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STABILITY OF TOTAL SCALAR CURVATURE AND THE CRITICAL POINT EQUATION

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Abstract. We consider the total scalar curvature functional, and show that if the second variation in the transverse traceless tensor direction is negative, then the metric is Einstein. We also find the relation between the second variation and the Lichnerowicz Laplacian.

1. Introduction

A symmetric 2-tensor h on a Riemannian manifold (M, g) is called *transverse* if its (negative) divergence is vanishing, that is, if $\delta h = 0$. One can find this condition from the Einstein field equation [2]

$$r_g - \frac{s_g}{2} g = T,$$

where r_g and s_g denote the Ricci curvature and scalar curvature of the metric g, respectively, and T is the stress-energy tensor. Since it is well-known that $\delta r_g = -\frac{1}{2} ds_g$, we have $\delta T = 0$. When we consider the momentum constraint in the initial data problem for the vacuum Einstein equation, we often assume that $\mathrm{tr}_g T = 0$ (cf. [3]). A symmetric 2-tensor h satisfying these two conditions is called *transverse-traceless* (TT-tensor for short), a designation introduced by Arnowitt, Deser, and Misner [1].

In this paper, we consider the total scalar curvature functional restricted to the metrics of unit volume on a compact manifold and its second variation in the TT-tensor direction. Denoting the set of smooth Riemannian structures on a manifold M of unit volume by $\mathcal{M}_1$, the total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ is defined by

$$\mathcal{S}(g) = \int_M s_g dv_g.$$

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Mutō [6] proved the instability of total scalar curvature restricted to the metrics of constant volume. Viaclovsky [10] showed that if (M, g) is a space form of positive constant sectional curvature K, then the second variation is strictly negative when restricted to transverse-traceless variations. If $K = 0$, then the second variation is strictly negative except for parallel h.

In this paper we derive a partial converse of the above results. Namely, if the second variation is negative for any TT-tensor direction on a compact Riemanian manifold (M, g) with constant scalar curvature, then the metric should be Einstein. As a result, in the case of the critical point equation (see below for exact definition), we show that (M, g) is isometric to a standard sphere. We also find the relation between the second variation and the Lichnerowicz Laplacian.

Convention and notations: Basically, we follow curvature conventions and operator conventions in [2] except only one the Laplace operator. Hereafter, for convenience and simplicity, we denote curvatures r_g, z_g, s_g, and the Hessian and Laplacian of f, $D_g df, \Delta_g$ by r, z, s, and Ddf, Δ, respectively, if there is no ambiguity. Here, z_g is the traceless Ricci tensor of the metric. We also use the notation $\langle\ ,\ \rangle$ for metric g or inner product induced by g on tensor spaces.

2. First and second variations of the total scalar curvature

Let M^n be a smooth compact n-dimensional manifold and $S^2(M)$ be the space of symmetric 2-tensors on M. Take a one parameter deformation of the metric g up to order two lying in $\mathcal{M}_1$ given by

$$g_t = g + th + \frac{t^2}{2}\xi$$

for $h, \xi \in S^2(M)$. It is well known that

$$\frac{d}{dt}\bigg|_{t=0} (g_t)^{ij} = -g^{ik}g^{jl}h_{kl} \tag{2.1}$$

and

$$\frac{d}{dt}\bigg|_{t=0} dv_{g_t} = \frac{1}{2}(tr_g h)dv_g. \tag{2.2}$$

Differentiating all parts of $\mathrm{vol}(g_t) = \int_M dv_{g_t} \equiv 1$ with respect to t, we have

$$0 = \frac{d}{dt}\bigg|_{t=0} \mathrm{vol}(g_t) = \frac{1}{2}\int_M (tr_g h)dv_g. \tag{2.3}$$

Now, since

$$\begin{aligned}\frac{d}{dt}dv_{g_t} &= \frac{1}{2}tr_{g_t}(h + t\xi)dv_{g_t} = \frac{1}{2}\langle g_t, h + t\xi\rangle_{g_t} dv_{g_t}\\ &= \frac{1}{2}(g_t)^{ij}(g_t)^{kl}(g_t)_{ik}(h + t\xi)_{jl}dv_{g_t} = \frac{1}{2}(g_t)^{ij}\delta_{il}(h + t\xi)_{jl}dv_{g_t}\end{aligned}$$

$$= \frac{1}{2}(g_t)^{ij}(h+t\xi)_{ij}dv_{g_t},$$

by (2.1) we obtain

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} dv_{g_t} &= -\frac{1}{2}g^{ik}g^{jl}h_{ij}h_{kl}dv_g + \frac{1}{2}g^{ij}\xi_{ij}dv_g + \frac{1}{4}g^{ij}h_{ij}(tr_g h)dv_g \\ &= -\frac{1}{2}|h|^2 dv_g + \frac{1}{2}(tr_g\,\xi)dv_g + \frac{1}{4}(tr_g h)^2 dv_g.\end{aligned}$$

Therefore, by (2.2) and (2.3) we have

$$\int_M (tr_g\,\xi)dv_g = \int_M \left[|h|^2 - \frac{1}{2}(tr_g h)^2\right] dv_g. \tag{2.4}$$

Note that the linearization of the scalar curvature is given by

$$s_g' \cdot h = -\Delta_g(tr_g h) + \delta\delta h - \langle r, h\rangle, \tag{2.5}$$

and the linearization of the Ricci tensor is given by

$$r_g' \cdot h = \frac{1}{2}D^*Dh + \frac{1}{2}(r\circ h + h\circ r) - \mathring{R}(h) - \delta^*\delta h - \frac{1}{2}Dd(tr_g h) \tag{2.6}$$

for any symmetric 2-tensor h (see p. 63 in [2]). Here, $\delta = -\mathrm{div}$ is the (negative) divergence defined by $\delta h(X) = -\sum_{i=1}^n D_{E_i}h(E_i, X)$ for any vector X and a local frame $\{E_i\}$ with the Riemannian connection D, and $D^*Dh = -D_{E_i}D_{E_i}h + D_{D_{E_i}E_i}h$. Also, for any vector fields X and Y, $\mathring{R}(h)(X,Y) = \sum_{i=1}^n h(R(X,E_i)Y, E_i)$ and $h\circ k(X,Y) = \sum_{i=1}^n h(X,E_i)k(E_i,Y)$.

From $\mathcal{S}(g_t) = \int_M s_{g_t}dv_{g_t}$, by the divergence theorem

$$\begin{aligned}\frac{d}{dt}\mathcal{S}(g_t) &= \int_M s_{g_t}'\cdot(h+t\xi)\,dv_{g_t} + \frac{1}{2}s_{g_t}tr_{g_t}(h+t\xi)\,dv_{g_t} \\ &= \int_M \left[-\Delta_g(tr_g(h+t\xi)) + \delta\delta(h+t\xi) - \langle r_{g_t}, h+t\xi\rangle\right] dv_{g_t} \\ &\quad + \int_M \frac{1}{2}s_{g_t}tr_{g_t}(h+t\xi)\,dv_{g_t} \\ &= \int_M \langle -r_{g_t} + \frac{1}{2}s_{g_t}g_t, h+t\xi\rangle_{g_t}dv_{g_t}.\end{aligned}$$

Thus,

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0}\mathcal{S}(g_t) &= \int_M \langle -r_g'\cdot h + \frac{1}{2}(s_g'\cdot h)g + \frac{1}{2}s_g h, h\rangle_g dv_g \\ &\quad + \int_M \langle -r_g + \frac{1}{2}s_g g, \xi\rangle_g dv_g + \int_M \langle -r_g + \frac{1}{2}s_g g, h\rangle \frac{1}{2}(tr_g h)dv_g \\ &\quad + \int_M \left.\frac{d}{dt}\right|_{t=0}\left[(g_t)^{ik}(g_t)^{jl}\right]\alpha_{ij}h_{kl}dv_g.\end{aligned}$$

Here, $\alpha_{ij} = -r_{ij} + \frac{1}{2}s_g g_{ij}$. Therefore, by (2.1)

$$\begin{aligned}
&\int_M \frac{d}{dt}\Big|_{t=0} \left[(g_t)^{ik}(g_t)^{jl}\right] \alpha_{ij} h_{kl} dv_g \\
&= \int_M -g^{ip}g^{kq}h_{pq}g^{jl}h_{kl}\left(-r_{ij} + \frac{1}{2}s_g g_{ij}\right) dv_g \\
&- \int_M g^{ik}g^{jp}g^{lq}h_{pq}h_{kl}\left(-r_{ij} + \frac{1}{2}s_g g_{ij}\right) dv_g \\
&= \int_M \left[2\langle r, h\circ h\rangle - s_g|h|^2\right] dv_g.
\end{aligned}$$

Hence we may conclude that

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{S}(g_t) = &\int_M \langle -r_g' \cdot h + \frac{1}{2}(s_g'\cdot h)g + \frac{1}{2}s_g h, h\rangle_g + \langle -r_g + \frac{1}{2}s_g g, \xi\rangle_g \\
&+ \frac{1}{2}\langle -r_g + \frac{1}{2}s_g g, (tr_g h)h\rangle + \left[2\langle r, h\circ h\rangle - s_g|h|^2\right] dv_g
\end{aligned} \tag{2.7}$$

(see p. 129 of [2]). In particular, if g is Einstein, we have the following.

Lemma 2.1 (Proposition 4.55 of [2])**.** *Assume g is an Einstein metric of unit volume with $g_t = g + th + \frac{t^2}{2}\xi \in \mathcal{M}_1$. Then we have*

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{S}(g_t) = &\int_M \Big[\langle -\frac{1}{2}D^*Dh + \mathring{R}(h) + \delta^*\delta h + \frac{1}{2}Dd(tr_g h) \\
&+ \frac{1}{2}\left(-\Delta(tr_g h) + \delta\delta h - \frac{s}{n}(tr_g h)\right) g, h\rangle\Big] dv_g.
\end{aligned}$$

Proof. Since $r_g = \frac{s}{n}g$, it follows from (2.4) that

$$\begin{aligned}
\int_M \langle -r_g + \frac{1}{2}sg, \xi\rangle dv_g &= \frac{(n-2)s}{2n}\int_M tr_g\, \xi dv_g \\
&= \frac{(n-2)s}{2n}\int_M \left[|h|^2 - \frac{1}{2}(tr_g h)^2\right] dv_g
\end{aligned}$$

and

$$\int_M \left[2\langle r, h\circ h\rangle - s_g|h|^2\right] dv_g = \frac{(2-n)s}{n}\int_M |h|^2 dv_g.$$

Also, we have

$$\frac{1}{2}\int_M \langle -r_g + \frac{1}{2}s_g g, (tr_g h)h\rangle = \frac{(n-2)s}{4n}\int_M (tr_g h)^2 dv_g.$$

Thus, by substituting (2.5) and (2.6) into (2.7), we obtain

$$\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{S}(g_t) = &\int_M \Big[\langle -\frac{1}{2}D^*Dh - \frac{1}{2}(r\circ h + h\circ r) + \mathring{R}(h) + \delta^*\delta h + \frac{1}{2}Dd(tr_g h) \\
&+ \frac{1}{2}\left(-\Delta(tr_g h) + \delta\delta h - \frac{s}{n}(tr_g h)\right) g + \frac{1}{2}sh, h\rangle\Big] dv_g
\end{aligned}$$

$$- \frac{(n-2)s}{2n} \int_M |h|^2 dv_g.$$

Our lemma follows from $r \circ h = h \circ r = \frac{s}{n} h$. □

In general, the second variation is given by the following.

Lemma 2.2. *Let (M^n, g) be an n-dimensional compact Riemannian manifold with constant scalar curvature. For $g_t = g + th + \frac{t^2}{2}\xi \in \mathcal{M}_1$ for $h, \xi \in S^2(M)$, we have*

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = & \int_M \Big[\langle -\frac{1}{2} D^* D h - \frac{1}{2}(r \circ h + h \circ r) + \mathring{R}(h) + \delta^* \delta h + \frac{1}{2} D d(tr_g h) \\ & + \frac{1}{2} \Big(-\Delta(tr_g h) + \delta\delta h - \frac{s}{n}(tr_g h) \Big) g, h \rangle \Big] dv_g \\ & + \int_M \Big[\langle -r, \xi \rangle + \frac{1}{2} \langle -r_g, (tr_g h) h \rangle + 2 \langle r, h \circ h \rangle \Big] dv_g. \end{aligned}$$

Proof. Substituting (2.5) and (2.6) into (2.7), we obtain

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = & \int_M \Big[\langle -\frac{1}{2} D^* D h - \frac{1}{2}(r \circ h + h \circ r) + \mathring{R}(h) + \delta^* \delta h + \frac{1}{2} D d(tr_g h) \\ & + \frac{1}{2} \Big(-\Delta(tr_g h) + \delta\delta h - \frac{s}{n}(tr_g h) \Big) g + \frac{1}{2} s h, h \rangle \Big] dv_g \\ & + \int_M \Big[\langle -r_g + \frac{1}{2} s_g g, \xi \rangle_g + \frac{1}{2} \langle -r_g + \frac{1}{2} s_g g, (tr_g h) h \rangle \Big] dv_g \\ & + \int_M \big[2 \langle r, h \circ h \rangle - s_g |h|^2 \big] dv_g. \end{aligned}$$

Since $g_t \in \mathcal{M}_1$, we have $\int_M (tr_g h) dv_g = 0$, and by (2.4)

$$\int_M \langle \frac{1}{2} s g, \xi \rangle dv_g = \frac{s}{2} \int_M (tr_g\, \xi) dv_g = \frac{s}{2} \int_M \Big[|h|^2 - \frac{1}{2}(tr_g h)^2 \Big] dv_g.$$

Therefore, we obtain

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = & \int_M \Big[\langle -\frac{1}{2} D^* D h - \frac{1}{2}(r \circ h + h \circ r) + \mathring{R}(h) + \delta^* \delta h + \frac{1}{2} D d(tr_g h) \\ & + \frac{1}{2} \Big(-\Delta(tr_g h) + \delta\delta h - \frac{s}{n}(tr_g h) \Big) g, h \rangle \Big] dv_g \\ & + \int_M \Big[\langle -r, \xi \rangle + \frac{1}{2} \langle -r_g, (tr_g h) h \rangle + 2 \langle r, h \circ h \rangle \Big] dv_g. \end{aligned}$$

□

Definition 2.3. A symmetric 2-tensor h is called transverse-traceless (TT for short) if $\delta h = 0$ and $tr_g h = 0$.

A variation $g_t = g + th + (t^2/2)\xi \in \mathcal{M}_1$ with a TT-tensor h is called a *TT-variation.* Note that if h is a TT-tensor, so that $\delta h = 0 = \mathrm{tr}_g h$ and g is Einstein, by Lemma 2.1 we have

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) &= \int_M \left[\langle -\frac{1}{2}D^*Dh + \mathring{R}(h), h\rangle\right] dv_g \\ &= \int_M \left[-\frac{1}{2}|Dh|^2 + \langle \mathring{R}(h), h\rangle\right] dv_g.\end{aligned}$$

Lemma 2.4. *Suppose that g has constant scalar curvature and g_t is a TT variation, so that $h = g_t'(0)$ is transverse-traceless. Then*

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = &\int_M \left[-\frac{1}{2}|Dh|^2 + \langle \mathring{R}(h) - \frac{1}{2}(r\circ h + h\circ r), h\rangle\right] dv_g \\ &- \int_M \langle r, \xi - 2h\circ h\rangle \, dv_g. \qquad (2.8)\end{aligned}$$

Proof. From Lemma 2.2, by integration by parts and the divergence theorem

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = &\int_M \left[|\delta h|^2 - \frac{1}{2}|Dh|^2 + \frac{1}{2}\langle d(tr_g h), \delta h\rangle\right] dv_g \\ &+ \int_M \langle \mathring{R}(h) - \frac{1}{2}(r\circ h + h\circ r), h\rangle dv_g \\ &+ \frac{1}{2}\int_M \left[|\nabla(tr_g h)|^2 + \langle \delta h, d(tr_g h)\rangle - \frac{s}{n}(tr_g h)^2\right] dv_g \\ &- \int_M \langle r, \xi + \frac{1}{2}(tr_g h)h - 2h\circ h\rangle dv_g.\end{aligned}$$

Our lemma follows from $\delta h = 0$ and $tr_g h = 0$. □

Remark 2.5. Note that for a symmetric 2-tensor h, we have

$$\langle r\circ h, h\rangle = \langle r, h\circ h\rangle = \langle h\circ r, h\rangle.$$

Thus, (2.8) becomes

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = \int_M \left[-\frac{1}{2}|Dh|^2 + \langle \mathring{R}(h), h\rangle + \langle r, h\circ h\rangle - \langle r, \xi\rangle\right] dv_g.$$

Moreover, since $r = z + \frac{s}{n}g$, where z is the traceless Ricci tensor,

$$\langle r, h\circ h\rangle = \langle z, h\circ h\rangle + \frac{s}{n}|h|^2 \quad \text{and} \quad \langle r, \xi\rangle = \langle z, \xi\rangle + \frac{s}{n}tr_g\, \xi.$$

Recalling (2.4), we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = &\int_M \left[-\frac{1}{2}|Dh|^2 + \langle \mathring{R}(h), h\rangle + \langle z, h\circ h\rangle\right] dv_g \\ &- \int_M \left[\langle z, \xi\rangle - \frac{s}{2n}(tr_g h)^2\right] dv_g. \qquad (2.9)\end{aligned}$$

Finally, if h is a TT-tensor, we obtain

$$\int_M tr_g\, \xi\, dv_g = \int_M |h|^2\, dv_g, \tag{2.10}$$

and

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = \int_M \left[-\frac{1}{2}|Dh|^2 + \langle \mathring{R}(h), h\rangle + \langle z, h\circ h\rangle - \langle z, \xi\rangle \right] dv_g. \tag{2.11}$$

3. Main results

As mentioned in the Introduction, if (M^n, g), $n \geqslant 2$, has positive constant sectional curvature, then the second variation is strictly negative in a TT direction (or transverse-traceless variations). Using (2.11), we have the following converse of this result.

Theorem 3.1. *Let (M^n, g) be an n-dimensional compact Riemannian manifold with constant scalar curvature. If $\frac{d^2}{dt^2}\big|_{t=0}\mathcal{S}(g_t) < 0$ for any TT variation g_t given by*

$$g_t = g + th + \frac{t^2}{2}\xi$$

for an arbitary symmetric 2-tensor ξ, then (M, g) is Einstein.

Proof. Suppose that (M, g) is not Einstein so that $\int_M |z|^2 dv_g > 0$. Since M is compact, for given TT-tensor h there exists a positive constant $C > 0$ such that

$$\int_M \left[\frac{1}{2}|Dh|^2 + \left|\langle \mathring{R}(h), h\rangle\right| \right] dv_g \leqslant C \int_M |z|^2 dv_g.$$

With this constant, let $\xi = -Cz + h\circ h$. Note that ξ satisfies (2.10). By (2.11),

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{S}(g_t) = \int_M \left[-\frac{1}{2}|Dh|^2 + \langle \mathring{R}(h), h\rangle \right] dv_g + C\int_M |z|^2 dv_g \geqslant 0,$$

which is a contradiction. □

Now we consider the critical point equation (CPE) on a compact smooth n-manifold M satisfying

$$(1+f)z = Ddf + \frac{sf}{n(n-1)}g.$$

It turns out that the CPE is the Euler-Lagrange equation of the total scalar curvature functional $\mathcal{S}$ restricted to the set $\mathcal{C}$ of constant scalar curvature metrics in $\mathcal{M}_1$. Recall that a critical point of $\mathcal{S}$ on $\mathcal{M}_1$ is Einstein. The Besse conjecture says that a critical point of $\mathcal{S}$ restricted to $\mathcal{C}$ is Einstein (see [2] and [11]). It is clear from the definition that a non-trivial solution (g, f) of the CPE has constant scalar curvature. As a consequence of Theorem 3.1, we have the following.

Corollary 3.2. *Let (g, f) be a non-trivial solution of the CPE on a compact manifold M. If $\frac{d^2}{dt^2}\big|_{t=0}\mathcal{S}(g_t) < 0$ for any TT variation g_t given by*

$$g_t = g + th + \frac{t^2}{2}\xi$$

for an arbitrary symmetric 2-tensor ξ, then (M, g) is isometric to a standard sphere $\mathbb{S}^n$.

Proof. By Theorem 3.1, (M, g) is Einstein. It follows from Obata's theorem [7] that (M, g) is isometric to a standard sphere $\mathbb{S}^n$. □

Now we consider the Lichnerowicz Laplacian defined on symmetric 2-tensors introduced in [5].

Definition 3.3. The Lichnerowicz Laplacian Δ_L acting on the space of symmetric 2-tensors is defined by

$$\Delta_L h = D^*Dh + r \circ h + h \circ r - 2\mathring{R}(h).$$

It is worth mentioning [4] that the Hessian of the total scalar curvature for TT-tensors has the form

$$\operatorname{Hess} \mathcal{S}_g(h, h) = -\frac{1}{2}\langle \Delta_L h - \frac{2}{n} sh, h\rangle.$$

It is also known [8] that for a standard sphere $\mathbb{S}^n$ with round metric, the smallest eigenvalue of the Lichnerowicz Laplacian on TT-tensors is $4n$. For a general $(0, 2)$-tensor not necessarily TT-tensor, some results on the eigenvalue estimation for the Lichenerowicz Lapalcain are also known [9].

Rewritting the formula in Lemma 2.2 using the Lichnerowicz Laplacian, we have

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{S}(g_t) = & \int_M \left[\langle -\frac{1}{2}\langle \Delta_L h, h\rangle + |\delta h|^2 + \frac{1}{2}\langle d(tr_g h), \delta h\rangle\right] dv_g \\ & + \frac{1}{2}\int_M \left[|d(tr_g h)|^2 + \langle \delta h, d(tr_g h)\rangle - \frac{s}{n}(tr_g h)^2\right] dv_g \\ & - \int_M \langle r, \xi + \frac{1}{2}(tr_g h)h - 2h \circ h\rangle dv_g.\end{aligned}$$

Lemma 3.4. *Let g_t be a TT variation of the metric g having constant scalar curvature. Then*

$$\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{S}(g_t) = \int_M \left[-\frac{1}{2}\langle \Delta_L h, h\rangle + 2\langle r, h \circ h\rangle - \langle r, \xi\rangle\right] dv_g.$$

Let us denote by $\mathfrak{T}$ the space of all transverse traceless symmetric 2-tensors on (M, g). Note that if (M, g) has constant scalar curvature, then the traceless Ricci tensor $z = r - \frac{s}{n}g$ is always contained in $\mathfrak{T}$. Thus, $\mathfrak{T}$ is not trivial unless (M, g) is Einstein.

Theorem 3.5. *Let (M,g) be a compact Riemannian manifold of constant scalar curvature. Assume that the smallest eigenvalue λ of the Lichnerowicz Laplacian is positive. If $\frac{d^2}{dt^2}\big|_{t=0}\mathcal{S}(g_t) < 0$ for any TT variation g_t given by*

$$g_t = g + th + \frac{t^2}{2}\xi$$

for an arbitrary symmetric 2-tensor ξ, then (M,g) is Einstein.

Proof. Let $\Delta_L h = \lambda h$ for a TT-tensor h so that

$$\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{S}(g_t) = \int_M \left[-\frac{1}{2}\lambda|h|^2 + 2\langle r, h\circ h\rangle - \langle r,\xi\rangle\right] dv_g \tag{3.1}$$

by Lemma 3.4. Suppose that (M,g) is not Einstein so that $\int_M |r|^2 dv_g \geqslant \int_M |z|^2 dv_g > 0$. Let $\max_M |r| \leqslant k$. Since M is compact, for the eigen-tensor h, there exists a positive constant $C > 0$ such that

$$\int_M \left[\frac{1}{2}\lambda|h|^2 + |\langle r, h\circ h\rangle|\right] dv_g \leqslant (\lambda + k)\int_M |h|^2 dv_g \leqslant C\int_M |z|^2 dv_g.$$

With this constant, let $\xi = -Cz + h\circ h$ so that $\int_M \mathrm{tr}_g \xi dv_g = \int_M |h|^2 dv_g$. Then, from (3.1) we have

$$\frac{d^2}{dt^2}\bigg|_{t=0}\mathcal{S}(g_t) = \int_M \left[-\frac{1}{2}\lambda|h|^2 + \langle r, h\circ h\rangle + C|z|^2\right] dv_g \geqslant 0,$$

which is a contradiction. □

As a result, for a CPE metric we derive the following result in a similar way.

Corollary 3.6. *Let (g,f) be a nontrivial solution to the CPE. Assume that the smallest eigenvalue λ of the Lichnerowicz Laplacian is positive. If $\frac{d^2}{dt^2}\big|_{t=0}\mathcal{S}(g_t) < 0$ for any TT variation g_t given by*

$$g_t = g + th + \frac{t^2}{2}\xi$$

for an arbitrary symmetric 2-tensor ξ, then (M,g) is isometric to a standard sphere $\mathbb{S}^n$.

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