J. Korean Math. Soc. **61** (2024), No. 2, pp. 207–226
https://doi.org/10.4134/JKMS.j210768
pISSN: 0304-9914 / eISSN: 2234-3008

MULTIPLE WEIGHTED ESTIMATES FOR MULTILINEAR COMMUTATORS OF MULTILINEAR SINGULAR INTEGRALS WITH GENERALIZED KERNELS

Liwen Gao, Yan Lin, and Shuhui Yang

Abstract. In this paper, the weighted L^p boundedness of multilinear commutators and multilinear iterated commutators generated by the multilinear singular integral operators with generalized kernels and BMO functions is established, where the weight is multiple weight. Our results are generalizations of the corresponding results for multilinear singular integral operators with standard kernels and Dini kernels under certain conditions.

1. Introduction

The multilinear singular integral operator theory is an important topic of modern harmonic analysis. It originated from the work [1–3] of Meyer and Coifman in the 1970s.

Before describing the main content of the theory, we give some notations that we need to use. $\mathbb{R}^n$ stands for the n-dimension Euclidean space and $C_c^\infty(\mathbb{R}^n)$ stands for all infinitely differentiable functions with compact support. We denote the support of f by $\mathrm{supp}(f)$, the p-th power integrable function by L^p and the essentially bounded function by L^∞. For $c>0$ and a ball B, the ball with the same center as B and the radius cr_B is denoted by cB, where the r_B stands for the radius of B.

The theory mainly studies the following operator:

Definition 1.1. Suppose $m \in \mathbb{N}^+$ and a function $K(y_0, y_1, \ldots, y_m)$ is defined away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. We define an m-linear operator T from m-tuples of test functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$ by

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} K(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j) dy_1 \cdots dy_m, \tag{1.1}$$

Received December 22, 2021; Revised March 28, 2023; Accepted December 7, 2023.
2020 *Mathematics Subject Classification.* Primary 42B25, 42B35.
Key words and phrases. Multilinear singular integral with generalized kernel, multiple weights, iterated commutator, BMO function.
This work was financially supported by the National Natural Science Foundation of China (No. 12071052).

where f_j $(j = 1, \dots, m) \in C_c^\infty(\mathbb{R}^n)$ such that $x \notin \cap_{j=1}^m \mathrm{supp}(f_j)$.

In particular, we call K a standard m-linear Calderón-Zygmund kernel if it satisfies the following size estimates:

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{(\sum_{k,\, l=0}^m |y_k - y_l|)^{mn}} \tag{1.2}$$

for some $A > 0$ and all $(y_0, y_1, \dots, y_m) \in (\mathbb{R}^n)^{m+1}$ away from the diagonal, and smooth estimates:

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y_j', \dots, y_m)| \leq \frac{C|y_j - y_j'|^\varepsilon}{(\sum_{k,\, l=0}^m |y_k - y_l|)^{mn+\varepsilon}} \tag{1.3}$$

for some $\varepsilon > 0$, whenever $0 \leq j \leq m$ and $|y_j - y_j'| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$.

Definition 1.2. Let T be an m-linear operator defined by (1.1) with a standard m-linear Calderón-Zygmund kernel K. We say that T is a standard m-linear Calderón-Zygmund operator if it satisfies either of the following conditions.

Given a group of numbers $t_1, t_2, \dots, t_m, t$ such that $1 \leq t_1, t_2, \dots, t_m \leq \infty$ and $1/t = 1/t_1 + \cdots + 1/t_m$,

(1) T maps $L^{t_1,1} \times \cdots \times L^{t_m,1}$ into $L^{t,\infty}$ if $t > 1$,

(2) T maps $L^{t_1,1} \times \cdots \times L^{t_m,1}$ into L^1 if $t = 1$,

where the signs $L^{t_1,1}, L^{t_2,1}, \dots, L^{t_m,1}$ and $L^{t_m,\infty}$ all stand for Lorentz spaces.

The initial interest in the study of commutators was related to the generalization of the classical factorization theorem for Hardy spaces according to [4]. Then Pérez and Torres generalized the definition to the commutators generated by multilinear singular integral operators and BMO functions in [16]. Now we recall the definitions of the multilinear commutators and multilinear iterated commutators as follows.

Definition 1.3. Let $\vec{b} = (b_1, \dots, b_m)$ be a family of locally integrable functions. The m-linear commutator generated by $\vec{b}$ and the m-linear operator T is defined by

$$T_{\vec{b}}(f_1, \dots, f_m) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f}),$$

where each summand $T_{\vec{b}}^j(\vec{f})$ means the commutator of b_j and the jth entry of T, that is

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \dots, f_j, \dots, f_m) - T(f_1, \dots, b_j f_j, \dots, f_m).$$

Definition 1.4. Suppose that T is an m-linear operator and $\vec{b} = (b_1, \dots, b_m)$ is a family of locally integrable functions. Then the m-linear iterated commutator generated by T and $\vec{b}$ is defined by

$$T_{\prod \vec{b}}(f_1, \dots, f_m) = [b_1, [b_2, \dots, [b_{m-1}, [b_m, T]_m]_{m-1} \dots]_2]_1(\vec{f}),$$

where $\vec{f} = (f_1, \ldots, f_m)$. If T is given as in Definition 1.1, then we can write

$$T_{\prod \vec{b}}(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} \Big(b_j(x) - b_j(y_j) \Big) K(x, y_1, \ldots, y_m) f_1(y_1) \cdots \times f_m(y_m) dy_1 \cdots dy_m.$$

In 2002, Grafakos-Kalton [6] and Grafakos-Torres [8–10] carried out a systematic study on the theory. In [8], the authors showed the boundedness of classical singular integral operators. They obtained that T was bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for every $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$.

Grafakos-Martell made some work about weighted estimates for standard m-linear Calderón-Zygmund operators in [7]. The weight that they discussed was the Muckenhoupt weight. They obtained that T was bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(w)$ for every $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, where $w_1, \ldots, w_m$ were a family of weight functions and $w = \prod_{j=1}^m w_j^{\frac{p}{p_j}}$.

In [10], Grafakos and Torres put forward a question: "Is there a multiple weight theory? The most appropriate multilinear maximal function and multiple weights to work with in this direction have not been yet clear." To answer the question, the multiple weight was given by Lerner, Ombrosi, Pérez, Torres and Trujillo-González in [11]. The definition of the multiple weight is as follows.

Definition 1.5. Let $1 \leq p_1, \ldots, p_m < \infty$ and p be numbers satisfying $1/p = 1/p_1 + \cdots + 1/p_m$. Given a family of weight functions $\vec{w} = (w_1, \ldots, w_m)$, set

$$v_{\vec{w}} = \prod_{j=1}^{m} w_j^{\frac{p}{p_j}}.$$

$\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{w}} \right)^{\frac{1}{p}} \prod_{j=1}^{m} \left(\frac{1}{|Q|} \int_Q w_j^{1-p_j'} \right)^{\frac{1}{p_j'}} < \infty.$$

When $p_j = 1$, the condition $\left(\inf_Q w_j \right)^{-1}$ takes place of the condition $\left(\frac{1}{|Q|} \int_Q w_j^{1-p_j'} \right)^{\frac{1}{p_j'}}$.

In [11], the authors not only gave the $A_{\vec{P}}$ condition for the multiple weight, but also introduced the following new maximal functions

$$\mathcal{M}(\vec{f})(x) := \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j,$$

$$\mathcal{M}_q(\vec{f})(x) := \sup_{Q \ni x} \left(\prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|^q dy_j \right)^{\frac{1}{q}}.$$

Making use of the above notions, the authors in [11] obtained a range of results for the boundedness of T and $T_{\vec{b}}$. More precisely, with the multilinear Calderón-Zygmund operator T and $\vec{w}$ satisfying the $A_{\vec{P}}$ condition, they proved that T and $T_{\vec{b}}$ are both bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(v_{\vec{w}})$ for every $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$.

In 2014, Zhang-Lu introduced multilinear singular integral operators with kernels of Dini type in [15]. They are multilinear operators satisfying the condition (1.3) in the foundation of standard multilinear Calderón-Zygmund operator into the following condition,

$$\begin{aligned} (1.4) \qquad & |K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \\ & \leq \frac{C}{(|y_0 - y_1| + \cdots + |y_0 - y_m|)^{mn}} h\left(\frac{|y_j - y_j'|}{|y_0 - y_1| + \cdots + |y_0 - y_m|}\right) \end{aligned}$$

for $|y_j - y_j'| \leq \frac{1}{2} \max_{1 \leq j \leq m} |y_0 - y_j|$, where $h(t)$ is a non-negative and non-decreasing function on $\mathbb{R}^+$.

We note that (1.4) implies the condition (1.3). One can easily check this by letting $h(t) = t^\varepsilon$. Thus the standard multilinear Calderón-Zygmund operator is a special case of multilinear singular integral operators with kernels of Dini type. Assume that $h(t) : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $0 < h(1) < \infty$. For $a > 0$, we say that $h \in \mathrm{Dini}(a)$ if

$$[h]_{\mathrm{Dini}(a)} := \int_0^1 \frac{h^a(t)}{t} \mathrm{d}t < \infty.$$

In [15], Lu and Zhang mainly studied the multilinear singular integrals with kernels of Dini type and the multilinear commutators with the multiple weight. Let $\vec{w} \in A_{\vec{P}}$ for $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Then the authors showed that if $\vec{b} \in \mathrm{BMO}^m$ and h satisfies

$$\int_0^1 \frac{h(t)}{t} \left(1 + \log \frac{1}{t}\right) dt < \infty, \tag{1.5}$$

then $T_{\vec{b}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(v_{\vec{w}})$, where $\vec{b} = (b_1, \ldots, b_m) \in \mathrm{BMO}^m$ is defined by $b_j \in \mathrm{BMO}$ for $j = 1, \ldots, m$. Also, it was shown in [18] that if $\vec{b} \in \mathrm{BMO}^m$ and h satisfies

$$\int_0^1 \frac{h(t)}{t} \left(1 + \log \frac{1}{t}\right)^m dt < \infty, \tag{1.6}$$

then $T_{\prod \vec{b}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(v_{\vec{w}})$. In [13], Lin-Xiao gave a kind of multilinear singular integrals with generalized kernels which changed the condition (1.4) into a weaker condition. For any positive integers $k_1, \ldots, k_m$,

$$(1.7) \qquad \Bigg(\int_{2^{k_m}|y_0-y_0'| \leq |y_1 - y_0| < 2^{k_m+1}|y_0 - y_0'|} \cdots \int_{2^{k_1}|y_0-y_0'| \leq |y_m - y_0| < 2^{k_1+1}|y_0 - y_0'|}$$

$$|K(y_0, y_1, \ldots, y_m) - K(y_0', y_1, \ldots, y_m)|^q dy_1 \ldots dy_m\Big)^{\frac{1}{q}}$$
$$\le C|y_0 - y_0'|^{-\frac{mn}{q'}} \prod_{i=1}^{m} C_{k_i} 2^{-\frac{n}{q'}k_i},$$

where $1 < q < \infty$, C_{k_i} is a positive constant about k_i and q' is the conjugate exponent of q.

Motivated by [7, 11, 13–15], we now discuss the weighted boundedness of multilinear commutators and multilinear iterated commutators generated by the multilinear singular integral operators with generalized kernels and BMO functions, where the weight is the multiple weight. However, the condition (1.7) can not be suited for the multiple weight condition. This fact forces us to search for another kind of generalized kernels as follows. For $k \in \mathbb{N}$,

$$\text{(1.8)} \quad \Big(\int_{2^k\sqrt{m}|y_0-y_0'|\le|\vec{y}-\vec{y_0}|<2^{k+1}\sqrt{m}|y_0-y_0'|} |K(y_0, y_1, \ldots, y_m) - K(y_0', y_1, \ldots, y_m)|^q dy_1 \cdots dy_m\Big)^{\frac{1}{q}}$$
$$\le CC_k|y_0 - y_0'|^{-\frac{mn}{q'}} 2^{-\frac{kmn}{q'}},$$

whenever $\vec{y_0} = (y_0, y_0, \ldots, y_0)$, $\vec{y} = (y_1, y_2, \ldots, y_m)$, (q, q') is a fixed pair of positive numbers satisfying $\frac{1}{q} + \frac{1}{q'} = 1$ and C_k is a positive constant about k.

Remark 1.6. By calculation, we can find that condition (1.4) implies condition (1.8) by putting $C_k = h(2^{-k})$. According to the contents that we mentioned in the foregoing, the work in [14] includes the classical case. So we can deduce naturally that our work contains both the cases of standard kernel and Dini type kernel under some conditions that relate to q.

This paper will be organized as follows. In Section 2, we will list the necessary definitions and lemmas. In Section 3, we will state our main theorem for the weighted estimates for $T_{\vec{b}}$ and $T_{\prod \vec{b}}$. In Section 4, we will give the proof of all theorems.

2. Necessary definitions and lemmas

In this section, we give some definitions and lemmas to be used later.

Definition 2.1. Fix positive integers k and m satisfying $1 \le k < m$ and we suppose C_k^m is a family of all finite subsets $\sigma = \{\sigma(1), \ldots, \sigma(k)\}$ of $\{1, 2, \ldots, m\}$ with k different elements. If $j < l$, then $\sigma(j) < \sigma(l)$. For any $\sigma \in C_k^m$, let $\sigma' = \{1, 2, \ldots, m\}\backslash\sigma$ be the complementary sequence. In particular, $C_0^m = \emptyset$. For an m-tuple $\vec{b}$ and $\sigma \in C_k^m$, we denote $\vec{b_\sigma} = (b_{\sigma(1)}, \ldots, b_{\sigma(k)})$. Given T the m-linear operator and $\sigma \in C_j^m$, the multilinear iterated commutator is defined by

$$T_{\prod \vec{b_\sigma}}(f_1, \ldots, f_m) = [b_{\sigma(1)}, [b_{\sigma(2)}, \ldots, [b_{\sigma(j-1)}, [b_{\sigma(j)}, T]_{\sigma(j)}]_{\sigma(j-1)} \cdots]_{\sigma(2)}]_{\sigma(1)}(\vec{f}).$$

According to (1.1), we can also represent the multilinear iterated commutator by

$$T_{\prod \vec{b_\sigma}}(f_1,\dots,f_m)(x) = \int_{(\mathbb{R}^n)^m} \prod_{i=1}^{m}\Big(b_{\sigma(i)}(x) - b_{\sigma(i)}(y_{\sigma(i)})\Big) K(x,y_1,\dots,y_m) \times f_1(y_1)\cdots f_m(y_m)dy_1\cdots dy_m.$$

Obviously, $T_{\prod \vec{b_\sigma}} = T_{\prod \vec{b}}$ when $\sigma = \{1,2,\dots,m\}$ and $T_{\prod \vec{b_\sigma}} = T^j_{b_j}$ when $\sigma = \{j\}$.

Definition 2.2. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|dy,$$

where the supremum is taken over all balls which contain x. We can also define the operator M_s by $M_s(f) = [M(|f|^s)]^{1/s}$, $s > 0$. The sharp maximal operator $M^\sharp$ is defined by

$$M^\sharp(f)(x) = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y) - f_B|dy \sim \sup_{B\ni x}\inf_{a\in\mathbb{C}} \frac{1}{|B|}\int_B |f(y) - a|dx,$$

and $f_B = \frac{1}{|B|}\int_B |f(y)|dy$. We define the l-sharp maximal operator $M^\sharp_l$ by $M^\sharp_l(f) = [M^\sharp(|f|^l)]^{1/l}$, $l > 0$.

Definition 2.3. For a locally integrable function f, we say that f has bounded mean oscillation if f satisfies $M^\sharp(f) \in L^\infty$. The function space consisting of all functions that have the property is denoted by BMO. Let

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|dx,$$

where the supreme is taken over all balls $B \subset \mathbb{R}^n$.

Lemma 2.4 ([11])**.** *Suppose that p, q satisfy $0 < p < q < \infty$. Then there is a positive constant $C = C_{p,q}$ such that for any measurable function f, the following inequality*

$$|Q|^{-1/p}\|f\|_{L^p(Q)} \le C|Q|^{-1/q}\|f\|_{L^{q,\infty}(Q)}$$

holds for any Q.

Lemma 2.5 (see [12])**.** *Suppose that f is a function in* BMO$(\mathbb{R}^n)$*, $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then there is a positive constant C independent of f, x, r_1, and r_2 such that*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p dy\right)^{1/p} \le C\Big(1 + \big|\ln\frac{r_1}{r_2}\big|\Big)\|f\|_{BMO}.$$

Lemma 2.6 ([5, 11])**.** *Let $0 < p, \delta < \infty$ and $w \in A_\infty$. Then there exists a constant $C > 0$ depending only on the A_∞ constant of w such that*

$$\int_{\mathbb{R}^n} [M_\delta(f)(x)]^p w(x)dx \le C\int_{\mathbb{R}^n} [M^\sharp_\delta(f)(x)]^p w(x)dx$$

for every function f such that the left-hand side is finite.

Lemma 2.7 ([11]). *Let $\vec{w} = (w_1, \dots, w_m)$ and $1 \le p_1, \dots, p_m < \infty$. Then $w \in A_{\vec{P}}$ if and only if*

$$\begin{cases} w_j^{1-p_j'} \in A_{mp_j'}, \quad j = 1, 2, \dots, m, \\ v_{\vec{w}} \in A_{mp}. \end{cases}$$

The condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is replaced by $w_j^{\frac{1}{m}} \in A_1$.

Lemma 2.8 ([11]). *Let $1 < p_j < \infty$, $j = 1, 2, \dots, m$, and $1/p = 1/p_1 + \cdots + 1/p_m$. Then the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

holds for any $\vec{f}$ if and only if $\vec{w}$ satisfies $A_{\vec{P}}$ condition.

Lemma 2.9 ([17]). *Let $m \ge 2$ and T be an m-linear singular integral operator defined by* (1.1) *with generalized kernel satisfying* (1.8) *and $\sum_{k=1}^{\infty} C_k < \infty$. Suppose for fixed $1 \le r_1, \dots, r_m \le q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $0 < \delta < 1/m$, then we have*

$$M_\delta^\sharp(T(\vec{f}))(x) \le C \mathcal{M}_{q'}(\vec{f})(x)$$

for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support.

3. Main results

Theorem 3.1. *Let $m \ge 2$ and T be an m-linear singular integral operator defined by* (1.1) *with generalized kernel satisfying* (1.8) *and $\sum_{k=1}^{\infty} kC_k < \infty$. Suppose for fixed $1 \le r_1, \dots, r_m \le q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in \mathrm{BMO}^m$, $0 < \delta < 1/m$, $\delta < \varepsilon_0 < \infty$ and $q' < s < \infty$, then for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support*

$$M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) \le C \|\vec{b}\|_{BMO^m} \Big(M_{\varepsilon_0}(T(\vec{f}))(x) + \mathcal{M}_s(\vec{f})(x) \Big),$$

where $\|\vec{b}\|_{BMO^m} = \max_{1 \le j \le m} \|b_j\|_{BMO}$.

Theorem 3.2. *Let m, T be as in Theorem 3.1. Suppose for fixed $1 \le r_1, \dots, r_m \le q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in \mathrm{BMO}^m$, then for $q' < p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, $T_{\vec{b}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(v_{\vec{w}})$, where $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}/q'}$ and $v_{\vec{w}} = \prod_{j=1}^m w_j^{\frac{p}{p_j}}$.*

Theorem 3.3. *Let $m \geq 2$ and T be an m-linear singular integral operator defined by* (1.1) *with generalized kernel satisfying* (1.8) *and* $\sum_{k=1}^{\infty} k^m C_k < \infty$. *Suppose for fixed $1 \leq r_1, \dots, r_m \leq q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in \mathrm{BMO}^m$, $0 < \delta < 1/m$, $\delta < \varepsilon_0 < \infty$ and $q' < s < \infty$, then for all m-tuples $\vec{f} = (f_1, \dots, f_m)$ of bounded measurable functions with compact support*

$$\begin{aligned} M_\delta^\sharp(T_{\prod \vec{b}}(\vec{f}))(x) \leq C \prod_{j=1}^{m} \|b_j\|_{BMO} \Big(M_{\varepsilon_0}(T(\vec{f}))(x) + \mathcal{M}_s(\vec{f})(x) \Big) \\ + C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \prod_{i=1}^{j} \|b_{\sigma(i)}\|_{BMO} M_{\varepsilon_0}(T_{\prod \vec{b}_{\sigma'}}(\vec{f}))(x). \end{aligned}$$

Theorem 3.4. *Let m, T be as in Theorem 3.3. Suppose for fixed $1 \leq r_1, \dots, r_m \leq q'$ with $1/r = 1/r_1 + \cdots + 1/r_m$, T is bounded from $L^{r_1} \times \cdots \times L^{r_m}$ into $L^{r,\infty}$. If $\vec{b} \in \mathrm{BMO}^m$, $q' < p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, then $T_{\prod \vec{b}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(v_{\vec{w}})$, where $\vec{w} = (w_1, \dots, w_m) \in A_{\vec{P}/q'}$ and $v_{\vec{w}} = \prod_{j=1}^{m} w_j^{\frac{p}{p_j}}$.*

Remark 3.5. It is easy to check that (1.5) implies $\sum_{k=1}^{\infty} kC_k < \infty$ and (1.6) implies $\sum_{k=1}^{\infty} k^m C_k < \infty$ by putting $C_k = h(2^{-k})$ in (1.8).

$$\sum_{k=1}^{\infty} kC_k = \sum_{k=1}^{\infty} k \cdot h(2^{-k}) \approx \int_0^1 \frac{h(t)}{t} \left(1 + \log \frac{1}{t}\right) dt < \infty,$$

and

$$\sum_{k=1}^{\infty} k^m C_k = \sum_{k=1}^{\infty} k^m \cdot h(2^{-k}) \approx \int_0^1 \frac{h(t)}{t} \left(1 + \log \frac{1}{t}\right)^m dt < \infty.$$

Thus the corresponding results of multilinear commutators and multilinear iterated commutators of multilinear singular integral operators with Dini type kernels in [15, 18] and the standard kernels can be deduced as special cases of our results in this paper under the exponent range $q' < p_1, \dots, p_m < \infty$.

4. Proof of the main results

Proof of Theorem 3.1. Fix $z \in \mathbb{R}^n$. For any ball $B \ni z$, we first consider

$$T_{b_1}^1(f_1, \dots, f_m)(z) = b_1(z) T(f_1, \dots, f_m)(z) - T(b_1 f_1, \dots, f_m)(z).$$

Let $B^* = 16\sqrt{m}B$, then for any $z \in B$,

$$\begin{aligned} T_{b_1}^1(f_1, \dots, f_m)(z) = (b_1(z) - b_{1B^*}) T(f_1, \dots, f_m)(z) \\ - T((b_1 - b_{1B^*}) f_1, \dots, f_m)(z), \end{aligned}$$

where $b_{1B^*} = \frac{1}{|B^*|}\int_{B^*} b_1(z)dz$. Since $0<\delta<1$, then for any $c \in \mathbb{C}$, we have

$$\begin{aligned}
&\bigg(\frac{1}{|B|}\int_B ||T^1_{b_1}(\vec{f})(z)|^\delta - |c|^\delta|dz\bigg)^{\frac{1}{\delta}}\\
&\le C\bigg(\frac{1}{|B|}\int_B |(b_1(z)-b_{1B^*})T(\vec{f})(z)|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1-b_{1B^*})f_1,\dots,f_m)(z) - |c|^\delta|dz\bigg)^{\frac{1}{\delta}}\\
&:= I + II.
\end{aligned}$$

We can find an l such that $1<l<\min\{\frac{\varepsilon_0}{\delta}, \frac{1}{1-\delta}\}$, then $l\delta < \varepsilon_0$ and $l'\delta > 1$. By Hölder's inequality and Lemma 2.5, we obtain

$$\begin{aligned}
I &\le C\bigg(\frac{1}{|B|}\int_B |b_1(z)-b_{1B^*}|^{\delta l'}dz\bigg)^{\frac{1}{\delta l'}}\bigg(\frac{1}{|B|}\int_B |T(\vec{f})(z)|^{\delta l}dz\bigg)^{\frac{1}{\delta l}}\\
&\le C(1+\ln 16\sqrt{m})\|b_1\|_{BMO} M_{\delta l}(T(\vec{f}))(x)\\
&\le C\|b_1\|_{BMO} M_{\varepsilon_0}(T(\vec{f}))(x).
\end{aligned}$$

For each j, $f_j = f_j^0 + f_j^\infty$ where $f_j^0 = f_j\chi_{B^*}$. Then

$$\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m f_j^0(y_j) + \sum_{(\alpha_1,\dots,\alpha_m)\in\Gamma} f_1^{\alpha_1}(y_1)\cdots f_m^{\alpha_m}(y_m),$$

where $\Gamma = \{(\alpha_1,\alpha_2,\dots,\alpha_m) : \text{there is at least one } \alpha_j \ne 0\}$. Denote $\vec{z_0} = (z_0,z_0,\dots,z_0)$, $\vec{y} = (y_1,y_2,\dots,y_m)$, the center of B by x_0 and the radius of B by r_B. We choose $z_0 \in 4B\backslash 3B$. Let

$$c = \sum_{(\alpha_1,\dots,\alpha_m)\in\Gamma} T((b_1-b_{1B^*})f_1^{\alpha_1}, f_2^{\alpha_2},\dots,f_m^{\alpha_m})(z_0).$$

Then

$$\begin{aligned}
II &\le C\bigg(\frac{1}{|B|}\int_B |T((b_1-b_{1B^*})f_1^0,f_2^0,\dots,f_m^0)(z)|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&\quad + C\sum_{(\alpha_1,\dots,\alpha_m)\in\Gamma}\bigg(\frac{1}{|B|}\int_B |T((b_1-b_{1B^*})f_1^{\alpha_1},f_2^{\alpha_2},\dots,f_m^{\alpha_m})(z)\\
&\quad - T((b_1-b_{1B^*})f_1^{\alpha_1},f_2^{\alpha_2},\dots,f_m^{\alpha_m})(z_0)|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&:= II_0 + \sum_{(\alpha_1,\dots,\alpha_m)\in\Gamma} II_{\alpha_1,\dots,\alpha_m}.
\end{aligned}$$

Let $t = s/q'$, then it follows from $s > q'$ that $t>1$. Since $r_j \le q'$ for any j, we get $r_jt \le s$. Using Lemmas 2.4, 2.5, and the assumption that

$T : L^{r_1} \times \cdots \times L^{r_m} \to L^{r,\infty}$ is bounded, we have

$$\begin{aligned}
II_0 &\le C|B|^{-1/\delta}\|T((b_1 - b_{1B^*})f_1^0, f_2^0, \ldots, f_m^0)\|_{L^\delta(B)} \\
&\le C|B|^{-1/r}\|T((b_1 - b_{1B^*})f_1^0, f_2^0, \ldots, f_m^0)\|_{L^{r,\infty}(\mathbb{R}^n)} \\
&\le C\bigg(\frac{1}{|B^*|}\int_{B^*}|(b_1(z) - b_{1B^*})f_1(y_1)|^{r_1}dy_1\bigg)^{\frac{1}{r_1}} \\
&\quad \times \prod_{j=2}^m\bigg(\frac{1}{|B^*|}\int_{B^*}|f_j(y_j)|^{r_j}dy_j\bigg)^{\frac{1}{r_j}} \\
&\le C\bigg(\frac{1}{|B^*|}\int_{B^*}|b_1(z) - b_{1B^*}|^{r_1t'}dy_1\bigg)^{\frac{1}{r_1t'}}\bigg(\frac{1}{|B^*|}\int_{B^*}|f_1(y_1)|^{r_1t}dy_1\bigg)^{\frac{1}{r_1t}} \\
&\quad \times \prod_{j=2}^m\bigg(\frac{1}{|B^*|}\int_{B^*}|f_j(y_j)|^{r_j}dy_j\bigg)^{\frac{1}{r_j}} \\
&\le C\|b_1\|_{BMO}\prod_{j=1}^m\bigg(\frac{1}{|B^*|}\int_{B^*}|f_j(y_j)|^{s}dy_j\bigg)^{\frac{1}{s}} \\
&\le C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

For any $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Gamma$, we can find a $j \in \{1, \ldots, m\}$ that makes $\alpha_j = \infty$. Then for any $\vec{y} = (y_1, y_2, \ldots, y_m) \in \mathrm{supp} f_1^{\alpha_1} \times \cdots \times \mathrm{supp} f_m^{\alpha_m}$ and $z \in B$, $|\vec{y} - \vec{z_0}| \ge 2\sqrt{m}|z - z_0|$, and $2r_B \le |z - z_0| \le 5r_B$. Thus

$$\begin{aligned}
&II_{\alpha_1,\alpha_2,\ldots,\alpha_m} \\
&\le C\frac{1}{|B|}\int_B\int_{|\vec{y}-\vec{z_0}|\ge 2\sqrt{m}|z-z_0|}|K(z,\vec{y}) - K(z_0,\vec{y})||b_1(y_1) - b_{1B^*}| \\
&\quad \times \prod_{j=1}^m|f_j(y_j)|d\vec{y}dz \\
&\le C\frac{1}{|B|}\int_B\sum_{k=1}^\infty\int_{2^k\sqrt{m}|z-z_0|\le|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{m}|z-z_0|}|K(z,\vec{y}) - K(z_0,\vec{y})| \\
&\quad \times |b_1(y_1) - b_{1B^*}|\prod_{j=1}^m|f_j(y_j)|d\vec{y}dz \\
&\le C\frac{1}{|B|}\int_B\sum_{k=1}^\infty\bigg(\int_{2^k\sqrt{m}|z-z_0|\le|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{m}|z-z_0|}|K(z,\vec{y}) - K(z_0,\vec{y})|^q d\vec{y}\bigg)^{\frac{1}{q}} \\
&\quad \times \bigg(\int_{|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{m}|z-z_0|}|b_1(y_1) - b_{1B^*}|^{q'}\prod_{j=1}^m|f_j(y_j)|^{q'}d\vec{y}\bigg)^{\frac{1}{q'}}dz \\
&\le C\frac{1}{|B|}\int_B\sum_{k=1}^\infty\bigg(\int_{2^k\sqrt{m}|z-z_0|\le|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{m}|z-z_0|}|K(z,\vec{y}) - K(z_0,\vec{y})|^q
\end{aligned}$$

$$
\begin{aligned}
&\times d\vec{y}\Big)^{\frac{1}{q}}\bigg(\frac{1}{|B(z_0,2^{k+1}\sqrt{m}|z-z_0|)|}\int_{|y_1-z_0|<2^{k+1}\sqrt{m}|z-z_0|}|(b_1(y_1)-b_{1B^*})\\
&\times f_1(y_1)|^{q'}dy_1\bigg)^{\frac{1}{q'}}\\
&\times\prod_{j=2}^{m}\bigg(\frac{1}{|B(z_0,2^{k+1}\sqrt{m}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+1}\sqrt{m}|z-z_0|}|f_j(y_j)|^{q'}dy_j\bigg)^{\frac{1}{q'}}\\
&\times(2^{k+1}\sqrt{m}|z-z_0|)^{\frac{mn}{q'}}dz\\
\le{}& C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\bigg(\int_{2^k\sqrt{m}|z-z_0|\le|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{m}|z-z_0|}|K(z,\vec{y})-K(z_0,\vec{y})|^q d\vec{y}\bigg)^{\frac{1}{q}}\\
&\times\bigg(\frac{1}{|B(x_0,2^{k+2}\sqrt{m}|z-z_0|)|}\int_{|y_1-z_0|<2^{k+2}\sqrt{m}|z-z_0|}|b_1(y_1)-b_{1B^*}|^{q't'}dy_1\bigg)^{\frac{1}{q't'}}\\
&\times\bigg(\frac{1}{|B(z_0,2^{k+1}\sqrt{m}|z-z_0|)|}\int_{|y_1-z_0|<2^{k+1}\sqrt{m}|z-z_0|}|f_1(y_1)|^{q't}dy_1\bigg)^{\frac{1}{q't}}\\
&\times\prod_{j=2}^{m}\bigg(\frac{1}{|B(z_0,2^{k+1}\sqrt{m}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+1}\sqrt{m}|z-z_0|}|f_j(y_j)|^{q't}dy_j\bigg)^{\frac{1}{q't}}\\
&\times(2^{k+1}\sqrt{m}|z-z_0|)^{\frac{mn}{q'}}dz\\
\le{}& C\|b_1\|_{BMO}\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}|z-z_0|^{-\frac{mn}{q'}}C_k2^{-\frac{kmn}{q'}}k(2^{k+1}\sqrt{m}|z-z_0|)^{\frac{mn}{q'}}\\
&\times\prod_{j=1}^{m}\bigg(\frac{1}{|B(z_0,2^{k+1}\sqrt{m}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+1}\sqrt{m}|z-z_0|}|f_j(y_j)|^{s}dy_j\bigg)^{\frac{1}{s}}dz\\
\le{}& C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}
$$

Thus

$$\sum_{(\alpha_1,\dots,\alpha_m)\in\Gamma}II_{\alpha_1,\dots,\alpha_m}\le C\|b_1\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

Combining the above estimates we get the result

$$
\begin{aligned}
M^{\sharp}_{\delta}(T^1_{\vec{b}}(\vec{f}))(x)&\le C\|b_1\|_{BMO}\Big(M_{\varepsilon_0}(T(\vec{f}))(x)+\mathcal{M}_s(\vec{f})(x)\Big)\\
&\le C\|\vec{b}\|_{BMO}\Big(M_{\varepsilon_0}(T(\vec{f}))(x)+\mathcal{M}_s(\vec{f})(x)\Big).
\end{aligned}
$$

Similarly, we obtain for any $j=1,2,\dots,m$,

$$M^{\sharp}_{\delta}(T^j_{\vec{b}}(\vec{f}))(x)\le C\|\vec{b}\|_{BMO}\Big(M_{\varepsilon_0}(T(\vec{f}))(x)+\mathcal{M}_s(\vec{f})(x)\Big).$$

So we have

$$\begin{aligned} M_\delta^\sharp(T_{\vec{b}}(\vec{f}))(x) &= M_\delta^\sharp\left(\sum_{j=1}^m T_{\vec{b}}^j(\vec{f})\right)(x) \\ &\le \sum_{j=1}^m M_\delta^\sharp(T_{\vec{b}}^j(\vec{f}))(x) \\ &\le C\|\vec{b}\|_{BMO}\Big(M_{\varepsilon_0}(T(\vec{f}))(x) + \mathcal{M}_s(\vec{f})(x)\Big), \end{aligned}$$

which completes the proof of the theorem. □

Proof of Theorem 3.2. Let $\vec{q} = \frac{\vec{P}}{q'} = (\frac{p_1}{q'}, \dots, \frac{p_m}{q'})$. Since $w \in A_{\vec{q}}$, by Lemma 2.7, each $\psi_j = w_j^{-\frac{1}{q_j-1}}$ belongs to A_∞ where $q_j = \frac{p_j}{q'}$, $j = 1, \dots, m$. By inverse Hölder's inequality, we can find constants $c_j, t_j > 1$ depending on the A_∞ constant of ψ_j such that

$$\left(\frac{1}{|B|}\int_B w_j^{-\frac{t_j}{q_j-1}}\right)^{\frac{1}{t_j}} \le \frac{c_j}{|B|}\int_B w_j^{-\frac{1}{q_j-1}}$$

for any ball B. Pick a $d_j > 1$ that makes

$$\frac{t_j}{q_j-1} = \frac{1}{\frac{q_j}{d_j}-1}.$$

Then $q_j > d_j > 1$, $j = 1, \dots, m$.

Let $d = \min\{d_1, \dots, d_m\}$ and $c = \max\{c_1, \dots, c_m\}$. We have for $q_0 = \frac{p}{q'}$

$$\begin{aligned} &\left(\frac{1}{|B|}\int_B v_{\vec{w}}\right)^{1/\frac{q_0}{d}} \prod_{j=1}^m \left(\frac{1}{|B|}\int_B w_j^{-\frac{1}{\frac{q_j}{d}-1}}\right)^{1-1/\frac{q_j}{d}} \\ &= \left(\frac{1}{|B|}\int_B v_{\vec{w}}\right)^{\frac{d}{q_0}} \prod_{j=1}^m \left(\frac{1}{|B|}\int_B w_j^{-\frac{1}{\frac{q_j}{d}-1}}\right)^{(\frac{q_j}{d}-1)\frac{d}{q_j}} \\ &\le \left(\frac{1}{|B|}\int_B v_{\vec{w}}\right)^{\frac{d}{q_0}} \prod_{j=1}^m \left(\frac{1}{|B|}\int_B w_j^{-\frac{1}{\frac{q_j}{d_j}-1}}\right)^{(\frac{q_j}{d_j}-1)\frac{d}{q_j}} \\ &= \left(\frac{1}{|B|}\int_B v_{\vec{w}}\right)^{\frac{d}{q_0}} \prod_{j=1}^m \left(\frac{1}{|B|}\int_B w_j^{-\frac{t_j}{q_j-1}}\right)^{(\frac{q_j-1}{t_j})\frac{d}{q_j}} \\ &\le c^{dm}\left(\frac{1}{|B|}\int_B v_{\vec{w}}\right)^{\frac{d}{q_0}} \prod_{j=1}^m \left(\frac{1}{|B|}\int_B w_j^{-\frac{1}{q_j-1}}\right)^{(q_j-1)\frac{d}{q_j}} \\ &\le c^{dm}[w]_{A_{\vec{q}}}^d. \end{aligned}$$

Thus $\vec{w} \in A_{\vec{q}/d}$. Let $s = q'd$, then we can find that $s > q'$ and $s < p_j$, $j = 1, \ldots, m$ from $d > 1$. We can get the result

$$\vec{w} \in A_{\vec{q}/d} = A_{\vec{P}/s}.$$

Take ε_0, δ so that $0 < \delta < \varepsilon_0 < \frac{1}{m}$. We can deduce the following conclusion from Lemma 2.6, Theorem 3.1, Lemma 2.8, Lemma 2.9 and $\vec{w} \in A_{\vec{P}/s}$.

$$\begin{aligned}
\|T_{\vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} &\le \|M_\delta(T_{\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\le \|M^\sharp_\delta(T_{\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\le C\|\vec{b}\|_{BMO^m}\|M_{\varepsilon_0}(T(\vec{f})) + \mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})} \\
&\le C\|\vec{b}\|_{BMO^m}(\|M^\sharp_{\varepsilon_0}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})}) \\
&\le C\|\vec{b}\|_{BMO^m}\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})} \\
&= C\|\vec{b}\|_{BMO^m}\|\mathcal{M}(\vec{f}^s)\|^{\frac{1}{s}}_{L^{\frac{p}{s}}(v_{\vec{w}})} \\
&\le C\|\vec{b}\|_{BMO^m}\prod_{j=1}^m \||f_j|^s\|^{\frac{1}{s}}_{L^{\frac{p_j}{s}}(w_j)} \\
&= C\|\vec{b}\|_{BMO^m}\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.
\end{aligned}$$

This finishes the proof of Theorem 3.2. □

Proof of Theorem 3.3. For the sake of simplicity, we only consider the case $m = 2$. The proof of other cases is similar. Let f_1, f_2 be bounded measurable functions with compact support and $b_1, b_2 \in$ BMO. Then for any constants λ_1 and λ_2,

$$\begin{aligned}
T_{\prod \vec{b}}(\vec{f})(z) &= (b_1(z) - \lambda_1)(b_2(z) - \lambda_2)T(f_1, f_2)(z) \\
&\quad - (b_1(z) - \lambda_1)T(f_1, (b_2 - \lambda_2)f_2)(z) \\
&\quad - (b_2(z) - \lambda_2)T((b_1 - \lambda_1)f_1, f_2)(z) \\
&\quad + T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(z) \\
&= \ - (b_1(z) - \lambda_1)(b_2(z) - \lambda_2)T(f_1, f_2)(z) \\
&\quad + (b_1(z) - \lambda_1)T^2_{b_2-\lambda_2}(f_1, f_2)(z) \\
&\quad + (b_2(z) - \lambda_2)T^1_{b_1-\lambda_1}(f_1, f_2)(z) \\
&\quad + T((b_1 - \lambda_1)f_1, (b_2 - \lambda_2)f_2)(z).
\end{aligned}$$

Let C_0 be a constant determined later. Fixed $x \in \mathbb{R}^n$, for any ball $B(x_0, r_B)$ containing x and $0 < \delta < \frac{1}{2}$, we have

$$\left(\frac{1}{|B|}\int_B ||T_{\prod \vec{b}}(\vec{f})(z)|^\delta - |C_0|^\delta| dz\right)^{\frac{1}{\delta}}$$

$$
\begin{aligned}
&\le \left(\frac{1}{|B|}\int_B |T_{\prod \vec{b}}(\vec{f})(z) - C_0|^\delta dz\right)^{\frac{1}{\delta}}\\
&\le C\bigg(\frac{1}{|B|}\int_B |(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&\quad + C\bigg(\frac{1}{|B|}\int_B |(b_1(z)-\lambda_1)T^2_{b_2-\lambda_2}(f_1,f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&\quad + C\bigg(\frac{1}{|B|}\int_B |(b_2(z)-\lambda_2)T^1_{b_1-\lambda_1}(f_1,f_2)(z)|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&\quad + C\bigg(\frac{1}{|B|}\int_B |T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z) - C_0|^\delta dz\bigg)^{\frac{1}{\delta}}\\
&:= I + II + III + IV.
\end{aligned}
$$

Then we calculate each term, respectively. Denote $B^* = 16\sqrt{2}B$ and $\lambda_j = (b_j)_{B^*} = \frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B} b_j(x)dx$, $j = 1,2$. Since $0 < \delta < \frac{1}{2}$ and $0 < \delta < \varepsilon_0 < \infty$, it is easy to seek an l such that $1 < l < \min\{\frac{\varepsilon_0}{\delta}, \frac{1}{1-\delta}\}$. We can deduce that $l\delta < \varepsilon_0$ and $l'\delta > 1$. Choose $q_1, q_2 \in (1,\infty)$ satisfying $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{l'}$, then $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{l} = 1$, $q_1\delta > 1$ and $q_2\delta > 1$. By Lemma 2.5, we have

$$
\begin{aligned}
I &\le C\bigg(\frac{1}{|B|}\int_B |T(f_1,f_2)(z)|^{l\delta} dz\bigg)^{\frac{1}{l\delta}}\left(\frac{1}{|B|}\int_B |b_1(z)-\lambda_1|^{q_1\delta} dz\right)^{\frac{1}{q_1\delta}}\\
&\quad \times \left(\frac{1}{|B|}\int_B |b_2(z)-\lambda_2|^{q_2\delta} dz\right)^{\frac{1}{q_2\delta}}\\
&\le C\|b_1\|_{BMO}\|b_2\|_{BMO} M_{\delta l}(T(f_1,f_2))(x)\\
&\le C\|b_1\|_{BMO}\|b_2\|_{BMO} M_{\varepsilon_0}(T(f_1,f_2))(x).
\end{aligned}
$$

It follow from Hölder's inequality that

$$
\begin{aligned}
II &\le C\bigg(\frac{1}{|B|}\int_B |T^2_{b_2-\lambda_2}(f_1,f_2)(z)|^{l\delta} dz\bigg)^{\frac{1}{l\delta}}\left(\frac{1}{|B|}\int_B |b_1(z)-\lambda_1|^{l'\delta} dz\right)^{\frac{1}{l'\delta}}\\
&\le C\|b_1\|_{BMO} M_{\delta l}(T^2_{b_2-\lambda_2}(f_1,f_2))(x)\\
&\le C\|b_1\|_{BMO} M_{\varepsilon_0}(T^2_{b_2}(f_1,f_2))(x).
\end{aligned}
$$

Similarly we can get

$$III \le C\|b_2\|_{BMO} M_{\varepsilon_0}(T^1_{b_1}(f_1,f_2))(x).$$

Next, we estimate IV. Split f_i into two parts $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f\chi_{B^*}$ and $f_i^\infty = f_i - f_i^0$, $i = 1,2$. Choose $z_0 \in 4B\backslash 3B$. Denote $\vec{z_0} = (z_0, z_0)$ and $\vec{y} = (y_1, y_2)$. Let

$$C_0 = \sum_{(\alpha_1,\alpha_2)\in\Gamma} T((b_1-\lambda_1)f_1^{\alpha_1}, (b_2-\lambda_2)f_2^{\alpha_2})(z_0),$$

where $\Gamma = \{(\alpha_1, \alpha_2) : \text{there is at least one } \alpha_j \neq 0,\ j = 1, 2\}$, then

$$\begin{aligned}
IV &\leq C\bigg(\frac{1}{|B|}\int_B |T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\quad + C\sum_{(\alpha_1,\alpha_2)\in\Gamma}\bigg(\frac{1}{|B|}\int_B |T((b_1-\lambda_1)f_1^{\alpha_1},(b_2-\lambda_2)f_2^{\alpha_2})(z) \\
&\quad - T((b_1-\lambda_1)f_1^{\alpha_1},(b_2-\lambda_2)f_2^{\alpha_2})(z_0)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&:= IV_0 + \sum_{(\alpha_1,\alpha_2)\in\Gamma} IV_{\alpha_1,\alpha_2}.
\end{aligned}$$

Let $t = s/q'$, then it follows from $s > q'$ that $t > 1$. Since $r_j \leq q' < s$ for any $j = 1, 2$, we can get $r_j t \leq s$. By Lemma 2.4 and Hölder's inequality,

$$\begin{aligned}
IV_0 &\leq C|B|^{-1/\delta}\|T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)\|_{L^\delta(B)} \\
&\leq C|B|^{-1/r}\|T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)\|_{L^{r,\infty}(B)} \\
&\leq C\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|b_1(y_1)-\lambda_1|^{r_1}|f_1(y_1)|^{r_1}dy_1\bigg)^{\frac{1}{r_1}} \\
&\quad\times\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|b_2(y_2)-\lambda_2|^{r_2}|f_2(y_2)|^{r_2}dy_2\bigg)^{\frac{1}{r_2}} \\
&\leq C\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|f_1(y_1)|^{r_1t}dy_1\bigg)^{\frac{1}{r_1t}} \\
&\quad\times\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|b_1(y_1)-\lambda_1|^{r_1t'}dy_1\bigg)^{\frac{1}{r_1t'}} \\
&\quad\times\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|f_2(y_2)|^{r_2t}dy_2\bigg)^{\frac{1}{r_2t}} \\
&\quad\times\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|b_2(y_2)-\lambda_2|^{r_2t'}dy_2\bigg)^{\frac{1}{r_2t'}} \\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|f_1(y_1)|^{s}dy_1\bigg)^{\frac{1}{s}} \\
&\quad\times\bigg(\frac{1}{|16\sqrt{2}B|}\int_{16\sqrt{2}B}|f_2(y_2)|^{s}dy_2\bigg)^{\frac{1}{s}} \\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).
\end{aligned}$$

For any $(\alpha_1, \alpha_2) \in \Gamma$, we can find a $j \in \{1, 2\}$ that makes $\alpha_j = \infty$. Then for any $\vec{y} = (y_1, y_2) \in \operatorname{supp} f_1^{\alpha_1} \times \operatorname{supp} f_2^{\alpha_2}$ and $z \in B$, $|\vec{y} - \vec{z_0}| \geq 2\sqrt{2}|z - z_0|$ and

$2r_B \leq |z-z_0| \leq 5r_B$. Thus

$$
\begin{aligned}
&IV_{\alpha_1,\alpha_2}\\
&\leq C\frac{1}{|B|}\int_B\int_{|\vec{y}-\vec{z_0}|\geq 2\sqrt{2}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|\\
&\quad\times\prod_{j=1}^{2}|b_j(y_j)-\lambda_j||f_j(y_j)|d\vec{y}dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\int_{2^k\sqrt{2}|z-z_0|\leq|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{2}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|\\
&\quad\times\prod_{j=1}^{2}|b_j(y_j)-\lambda_j||f_j(y_j)|d\vec{y}dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\Big(\int_{2^k\sqrt{2}|z-z_0|\leq|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{2}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|^q\\
&\quad\times d\vec{y}\Big)^{\frac{1}{q}}\Big(\int_{|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{2}|z-z_0|}\prod_{j=1}^{2}|b_j(y_j)-\lambda_j|^{q'}|f_j(y_j)|^{q'}d\vec{y}\Big)^{\frac{1}{q'}}dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\Big(\int_{2^k\sqrt{2}|z-z_0|\leq|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{2}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|^q\\
&\quad\times d\vec{y}\Big)^{\frac{1}{q}}\prod_{j=1}^{2}\Big(\frac{1}{|B(z_0,2^{k+1}\sqrt{2}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+1}\sqrt{2}|z-z_0|}|b_j(y_j)-\lambda_j|^{q'}\\
&\quad\times|f_j(y_j)|^{q'}dy_j\Big)^{\frac{1}{q'}}|B(z_0,2^{k+1}\sqrt{2}|z-z_0|)|^{\frac{2}{q'}}dz\\
&\leq C\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\Big(\int_{2^k\sqrt{2}|z-z_0|\leq|\vec{y}-\vec{z_0}|<2^{k+1}\sqrt{2}|z-z_0|}|K(z,y_1,y_2)-K(z_0,y_1,y_2)|^q\\
&\quad\times d\vec{y}\Big)^{\frac{1}{q}}\prod_{j=1}^{2}\Big(\frac{1}{|B(x_0,2^{k+2}\sqrt{2}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+2}\sqrt{2}|z-z_0|}|b_j(y_j)-\lambda_j|^{q't'}\\
&\quad\times dy_j\Big)^{\frac{1}{q't'}}\prod_{j=1}^{2}\Big(\frac{1}{|B(z_0,2^{k+1}\sqrt{2}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+1}\sqrt{2}|z-z_0|}|f_j(y_j)|^{q't}dy_j\Big)^{\frac{1}{q't}}\\
&\quad\times|B(z_0,2^{k+1}\sqrt{2}|z-z_0|)|^{\frac{2}{q'}}dz\\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}|z-z_0|^{-\frac{2n}{q'}}C_k2^{-\frac{2kn}{q'}}k^2(2^{k+1}\sqrt{2}|z-z_0|)^{\frac{2n}{q'}}\\
&\quad\times\prod_{j=1}^{2}\Big(\frac{1}{|B(z_0,2^{k+1}\sqrt{2}|z-z_0|)|}\int_{|y_j-z_0|<2^{k+1}\sqrt{2}|z-z_0|}|f_j(y_j)|^{s}dy_j\Big)^{\frac{1}{s}}dz
\end{aligned}
$$

$\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$

Thus

$$\sum_{(\alpha_1,\alpha_2)\in\Gamma} IV_{\alpha_1,\alpha_2} \leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

So

$$IV \leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\mathcal{M}_s(\vec{f})(x).$$

Finally, we can get the result,

$$\begin{aligned}
M_\delta^\sharp(T_{\prod\vec{b}}(\vec{f}))(x) &= M^\sharp(|T_{\prod\vec{b}}(\vec{f})|^\delta)^{\frac{1}{\delta}}(x)\\
&\leq C\sup_{B\ni x}\left(\frac{1}{|B|}\int_B ||T_{\prod\vec{b}}(\vec{f})(z)|^\delta - |C_0|^\delta|dz\right)^{\frac{1}{\delta}}\\
&\leq C\|b_1\|_{BMO}\|b_2\|_{BMO}\Big(M_{\varepsilon_0}(T(\vec{f}))(x) + \mathcal{M}_s(\vec{f})(x)\Big)\\
&\quad + C(\|b_1\|_{BMO}M_{\varepsilon_0}(T_{b_2}^2(\vec{f}))(x) + \|b_2\|_{BMO}M_{\varepsilon_0}(T_{b_1}^1(\vec{f}))(x)).
\end{aligned}$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.4. Proceeding as in the proof of Theorem 3.2, we can find an s such that $q' < s$, $s < p_j$, $j = 1, \dots, m$ and $\vec{w} \in A_{\vec{P}/s}$.

Choose $\delta, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ satisfying $0 < \delta < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m < \frac{1}{m}$. We can deduce the conclusion by Lemma 2.6 and Lemma 2.9.

$$\begin{aligned}
\|M_{\varepsilon_j}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} &\leq C\|M_{\varepsilon_j}^\sharp(T(\vec{f}))\|_{L^p(v_{\vec{w}})}\\
&\leq C\|\mathcal{M}_{q'}(\vec{f})\|_{L^p(v_{\vec{w}})} \leq C\|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})}.
\end{aligned}$$

By Theorem 3.3, we have

$$\begin{aligned}
&\|M_\delta^\sharp(T_{\prod\vec{b}}(\vec{f}))\|_{L^p(v_{\vec{w}})}\\
&\leq C\prod_{j=1}^m\|b_j\|_{BMO}\Big(\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})}\Big)\\
&\quad + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i=1}^j\|b_{\sigma(i)}\|_{BMO}\|M_{\varepsilon_1}(T_{\prod\vec{b}_{\sigma'}}(\vec{f}))\|_{L^p(v_{\vec{w}})}\\
&\leq C\prod_{j=1}^m\|b_j\|_{BMO}\Big(\|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})}\Big)\\
&\quad + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i=1}^j\|b_{\sigma(i)}\|_{BMO}\|M_{\varepsilon_1}^\sharp(T_{\prod\vec{b}_{\sigma'}}(\vec{f}))\|_{L^p(v_{\vec{w}})}.
\end{aligned}$$

For the purpose of reducing the dimension of BMO functions in the commutators, we apply Theorem 3.3 to $\|M_{\varepsilon_1}^\sharp(T_{\prod\vec{b}_{\sigma'}}(\vec{f}))(x))\|_{L^p(v_{\vec{w}})}$.

Let $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ and $\sigma' = \{\sigma(j+1), \ldots, \sigma(m)\}$, A_h=$\{\sigma_1$: subsequence of σ' with $\sigma_1 \neq \sigma'\}$.

$$\begin{aligned}
&\|M^{\sharp}_{\varepsilon_1}(T_{\prod \vec{b}_{\sigma'}}(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\le C \prod_{i=j+1}^{m} \|b_{\sigma(i)}\|_{BMO} \Big(\|M_{\varepsilon_2}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})} \Big) \\
&\quad + C \sum_{h=1}^{m-j-1} \sum_{\sigma_1 \in A_h} \prod_{i=1}^{h} \|b_{\sigma_1(i)}\|_{BMO} \|M_{\varepsilon_2}(T_{\prod \vec{b}_{\sigma' \setminus \sigma_1}}(\vec{f}))\|_{L^p(v_{\vec{w}})}.
\end{aligned}$$

Repeating the process above and using Theorem 3.1, we can get

$$\begin{aligned}
&\|M^{\sharp}_{\delta}(T_{\prod \vec{b}}(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\le C \prod_{j=1}^{m} \|b_j\|_{BMO} \Big(A_{m+1}(m,n) \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})} \\
&\quad + A_1(m,n) \|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + A_2(m,n) \|M_{\varepsilon_2}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\quad + \cdots + A_m(m,n) \|M_{\varepsilon_m}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} \Big),
\end{aligned}$$

where $A_1(m,n), A_2(m,n), \ldots, A_{m+1}(m,n)$ are positive constants depending on m and n.

We can deduce the following conclusion from Lemma 2.6, Lemma 2.8 and $\vec{w} \in A_{\vec{P}/s}$.

$$\begin{aligned}
\|T_{\prod \vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})} &\le \|M_{\delta}(T_{\prod \vec{b}}(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\le C \|M^{\sharp}_{\delta}(T_{\prod \vec{b}}(\vec{f}))\|_{L^p(v_{\vec{w}})} \\
&\le C \prod_{j=1}^{m} \|b_j\|_{BMO} \Big(A_{m+1}(m,n) \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})} + A_1(m,n) \\
&\quad \times \|M_{\varepsilon_1}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + A_2(m,n) \|M_{\varepsilon_2}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} + \cdots \\
&\quad + A_m(m,n) \|M_{\varepsilon_m}(T(\vec{f}))\|_{L^p(v_{\vec{w}})} \Big) \\
&\le C \prod_{j=1}^{m} \|b_j\|_{BMO} \|\mathcal{M}_s(\vec{f})\|_{L^p(v_{\vec{w}})} \\
&\le C \prod_{j=1}^{m} \|b_j\|_{BMO} \|\|f_j\|_{L^{p_j}(w_j)}.
\end{aligned}$$

The proof is completed. □

References

[1] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331. https://doi.org/10.2307/1998628

[2] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, 57, Société Mathématique de France, Paris, 1978.

[3] R. R. Coifman and Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 3, xi, 177–202.

[4] R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635. https://doi.org/10.2307/1970954

[5] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. https://doi.org/10.1007/BF02392215

[6] L. Grafakos and N. J. Kalton, *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. **52** (2001), no. 2, 169–179.

[7] L. Grafakos and J. M. Martell, *Extrapolation of weighted norm inequalities for multivariable operators and applications*, J. Geom. Anal. **14** (2004), no. 1, 19–46. https://doi.org/10.1007/BF02921864

[8] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math. **165** (2002), no. 1, 124–164. https://doi.org/10.1006/aima.2001.2028

[9] L. Grafakos and R. H. Torres, *Discrete decompositions for bilinear operators and almost diagonal conditions*, Trans. Amer. Math. Soc. **354** (2002), no. 3, 1153–1176. https://doi.org/10.1090/S0002-9947-01-02912-9

[10] L. Grafakos and R. H. Torres, *On multilinear singular integrals of Calderón-Zygmund type*, Publ. Mat. **2002** (2002), Vol. Extra, 57–91. https://doi.org/10.5565/PUBLMAT_Esco02_04

[11] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. **220** (2009), no. 4, 1222–1264. https://doi.org/10.1016/j.aim.2008.10.014

[12] Y. Lin and S. Z. Lu, *Strongly singular Calderón-Zygmund operators and their commutators*, Jordan J. Math. Stat. **1** (2008), 31–49.

[13] Y. Lin and Y. Xiao, *Multilinear singular integral operators with generalized kernels and their multilinear commutators*, Acta Math. Sin. (Engl. Ser.) **33** (2017), no. 11, 1443–1462. https://doi.org/10.1007/s10114-017-7051-0

[14] Y. Lin and N. Zhang, *Sharp maximal and weighted estimates for multilinear iterated commutators of multilinear integrals with generalized kernels*, J. Inequal. Appl. **2017** (2017), Paper No. 276, 15 pp. https://doi.org/10.1186/s13660-017-1553-2

[15] G. Lu and P. Zhang, *Multilinear Calderón-Zygmund operators with kernels of Dini's type and applications*, Nonlinear Anal. **107** (2014), 92–117. https://doi.org/10.1016/j.na.2014.05.005

[16] C. Pérez and R. H. Torres, *Sharp maximal function estimates for multilinear singular integrals*, in Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 323–331, Contemp. Math., 320, Amer. Math. Soc., Providence, RI, 2003. https://doi.org/10.1090/conm/320/05615

[17] S. H. Yang, P. F. Li and Y. Lin, *Multiple weight inequalities for multilinear singular integral operators with generalized kernels*, Adv. Math. (China), accepted, to appear.

[18] P. Zhang and J. Sun, *Commutators of multilinear Calderón-Zygmund operators with kernels of Dini's type and applications*, J. Math. Inequal. **13** (2019), no. 4, 1071–1093. https://doi.org/10.7153/jmi-2019-13-76

Liwen Gao
School of Science
China University of Mining and Technology, Beijing
Beijing 100083, P. R. China
Email address: gaoliwen1206@163.com

Yan Lin
School of Science
China University of Mining and Technology, Beijing
Beijing 100083, P. R. China
Email address: linyan@cumtb.edu.cn

Shuhui Yang
School of Science
China University of Mining and Technology, Beijing
Beijing 100083, P. R. China
Email address: yangshuhui0601@163.com