$$S^{-1}\mathfrak{q}_k \in \operatorname{Ass}_{S^{-1}R} S^{-1}(\operatorname{Ext}^j_R(R/I, M)/W)$$

for $k = 1, 2, \ldots$, which is a contradiction. □

The following theorem is the first main result of this paper.

Theorem 2.6. *Let I be an ideal of a semi-local Noetherian ring R and M be an R-module such that $\dim M \leq 2$ and $\operatorname{Supp}_R M \subseteq V(I)$. Then the following statements are equivalent:*

(i) *M is I-weakly cofinite.*
(ii) *The R-modules $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}^1_R(R/I, M)$ are weakly Laskerian.*

Proof. By using the localization at the maximal ideals of R, the assertion easily follows from Lemma 2.5. □

Now, we are ready to state and prove our second main result.

Theorem 2.7. *Let R be a Noetherian semi-local ring and I be an ideal of R. Then $\mathscr{C}^2(R, I)_{wcof}$ is Abelian.*

Proof. Suppose that M, $N \in \mathscr{C}^2(R,I)_{wcof}$ and let $f : M \longrightarrow N$ be an R-homomorphism. By the exact sequences

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \operatorname{im} f \longrightarrow 0, \tag{2.1}$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow N \longrightarrow \operatorname{coker} f \longrightarrow 0, \tag{2.2}$$

it is enough to prove that the R-module $\ker f$ is I-weakly cofinite. The exact sequence (2.1) yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/I, \ker f) \longrightarrow \operatorname{Hom}_R(R/I, M),$$

which implies that the R-module $\operatorname{Hom}_R(R/I, \ker f)$ is weakly Laskerian. Also, the exact sequence (2.2) induces the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/I, \operatorname{im} f) \longrightarrow \operatorname{Hom}_R(R/I, N),$$

which shows that the R-module $\operatorname{Hom}_R(R/I, \operatorname{im} f)$ is weakly Laskerian likewise.

Furthermore, from the exact sequence (2.1) we achieve the exact sequence

$$\operatorname{Hom}_R(R/I, \operatorname{im} f) \longrightarrow \operatorname{Ext}^1_R(R/I, \ker f) \longrightarrow \operatorname{Ext}^1_R(R/I, M),$$

which implies that the R-module $\operatorname{Ext}^1_R(R/I, \ker f)$ is weakly Laskerian too. Now, the assertion follows from Theorem 2.6. □

Corollary 2.8. *Let R be a Noetherian semi-local ring and I be an ideal of R such that* $\dim R/I \leq 2$. *Then $\mathscr{C}(R,I)_{wcof}$ is Abelian.*

Proof. From the assumption $\dim R/I \leq 2$ we get the relation $\mathscr{C}(R,I)_{wcof} = \mathscr{C}^2(R,I)_{wcof}$ and hence the assertion follows from Theorem 2.7. □

Corollary 2.9. *For each ideal I of a Noetherian semi-local ring R the following statements hold*:

(i) *Suppose that*

$$X^\bullet : \cdots \longrightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \longrightarrow \cdots$$

is a complex such that $X^i \in \mathscr{C}^2(R,I)_{wcof}$ for all $i \in \mathbb{Z}$. Then for each $i \in \mathbb{Z}$ the ith cohomology module $H^i(X^\bullet)$ is in $\mathscr{C}^2(R,I)_{wcof}$.

(ii) *Assume that $M \in \mathscr{C}^2(R,I)_{wcof}$ and N is a finitely generated R-module. Then for each $i \in \mathbb{N}_0$, the R-modules $\operatorname{Tor}^R_i(N,M)$ and $\operatorname{Ext}^i_R(N,M)$ are in $\mathscr{C}^2(R,I)_{wcof}$.*

Proof. (i) The assertion follows easily from Corollary 2.8.

(ii) Since N is finitely generated it follows that N has a free resolution with finitely generated free R-modules. Now the assertion follows from applying part (i) and computing the R-modules $\operatorname{Tor}^R_i(N,M)$ and $\operatorname{Ext}^i_R(N,M)$ by this free resolution. □

Lemma 2.10 (see [1, Lemma 2.3]). *Let I be an ideal of a Noetherian ring R and $\mathscr{M}$ be a Serre subcategory of the category of R-modules. Let $n \in \mathbb{N}_0$ and M be an R-module such that $\mathrm{Ext}^j_R(R/I, H^i_I(M)) \in \mathscr{M}$ for all $0 \leq i < n$ and all $j \in \mathbb{N}_0$. If the R-modules $\mathrm{Ext}^n_R(R/I, M)$ and $\mathrm{Ext}^{n+1}_R(R/I, M)$ are in $\mathscr{M}$, then the R-modules $\mathrm{Hom}_R(R/I, H^n_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^n_I(M))$ are in $\mathscr{M}$.*

Proposition 2.11. *For each ideal I of a Noetherian semi-local ring R with $\dim R/I \leq 2$, and each R-module M, the following statements are equivalent:*

(i) *For each $i \geq 0$, the R-module $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian.*
(ii) *For each $0 \leq i \leq \dim M$, the R-module $\mathrm{Ext}^i_R(R/I, M)$ is weakly Laskerian.*
(iii) *For each $i \geq 0$, the R-module $H^i_I(M)$ is I-weakly cofinite.*

Proof. (i)$\Longrightarrow$(ii) The assertion is clear.

(ii)$\Longrightarrow$(iii) By *Grothendieck's Vanishing Theorem*, one sees that $H^i_I(M) = 0$ for each $i \geq \dim M$. Therefore, it is enough to prove the assertion for each $0 \leq i \leq \dim M$. In order to prove this assertion, first we use induction on i for all $0 \leq i < \dim M$.

For $i = 0$, by Lemma 2.10, the R-modules

$$\mathrm{Hom}_R(R/I, \Gamma_I(M)),\ \mathrm{Ext}^1_R(R/I, \Gamma_I(M)),$$

are weakly Laskerian. Hence, by Theorem 2.6 the R-module $\Gamma_I(M)$ is I-weakly cofinite.

Suppose, inductively, that $0 < i < \dim M$ and the result has been proved for smaller values of i. Then by Lemma 2.10, the R-modules $\mathrm{Hom}_R(R/I, H^i_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^i_I(M))$ are weakly Laskerian and so by Theorem 2.6 the R-module $H^i_I(M)$ is I-weakly cofinite. This completes the inductive step. Now by [7, Lemma 2.1] the R-modules $\mathrm{Hom}_R(R/I, H^{\dim M}_I(M))$ is weakly Laskerian. Also, using the *Grothendieck's Vanishing Theorem* it can be seen that the R-module $H^{\dim M}_I(M)$ is of dimension not exceeding 0. Therefore,

$$\mathrm{Supp}\, H^{\dim M}_I(M) = \mathrm{Ass}_R \mathrm{Hom}_R(R/I, H^{\dim M}_I(M))$$

is a finite set, which means that $H^{\dim M}_I(M)$ is I-weakly cofinite as well.

(iii)$\Longrightarrow$(i) The assertion follows from [22, Proposition 3.9]. □

The following theorem is the final main result of this paper.

Theorem 2.12. *Let I be an ideal of a Noetherian semi-local ring R with $\dim R/I \leq 2$, and let M, N be two finitely generated R-modules. Then for each pair of integers $i, j \geq 0$, the R-modules $\mathrm{Tor}^R_i(N, H^j_I(M))$ and $\mathrm{Ext}^i_R(N, H^j_I(M))$ are in $\mathscr{C}(R, I)_{wcof}$.*

Proof. The assertion follows from Corollary 2.9 and Proposition 2.11. □

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