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STRESS-ENERGY TENSOR OF THE TRACELESS RICCI TENSOR AND EINSTEIN-TYPE MANIFOLDS

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Abstract. In this paper, we introduce the notion of stress-energy tensor Q of the traceless Ricci tensor for Riemannian manifolds (M^n, g), and investigate harmonicity of Riemannian curvature tensor and Weyl curvature tensor when (M, g) satisfies some geometric structure such as critical point equation or vacuum static equation for smooth functions.

1. Introduction

A smooth map $\varphi : (M^n, g) \to (N^m, h)$ between Riemannian manifolds is said to be harmonic if it is a critical point of the energy functional defined by

$$\mathcal{E}(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 dv_g.$$

Here dv_g denotes the volume element of (M, g). The Euler-Lagrange equation associated with $\mathcal{E}$ is written by $\tau(\varphi) := \operatorname{div}(d\varphi) = 0$ and $\tau(\varphi)$ is called the tension field of φ. So φ is harmonic if and only if its tension field vanishes identically.

Now fix a smooth map $\varphi : M \to (N, h)$ from an n-dimensional smooth manifold M into a Riemannian m-manifold (N^m, h). For each Riemannian metric g on M, define the functional $\mathcal{F}$ by

$$\mathcal{F}(g) = \frac{1}{2}\int_M |d\varphi|_g^2 dv_g,$$

where $|d\varphi|_g$ is the norm of $d\varphi$ with respect to the metric g and h. Then it is well known [1] that the Euler-Lagrange equation for $\mathcal{F}$ is given by

$$S_\varphi := \frac{1}{2}|d\varphi|^2 g - \varphi^*(h) = 0,$$

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and S_φ is called the stress-energy tensor of φ. Baird and Eells [1] showed that $\mathrm{div}S_\varphi = -\langle \tau(\varphi), d\varphi\rangle$ and so, if $\varphi : (M, g) \to (N, h)$ is harmonic, then $\mathrm{div}S_\varphi = 0$. We say $\varphi : (M, g) \to (N, h)$ is conservative if $\mathrm{div}S_\varphi = 0$.

Xin [9] introduced the stress-energy tensor of vector bundle valued p-forms ω as follows:

$$S_\omega = \frac{1}{2}|\omega|^2 g - \omega \odot \omega,$$

where $\omega \odot \omega$ is defined by

$$\omega \odot \omega(X, Y) = \langle i_X\omega, i_Y\omega\rangle$$

and i_X is the interior product, and he proved that a closed and co-closed form satisfies the conservation law.

As we did for smooth maps between Riemannian manifolds and differential forms, we introduce the stress-energy tensor Q of the traceless Ricci tensor for Riemannian manifolds. Namely, for the traceless Ricci tensor $z = \mathrm{Ric} - \frac{s}{n}g$ of a Riemannian n-manifold (M^n, g), the stress-energy tensor Q of z is defined by

$$Q = \frac{1}{2}|z|^2 g - z \circ z. \tag{1.1}$$

Here Ric and s denote the Ricci tensor and the scalar curvature of the metric g, respectively, and $z \circ z$ is defined as

$$z \circ z(X, Y) = \sum_i z(X, e_i)z(Y, e_i)$$

for a local orthonormal frame $\{e_i\}$. It is easy to see that a Riemannian n-manifold with $n \geq 3$ is Einstein if and only if the stress-energy tensor Q of the traceless Ricci tensor vanishes identically (Lemma 2.1). We can also see later that the divergence of stress-energy tensor Q is deeply related to the divergence of the Bach tensor (see Section 2 for definition of Bach tensor). We say that Q is *conservative* if the divergence of Q vanishes identically. In case of dimension 4, if (M^4, g) has constant scalar curvature, then Q is always conservative (Corollary 2.7).

In this paper, we first study the stress-energy tensor Q of the traceless Ricci tensor, and investigate its relation to Cotton tensor C and Bach tensor B (see Section 2 below for definitions of Cotton tensor and Bach tensor). In case a Riemannian manifold (M^n, g) has constant scalar curvature, we have the following results.

Theorem 1.1. *Let (M^n, g) be an n-dimensional Riemannian manifold with constant scalar curvature. Then Q is conservative if and only if $\langle i_X C, z\rangle = 0$ for any vector X.*

Theorem 1.2. *Let (M^n, g), $n \geq 5$, be an n-dimensional Riemannian manifold with constant scalar curvature. Then Q is conservative if and only if $\delta B = 0$. Here δ denotes the (negative) divergence operator.*

Next, we study the relation of stress-energy tensor Q to harmonicity of Riemannian curvature tensor or Weyl curvature tensor for Einstein-type manifolds. A smooth Riemannian n-manifold (M^n, g) is called an *Einstein-type manifold* if there exist smooth functions $f, \lambda : M \to \mathbb{R}$ satisfying

$$f\mathrm{Ric} = Ddf + \lambda g, \tag{1.2}$$

where Ddf is the Hessian of f. The equation (1.2) is called *Einstein-type equation.* Note that if $\lambda = \frac{sf}{n-1}$, (1.2) becomes a vacuum static equation $s_g'^*(f) = 0$, or

$$f\mathrm{Ric} = Ddf + \frac{sf}{n-1} g. \tag{1.3}$$

Here $s_g'^*$ is the L^2-adjoint of the linearization s_g' of the scalar curvature s with respect to the metric g. When $\lambda = \frac{s(nf-1)}{n(n-1)}$, the Einstein-type equation is reduced to the critical point equation $z = s_g'^*(f)$, or

$$(1+f)z = Ddf + \frac{sf}{n(n-1)} g. \tag{1.4}$$

Hwang and the author [4] proved that if (M^n, g, f), $n \geq 5$, is an n-dimensional complete vacuum static space with compact level sets of f, and if $\delta^4\mathcal{W} = 0$ and $\delta^2 B \geq 0$, then (M, g) has harmonic curvature. Here $\mathcal{W}$ and B denote the Weyl curvature tensor and Bach tensor, respectively. We say a Riemannian manifold (M, g) has harmonic curvature if $\delta R = 0$, where R is the Riemannian curvature tensor. We would like to mention that $\delta^4\mathcal{W} = -\frac{n-3}{n-2}\delta^3 C$, where C is the Cotton tensor of (M, g) (see Section 2 for definition), and follow the convention in [2] so that $\delta = -\mathrm{div}$, the negative divergence operator. Throughout this paper, the dimension n of any manifold is assumed to be greater than or equal to 4 otherwise stated.

Related to the vacuum static equation (1.3), we first have the following.

Theorem 1.3. *Let (M^n, g, f) be a vacuum static space satisfying (1.3) with compact level sets of f. Assume that (M, g) has zero radial Weyl curvature, i.e., $\tilde{i}_{\nabla f}\mathcal{W} = 0$, and $\delta^4\mathcal{W} \leq 0$. Then (M, g) has harmonic curvature and Q is conservative.*

In case $n \geq 5$, we can obtain the following result which can be considered as an extension of a result in [4] to a little more general situation involving the stress-energy tensor Q of the traceless Ricci tensor.

Theorem 1.4. *Let (M^n, g, f) be a vacuum static space satisfying (1.3) with compact level sets of f. Assume that $\delta^4\mathcal{W} \leq 0$ and $\delta^2 Q \geq 0$. Then (M, g) has harmonic curvature and Q is conservative.*

For Riemannian manifolds satisfying the critical point equation (1.4), we have the following.

Theorem 1.5. *Let (g, f) be a non-trivial solution of the critical point equation* (1.4) *on an n-dimensional compact manifold M. Assume that $\delta^4\mathcal{W} \leq 0$ and $\delta^2 Q \geq 0$. Then (M, g) is isometric to a standard sphere $\mathbb{S}^n$.*

This result can be considered as a generalization of Corollary 1 in [6] since zero radial Weyl curvature condition is stronger than our conditions. In fact, we can also show the following.

Corollary 1.6. *Let (g, f) be a non-trivial solution of the critical point equation on an n-dimensional compact manifold M with zero radial Weyl curvature. If $\delta^4\mathcal{W} \leq 0$, then (M, g) is isometric to a standard sphere $\mathbb{S}^n$.*

Related to the Einstein-type equation (1.2), we obtain the following.

Theorem 1.7. *Let (g, f, λ) be a solution of an Einstein-type equation* (1.2) *on an n-dimensional manifold M. Assume that (M, g) has harmonic Weyl tensor. Then Q is conservative if and only if the scalar curvature s is constant.*

Next, Leandro [6] proved that if (M^n, g, f), $n \geq 4$, is a smooth Riemannian n-manifold satisfying the Einstein-type equation (1.2) with zero radial Weyl curvature and $\mathrm{div}^4\mathcal{W} = 0$, and each level set of f is compact, then (M, g) has harmonic Weyl curvature. We say a Riemannian manifold (M, g) has harmonic Weyl curvature if $\delta\mathcal{W} = 0$, or equivalently $C = 0$. We generalize this result as follows.

Theorem 1.8. *Let (g, f, λ) be a solution of an Einstein-type equation* (1.2) *on an n-dimensional manifold M with compact level sets of f. If (M, g) has zero radial Weyl curvature and $\delta^4\mathcal{W} \leq 0$, then (M, g) has harmonic Weyl curvature and Q is conservative.*

Theorem 1.9. *Let (g, f, λ) be a solution of an Einstein-type equation* (1.2) *on an n-dimensional manifold M with compact level sets of f. Assume that (M, g) has zero radial Weyl curvature and $\delta^4\mathcal{W} \leq 0$. Then Q is conservative if and only if the scalar curvature s is constant.*

This paper is organized as follows. In Section 2, we study basic properties on the stress-energy tensor Q of the traceless Ricci tensor, and then investigate its relations to Cotton tensor and Bach tensor including proofs for Theorem 1.1 and Theorem 1.2. Section 3 is devoted to introduce vacuum static spaces and critical point equation, and then prove Theorem 1.3, Theorem 1.4 and Theorem 1.5. In Section 4, we study Einstein-type manifolds and Einstein-type equation, and prove Theorem 1.7, Theorem 1.8 and Theorem 1.9.

2. Stress-energy tensor and Cotton tensor

First of all, we have the following observation by taking the trace of Q.

Lemma 2.1. *Let (M^n, g) be a Riemannian n-manifold with $n \geq 3$. Then (M, g) is Einstein if and only if $Q = 0$, or equivalently $\mathrm{tr}_g Q = 0$.*

As the stress-energy tensor of a smooth map from a smooth manifold into a Riemannian manifold, we can handle the stress-energy tensor of the traceless Ricci tensor as variational view point. Fix a Riemannian metric g on a smooth n-manifold M^n and consider the traceless Ricci tensor $z = r - \frac{r}{n}g$ with respect to the metric g. Denote by $\mathcal{M}$ the set of all smooth Riemannian metrics on M and define a functional $\mathcal{Q} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{Q}(\bar{g}) = \frac{1}{2}\int_M \left[|z|^2_{\bar{g}} + |z|^2_g\right] dv_{\bar{g}}.$$

Note that, for a variation $g_t = g + th$ with $h \in S^2(M)$ of the metric g, we have $|z|^2_{g_t} = g_t^{ik} g_t^{jl} z_{ij} z_{kl}$ and

$$\frac{d}{dt}\bigg|_{t=0} (dv_{g_t}) = \frac{1}{2}\mathrm{tr}_g(h), \quad \frac{d}{dt}\bigg|_{t=0} \left(|z|^2_{g_t}\right) = -2\langle z \circ z, h\rangle_g.$$

Here $S^2(M)$ denotes the space of all symmetric 2-tensor fields on M. So,

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{Q}(g + th) = \int_M \left\langle -z \circ z + \frac{1}{2}|z|^2 g, h\right\rangle dv_g. \tag{2.1}$$

Hence the Euler-Lagrange equation for $\mathcal{Q}$ is given by

$$Q = \frac{1}{2}|z|^2 g - z \circ z = 0.$$

Lemma 2.1 can be rephrased as follows. For a smooth n-manifold M^n with $n \geq 3$, a Riemannian metric g on M is critical for $\mathcal{Q}$ if and only if g is Einstein.

Let (M^n, g) be a Riemannian manifold of dimension n with the Levi-Civita connection D, and let h be a symmetric 2-tensor on M. The differential $d^D h$ is defined by

$$d^D h(X, Y, Z) = D_X h(Y, Z) - D_Y h(X, Z)$$

for any vectors X, Y and Z. From now on, let us denote by r the Ricci tensor for convenience if there is no ambiguity.

Definition 2.2. The Cotton tensor $C \in \Gamma(\Lambda^2 M \otimes T^*M)$ is defined by

$$C = d^D\left(\mathrm{Ric} - \frac{s}{2(n-1)}g\right) = d^D r - \frac{1}{2(n-1)}ds \wedge g. \tag{2.2}$$

Here for a smooth function φ and a symmetric 2-tensor h, $d\varphi \wedge h$ is defined as follows:

$$d\varphi \wedge h(X, Y, Z) = d\varphi(X)h(Y, Z) - d\varphi(Y)h(X, Z)$$

for any vector fields X, Y, Z.

Related to the Cotton tensor C, the followings are well-known.

- The Weyl curvature tensor $\mathcal{W}$ satisfies

$$\delta\mathcal{W} = -\frac{n-3}{n-2}d^D\left(\mathrm{Ric} - \frac{s_g}{2(n-1)}g\right) = -\frac{n-3}{n-2}C \tag{2.3}$$

under the following identification

$$\Gamma(T^*M \otimes \Lambda^2 M) \equiv \Gamma(\Lambda^2 M \otimes T^*M).$$

From this, we have $\mathrm{div}^4 \mathcal{W} = \frac{n-3}{n-2}\mathrm{div}^3 C$.

- Using $r = z + \frac{s}{n}g$, the Cotton tensor can be written as

$$C = d^D z + \frac{n-2}{2n(n-1)} ds \wedge g. \tag{2.4}$$

- Since $\delta(ds \wedge g) = -(\Delta s)g + Dds$, we have

$$\delta C = \delta d^D z + \frac{n-2}{2n(n-1)} Dds - \frac{n-2}{2n(n-1)}(\Delta s)g.$$

  Moreover, introducing a local frame $\{e_i\}$, and denoting $C_{ijk} = C(e_i, e_j, e_k)$, we have

$$\begin{aligned} \langle \delta C, z\rangle &= -C_{ijk;i} z_{jk} \\ &= -(C_{ijk} z_{jk})_{;i} + C_{ijk} z_{jk;i} \\ &= -(C_{ijk} z_{jk})_{;i} + \frac{1}{2}|C|^2, \end{aligned}$$

  where the semi-colon denotes covariant derivative.
- The cyclic summation of indices in C vanishes: $C_{ijk} + C_{jki} + C_{kij} = 0$, and trace of C in any two summands also vanishes.

Also it is easy to see that the following identities hold (cf. [2]).

$$\delta r = -\frac{1}{2} ds \quad \text{and} \quad \delta z = -\frac{n-2}{2n} ds. \tag{2.5}$$

Lemma 2.3. *Let (M, g) be a Riemannian n-manifold. Then for any vector field X, we have*

$$\langle i_X C, z\rangle = \frac{1}{2}X(|z|^2) + \delta(z \circ z)(X) + \frac{(n-2)^2}{2n(n-1)} z(\nabla s, X). \tag{2.6}$$

Proof. Let $\{e_i\}$ be a local frame which is normal at a point. Then, at the point, it follows from (2.4) and $\mathrm{tr}(z) = 0$ that

$$C(X, e_i, e_j)z(e_i, e_j) = \frac{1}{2}X(|z|^2) - D_{e_i} z(X, e_j)z(e_i, e_j) - \frac{n-2}{2n(n-1)} z(\nabla s, X).$$

It also follows from definition together with (2.5) that

$$\delta(z \circ z)(X) = -\frac{n-2}{2n} z(\nabla s, X) - z(e_i, e_j) D_{e_i} z(e_j, X). \tag{2.7}$$

So, since

$$-D_{e_i} z(X, e_j)z(e_i, e_j) - \frac{n-2}{2n(n-1)} z(\nabla s, X) = \delta(z \circ z)(X) + \frac{(n-2)^2}{2n(n-1)} z(\nabla s, X),$$

we obtain

$$\langle i_X C, z\rangle = \frac{1}{2}X(|z|^2) + \delta(z \circ z)(X) + \frac{(n-2)^2}{2n(n-1)} z(\nabla s, X).$$

□

Lemma 2.4. *Let (M, g) be a Riemannian n-manifold. Then for any vector field X, we have*

$$\delta Q(X) = -\langle i_X C, z\rangle + \frac{(n-2)^2}{2n(n-1)} z(\nabla s, X).$$

Proof. Since $\delta Q = -\frac{1}{2}d|z|^2 - \delta(z \circ z)$, this follows from Lemma 2.3. □

Corollary 2.5. *Let (M^n, g) be an n-dimensional Riemannian manifold with constant scalar curvature. Then Q is conservative if and only if $\langle i_X C, z\rangle = 0$ for any vector X.*

From Corollary 2.5 together with (2.3), we obtain directly the following.

Corollary 2.6. *Let (M^n, g) be an n-dimensional Riemannian manifold with constant scalar curvature. If (M, g) has harmonic Weyl tensor, i.e., $\delta\mathcal{W} = 0$, then Q is conservative.*

Now we consider the Bach tensor B and its relations to the stress-energy tensor Q of the traceless Ricci tensor. The *Bach tensor* B is defined by

$$B = \frac{1}{n-3}\delta^D \delta\mathcal{W} + \frac{1}{n-2}\mathring{\mathcal{W}}(r), \tag{2.8}$$

where δ^D is the adjoint of δ and $\mathring{\mathcal{W}}(r)$ is defined by

$$\mathring{\mathcal{W}}(r)(X, Y) = \mathrm{Ric}(\mathcal{W}(X, e_i)Y, e_i)$$

for a local frame $\{e_i\}$ (cf. [2]). Since

$$C = -\frac{n-2}{n-3}\delta\mathcal{W} \quad \text{and} \quad \delta C = -\frac{n-2}{n-3}\delta\delta\mathcal{W}, \tag{2.9}$$

the Bach tensor satisfies

$$B = \frac{1}{n-2}\left(-\delta C + \mathring{\mathcal{W}}(r)\right). \tag{2.10}$$

Moreover, since $\mathring{\mathcal{W}}(r) = \mathring{\mathcal{W}}(z)$, we have

$$B = \frac{1}{n-2}\left(-\delta C + \mathring{\mathcal{W}}(z)\right). \tag{2.11}$$

Here $\mathring{\mathcal{W}}(z)$ is defined similarly as $\mathring{\mathcal{W}}(r)$.

For $n \geq 4$ and for any vector field X, the following property holds in general (cf. [3] or [4]):

$$(n-2)\delta B(X) = -\frac{n-4}{n-2}\langle i_X C, z\rangle. \tag{2.12}$$

Because of dimension property, we have the following from Corollary 2.5 and (2.12).

Corollary 2.7. *Let (M^4, g) be a 4-dimensional Riemannian manifold with constant scalar curvature. Then Q is always conservative.*

Note that, in dimension $n = 3$, we just have $B = -\delta C$. The complete divergence of the Bach tensor has the following form for dimension $n \geq 4$:

$$\delta\delta B = \frac{n-4}{(n-2)^2}\left(\frac{1}{2}|C|^2 - \langle \delta C, z\rangle\right). \tag{2.13}$$

Combining Lemma 2.4 with (2.12), we obtain the following.

Lemma 2.8. *Let $n \geq 5$. Then the divergence of Q is given by*

$$\delta Q = \frac{(n-2)^2}{n-4}\delta B + \frac{(n-2)^2}{2n(n-1)} i_{\nabla s} z.$$

Here $i_{\nabla s}$ denotes the usual interior product to the first factor defined by $i_{\nabla s}z(X) = z(\nabla s, X)$ for any vector X.

Corollary 2.9. *Let (M^n, g), $n \geq 5$, be an n-dimensional Riemannian manifold with constant scalar curvature. Then Q is conservative if and only if $\delta B = 0$.*

Finally, we consider the complete divergence of the stress-energy tensor Q of the traceless Ricci tensor. First, it follows from Lemma 2.8 that

$$\begin{aligned}\delta\delta Q &= \frac{1}{2}\Delta|z|^2 - \delta\delta(z \circ z)\\ &= \frac{(n-2)^2}{n-4}\delta\delta B + \frac{(n-2)^2}{2n(n-1)}\delta i_{\nabla s} z. \qquad (2.14)\end{aligned}$$

Since, for a local frame $\{e_i\}$ normal at a point,

$$\begin{aligned}\delta i_{\nabla s} z &= -D_{e_i}(z(\nabla s, e_i))\\ &= -D_{e_i}z(\nabla s, e_i) - z(D_{e_i}\nabla s, e_i)\\ &= \delta z(\nabla s) - \langle Dds, z\rangle\\ &= -\frac{n-2}{2n}|\nabla s|^2 - \langle Dds, z\rangle, \qquad (2.15)\end{aligned}$$

we obtain

$$\delta\delta Q = \frac{(n-2)^2}{n-4}\delta\delta B - \frac{(n-2)^3}{4n^2(n-1)}|\nabla s|^2 - \frac{(n-2)^2}{2n(n-1)}\langle Dds, z\rangle. \tag{2.16}$$

Combining (2.13) with (2.16), we can also show the following identity for the complete divergence of Q.

Lemma 2.10. *Let (M^n, g) be a Riemannian n-manifold. Then*

$$\delta\delta Q = \frac{1}{2}|C|^2 - \langle \delta C, z\rangle - \frac{(n-2)^3}{4n^2(n-1)}|\nabla s|^2 - \frac{(n-2)^2}{2n(n-1)}\langle Dds, z\rangle. \tag{2.17}$$

Proof. The equation (2.17) can be deduced directly from Lemma 2.4. Let $\{e_i\}$ be a local frame which is normal at a point. Then, at the point, it follows from Lemma 2.4 that

$$\delta\delta Q = -e_i(\delta Q(e_i)) = -e_i\left(-\langle i_{e_i}C, z\rangle + \frac{(n-2)^2}{2n(n-1)}z(\nabla s, e_i)\right)$$

$$= -\langle \delta C, z\rangle + \frac{1}{2}|C|^2 + \frac{(n-2)^2}{2n(n-1)}\delta z(\nabla s) - \frac{(n-2)^2}{2n(n-1)}\langle Dds, z\rangle.$$

Substituting $\delta z = -\frac{n-2}{2n}ds$, we can obtain (2.17). □

For later use, we would like to mention some properties on divergence of the Cotton tensor and traceless Ricci tensor which hold in general for Riemannian manifolds.

Proposition 2.11 ([4]). *Let (M^n, g) be a Riemannian manifold. Then, we have*

$$\operatorname{div}^2 C(X) = \frac{1}{2}\langle \tilde{i}_X \mathcal{W}, C\rangle - \frac{1}{n-2}\langle i_X C, z\rangle$$

for any vector field X. Here, $\tilde{i}_X$ is the interior product to the last factor defined by $\tilde{i}_X \mathcal{W}(Y,Z,U) = \mathcal{W}(Y,Z,U,X)$ for any vectors Y, Z, and U.

Lemma 2.12. *Let (M^n, g) be a Riemannian manifold. Then*

$$\begin{aligned}\delta d^D z = D^*Dz + \frac{n}{n-2}z\circ z + \frac{s}{n-1}z \\ - \mathring{\mathcal{W}}(z) - \frac{1}{n-2}|z|^2 g + \frac{n-2}{2n}Dds.\end{aligned} \tag{2.18}$$

Proof. The following identity holds in general (cf. [2]):

$$\delta d^D r = D^*Dr + r\circ r - \mathring{R}(r) + \frac{1}{2}Dds. \tag{2.19}$$

Note that

(i) $D^*Dr = D^*Dz - \frac{1}{n}(\Delta s)g$,

(ii) $\delta d^D r = \delta d^D\left(z + \frac{s}{n}g\right) = \delta d^D z - \frac{1}{n}(\Delta s)g + \frac{1}{n}Dds$,

(iii) $r\circ r = z\circ z + \frac{2s}{n}z + \frac{s^2}{n^2}g$,

(iv) $\mathring{R}(r) = \mathring{W}(z) + \frac{1}{n-2}|z|^2 g + \frac{n-2}{n(n-1)}sz - \frac{2}{n-2}z\circ z + \frac{s^2}{n^2}g.$

Substituting these into (2.19), we obtain

$$\delta d^D z = D^*Dz + \frac{n}{n-2}z\circ z + \frac{1}{n-1}sz - \mathring{W}(z) - \frac{1}{n-2}|z|^2 g + \frac{n-2}{2n}Dds.$$ □

3. Vacuum static equation and critical point equation

In this section, we study the harmonicity of Reimannian curvature tensor and conservation law of the stress-energy tensor Q of traceless Ricci tensor for Riemannian manifolds satisfying vacuum static equation (1.3) or critical point equation (1.4). Recall that, using $z = r - \frac{s}{n}$, vacuum static equation can be rewritten as

$$fz = Ddf + \frac{sf}{n(n-1)}g. \tag{3.1}$$

It is well-known that a Riemannian manifold satisfying either vacuum static equation or critical point equation has positive constant scalar curvature. So, in these cases, we have

$$\delta Q = \frac{(n-2)^2}{n-4}\delta B \quad \text{and} \quad \delta\delta Q = \frac{(n-2)^2}{n-4}\delta\delta B$$

when $n \geq 5$. Furthermore, since $\Delta f = -\frac{s}{n-1}$ by taking the trace of (3.1), we always assume that the potential function f attains both positive and negative values.

To investigate the harmonicity of Riemannian curvature tensor, we introduce a 3-tensor T. Namely, let (M^n, g, f) be an n-dimensional Riemannian manifold whose potential function f satisfies the vacuum static equation (1.3) or critical point equation (1.4). Define T by

$$T = \frac{1}{n-2}\, df \wedge z + \frac{1}{(n-1)(n-2)}\, i_{\nabla f} z \wedge g, \tag{3.2}$$

where $i_{\nabla f}$ denotes the usual interior product to the first factor defined by $i_{\nabla f} z(X) = z(\nabla f, X)$ for any vector X.

Here, we will prove that nonnegative complete divergences of Cotton tensor and nonpositive complete divergence of stress-energy tensor on a Riemannian manifold satisfying vacuum static equation or critical point equation has harmonic curvature for $n \geq 5$ if the level sets of the potential function f are compact. To do this, we first invoke some identities on the tensor T and the Cotton tensor C in [4].

Lemma 3.1 ([4]). *Let (M, g, f) be a vacuum static space. Then*

$$f\, C = \tilde{i}_{\nabla f}\mathcal{W} - (n-1)\, T.$$

Lemma 3.2 ([4]). *On a vacuum static space we have*

$$\mathrm{div}^2 C(\nabla f) = \frac{1}{2} f|C|^2 + \langle i_{\nabla f} C, z\rangle. \tag{3.3}$$

Proposition 3.3 ([4]). *Let (M, g, f) be a vacuum static space. If $T = 0$, then $B = 0$ and $C = 0$.*

Lemma 3.4. *We have*

$$\langle T, C\rangle = \frac{2}{n-2}\langle i_{\nabla f} C, z\rangle. \tag{3.4}$$

In particular, if either $\delta Q = 0$, or $\delta B = 0$ and $n \geq 5$, then $\langle T, C\rangle = 0$ and so

$$(n-1)|T|^2 = \langle \tilde{i}_{\nabla f}\mathcal{W}, T\rangle \quad \textit{and} \quad f|C|^2 = \langle \tilde{i}_{\nabla f}\mathcal{W}, C\rangle.$$

Proof. Introducing a geodesic orthonormal frame $\{e_i\}$, and denoting $T_{ijk} = T(e_i, e_j, e_k)$ and $C_{ijk} = C(e_i, e_j, e_k)$, we can write, using Einstein convention on the sum,

$$(n-2)T_{ijk} = f_i z_{jk} - f_j z_{ik} + \frac{1}{n-1}\,(f_l z_{li}\delta_{jk} - f_l z_{lj}\delta_{ik})$$

and

$$C_{ijk}\delta_{ik} = C_{ijk}\delta_{jk} = 0, \quad C_{ijk} = -C_{jik}.$$

Thus,

$$\begin{aligned}(n-2)\langle T, C\rangle &= f_i z_{jk} C_{ijk} - f_j z_{ik} C_{ijk} + \frac{1}{n-1}\left(f_l z_{li} C_{ijk}\delta_{jk} - f_l z_{lj} C_{ijk}\delta_{ik}\right)\\ &= 2 f_i z_{jk} C_{ijk} = 2\langle i_{\nabla f} C, z\rangle.\end{aligned}$$

If $\delta Q = 0$, from Lemma 2.4, we have $\langle i_{\nabla f} C, z\rangle = 0$, and the last two equalities follow from Lemma 3.1. □

For vacuum static spaces (M, g, f), recall that vanishing of the Cotton tensor C is equivalent to the harmonicity of Riemannian curvature tensor since $\delta R = -d^D r = C$ when the scalar curvature is constant. Also, by Lemma 3.1, we can see that vanishing of the Cotton tensor C is equivalent to $\tilde{i}_{\nabla f}\mathcal{W} = 0$ when $T = 0$. We say that, on vacuum static spaces, (M, g) has radially zero Weyl curvature if $\tilde{i}_{\nabla f}\mathcal{W} = \mathcal{W}(\cdot,\cdot,\cdot,\nabla f) = 0$, or equivalently $i_{\nabla f}\mathcal{W} = 0$. The following result is a generalization of Proposition 3.3 in [6] in some sense.

Theorem 3.5. *Let (M^n, g, f) be a vacuum static space with compact level sets of f. Assume that $\mathrm{div}^3 C \le 0$ and $\tilde{i}_{\nabla f}\mathcal{W} = 0$. Then (M, g) has harmonic curvature and Q is conservative.*

Proof. By Lemma 3.1 together with our assumption, we have $fC = -(n-1)T$ and so

$$f|C|^2 = -(n-1)\langle T, C\rangle = -\frac{2(n-1)}{n-2}\langle i_{\nabla f} C, z\rangle.$$

The second equality follows from Lemma 3.4. Substituting this into (3.3), we obtain

$$\mathrm{div}^2 C(\nabla f) = \frac{1}{2(n-1)} f|C|^2.$$

So, applying the divergence theorem for two regular values t_1, t_2 $(t_1 < t_2)$ of f, we have

$$\begin{aligned}\int_{t_1\le f\le t_2} \mathrm{div}^3 C\, dv_g &= \int_{f=t_2} \mathrm{div}^2 C(N)\, d\sigma - \int_{f=t_1} \mathrm{div}^2 C(N)\, d\sigma\\ &= \frac{t_2}{2(n-1)}\int_{f=t_2}\frac{|C|^2}{|\nabla f|}d\sigma - \frac{t_1}{2(n-1)}\int_{f=t_1}\frac{|C|^2}{|\nabla f|}d\sigma.\end{aligned}$$

Since $\mathrm{div}^3 C \ge 0$, we obtain

$$t_2\int_{f=t_2}\frac{|C|^2}{|\nabla f|}d\sigma \le t_1\int_{f=t_1}\frac{|C|^2}{|\nabla f|}d\sigma. \tag{3.5}$$

Now by taking $t_1 = 0$, $t_2 > 0$ and $t_2 = 0$, $t_1 < 0$ in turns, we have $C = 0$, which shows (M, g) has harmonic curvature since the scalar curvature is constant. Finally, by Corollary 2.5, we can see that Q is conservative. □

Lemma 3.6. *Let (M^n, g, f) be an n-dimensional vacuum static space with compact level sets of f. Assume that* $\operatorname{div}^2 Q \geq 0$. *Then*

$$\int_{f=t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma$$

is monotone increasing with respect to regular values t's of f.

Proof. Since the scalar curvature is constant, it follows from Lemma 2.10 that

$$\frac{1}{2}|C|^2 \geq -\langle \operatorname{div} C, z\rangle. \tag{3.6}$$

For a geodesic orthonormal frame $\{e_i\}_{i=1}^n$, we claim

$$C_{ijk} z_{jk;i} = C(e_i, e_j, e_k) D_{e_i} z(e_j, e_k) = \frac{1}{2}|C|^2.$$

In fact, since $C_{ijk} z_{ik;j} = -C_{jik} z_{ik;j} = -C_{ijk} z_{jk;i}$, we have

$$|C|^2 = C_{ijk}^2 = C_{ijk}\left(z_{jk;i} - z_{ik;j}\right) = 2C_{ijk} z_{jk;i}.$$

Thus, we compute

$$\langle \operatorname{div} C, z\rangle = C_{ijk;i} z_{jk} = (C_{ijk} z_{jk})_{;i} - C_{ijk} z_{jk;i} = (C_{ijk} z_{jk})_{;i} - \frac{1}{2}|C|^2.$$

Thus, by (3.6) we have

$$\operatorname{div}(C(\cdot, e_j, e_k) z_{jk}) = (C_{ijk} z_{jk})_{;i} \geq 0$$

on M. This implies that, for two regular values t_1, t_2 $(t_1 < t_2)$ of f,

$$\begin{aligned} 0 &\leq \int_{t_1 \leq f \leq t_2} \operatorname{div}(C(\cdot, e_j, e_k) z_{jk}) \, dv_g \\ &= \int_{f=t_2} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma - \int_{f=t_1} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma. \end{aligned}$$

□

Theorem 3.7. *Let (M^n, g, f) be a vacuum static space with compact level sets of f satisfying* $\operatorname{div}^3 C \leq 0$ *and* $\operatorname{div}^2 Q \geq 0$. *Then (M, g) has harmonic curvature and Q is conservative.*

Proof. First, suppose that $f^{-1}(0) = \emptyset$ and we may assume that $f > 0$; otherwise we may take $-f$ instead of f. In this case, it follows from Lemma 3.6 that

$$\int_{f=t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma \geq 0$$

for any value $t > 0$. For a regular value $t > 0$, the divergence theorem together with Lemma 3.2 and Lemma 3.6 shows

$$\begin{aligned} \int_{f \leq t} \operatorname{div}^3 C dv_g &= \int_{f=t} \operatorname{div}^2 C(N) d\sigma \\ &= \frac{1}{2}\int_{f=t} \frac{f|C|^2}{|\nabla f|} d\sigma + \int_{f=t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma \geq \frac{t}{2}\int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma. \end{aligned}$$

Since $\operatorname{div}^3 C \leq 0$ and $t > 0$, this implies that $C = 0$ on M.

Now assume that $f^{-1}(0) \neq \emptyset$. For regular values t_1 and t_2 of f with $t_1 < t_2$, by Lemma 3.2 and Lemma 3.6, we have

$$\begin{aligned}
\int_{t_1 \leq f \leq t_2} \operatorname{div}^3 C \, dv_g &= \int_{f=t_2} \operatorname{div}^2 C(N) \, d\sigma - \int_{f=t_1} \operatorname{div}^2 C(N) \, d\sigma \\
&= \frac{t_2}{2} \int_{f=t_2} \frac{|C|^2}{|\nabla f|} \, d\sigma + \int_{f=t_2} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z \rangle \, d\sigma \\
&\quad - \frac{t_1}{2} \int_{f=t_1} \frac{|C|^2}{|\nabla f|} \, d\sigma - \int_{f=t_1} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z \rangle \, d\sigma \\
&\geq \frac{t_2}{2} \int_{f=t_2} \frac{|C|^2}{|\nabla f|} \, d\sigma - \frac{t_1}{2} \int_{f=t_1} \frac{|C|^2}{|\nabla f|} \, d\sigma. \qquad (3.7)
\end{aligned}$$

Consequently, by assumption $\operatorname{div}^3 C \leq 0$, we have

$$\frac{t_2}{2} \int_{f=t_2} \frac{|C|^2}{|\nabla f|} \, d\sigma \leq \frac{t_1}{2} \int_{f=t_1} \frac{|C|^2}{|\nabla f|} \, d\sigma$$

for any two regular values t_1, t_2 with $t_1 < t_2$.

By taking $t_1 = 0$ we may conclude that $C = 0$ on the set $\{x : f(x) \geq 0\}$. Similarly, by taking $t_2 = 0$ we may conclude that $C = 0$ on $f = t_1$ for all regular values t_1 of f with $t_1 < t_2 = 0$. Hence, we have $C = 0$ on the whole M by continuity. In other words, M has harmonic curvature. Finally, we can see from Lemma 2.4 that Q is conservative. □

Now we consider a converse of Theorem 3.5 and Theorem 3.7.

Proposition 3.8. *Let (M^n, g, f) be an n-dimensional vacuum static space. If $\delta Q = 0$ and $\tilde{i}_{\nabla f} \mathcal{W} = 0$, then (M, g) is Bach-flat.*

Proof. The proof follows from Proposition 3.3 and Lemma 3.4. □

Theorem 3.9. *Let (M^n, g, f) be an n-dimensional vacuum static space with compact level sets $(n \geq 3)$. If $\delta Q = 0$ and $\tilde{i}_{\nabla f} \mathcal{W} = 0$, then, up to finite cover and appropriate scaling, either (M, g) is Ricci-flat, or isometric to one of $\mathbb{S}^n$, $\mathbb{H}^n$ or the warped product $\mathbb{S}^1 \times \Sigma$ or $\mathbb{R} \times \Sigma$, where Σ is a compact Einstein manifold of positive scalar curvature.*

Proof. The proof follows from the main result in [8] and Proposition 3.8. □

Example 3.10. Consider the product manifold $M = \mathbb{R} \times \mathbb{S}^{n-1}$ with the product metric $g = dt^2 + g_0$, where g_0 is the round metric on $\mathbb{S}^{n-1}$. Letting $(\theta_2, \ldots, \theta_n)$ be the spherical coordinates on $\mathbb{S}^{n-1}$ so that $\{\partial_t = \frac{\partial}{\partial t}, \partial_i = \partial_{\theta_i}\}$ becomes a frame on M, we have

$$s = (n-1)(n-2)$$

and

$$z(\partial_t, \partial_t) = -\frac{(n-1)(n-2)}{n}, \quad z(\partial_t, \partial_j) = 0, \quad z(\partial_i, \partial_j) = \frac{n-2}{n}\delta_{ij}.$$

Define the function f on M by

$$f(t, \theta_2, \ldots, \theta_n) = A \sin \sqrt{n-2}\, t + B \cos \sqrt{n-2}\, t$$

for two constants A, B. Then it is easy to see that f satisfies the following vacuum static equation

$$fz = Ddf + \frac{sf}{n(n-1)} g.$$

For a compact Einstein-type manifold, we consider

$$M = \mathbb{S}^1\left(\frac{1}{\sqrt{n-2}}\right) \times \mathbb{S}^{n-1}.$$

Then we can see that the function

$$f(t, \theta_2, \ldots, \theta_n) = \cos \sqrt{n-2}\, t$$

solves the vacuum static equation.

In both cases, we have

$$|z|^2 = \frac{(n-1)(n-2)^2}{n}, \quad |i_N z|^2 = \frac{(n-1)^2(n-2)^2}{n^2}$$

and

$$z \circ z(\partial_t, \partial_t) = \left(\frac{(n-1)(n-2)}{n}\right)^2, \ z \circ z(\partial_t, \partial_i) = 0, \ z \circ z(\partial_i, \partial_j) = \left(\frac{n-2}{n}\right)^2 \delta_{ij}.$$

So, $\delta Q = 0$, i.e., Q is conservative. One can also easily check that (M, g) is Bach flat and has harmonic curvature. In fact, since M is locally conformally flat, the conclusion follows from Proposition 3.8.

Now we consider Riemannian manifolds satisfying critical point equation (1.4). We call the critical point equation CPE in short. Whenever we consider the critical point equation, we always assume that M is compact without boundary. For a solution (g, f) of the CPE, we may assume that

$$\min_M f < -1.$$

In fact, if the minimum of f is greater than or equal to -1, it is easily see that (M, g) is Einstein and so due to a result [7] of Obata, (M, g) must be isometric to a standard sphere. In fact, for critical point equation, we have $\mathrm{div}(i_{\nabla f} z) = (1+f)|z|^2$ and so by integrating it over M, we have $z = 0$ by divergence theorem.

As in the proofs of Lemma 3.1 and Lemma 3.2, we can prove the following two identities for critical point equation.

Lemma 3.11. *Let (M, g, f) be a non-trivial solution of the CPE. Then*

$$(1+f)\, C = \tilde{i}_{\nabla f} \mathcal{W} - (n-1)\, T.$$

Lemma 3.12. *For a solution of the CPE, we have*

$$\operatorname{div}^2 C(\nabla f) = \frac{1}{2}(1+f)|C|^2 + \langle i_{\nabla f} C, z\rangle.$$

The exactly same proof of Lemma 3.6 shows the following in CPE case.

Lemma 3.13. *Assume that* $\operatorname{div}^2 Q \geq 0$ *for a solution* (M, g, f) *of the CPE. Then*

$$\int_{f=t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z\rangle \, d\sigma$$

is nonnegative and monotone increasing with respect to regular values $t's$ *of* f.

Theorem 3.14. *Let* (g, f) *be a non-trivial solution of the CPE on an* n*-dimensional compact manifold* M. *If* $\operatorname{div}^3 C \leq 0$ *and* $\operatorname{div}^2 Q \geq 0$, *then* (M, g) *is isometric to a standard sphere* $\mathbb{S}^n$.

Proof. As in the proof of Theorem 3.7, by Lemma 3.12 and Lemma 3.13 together with the assumption $\operatorname{div}^3 C \leq 0$, we have

$$\frac{1+t_2}{2} \int_{f=t_2} \frac{|C|^2}{|\nabla f|} \leq \frac{1+t_1}{2} \int_{f=t_1} \frac{|C|^2}{|\nabla f|}$$

for any two regular values t_1, t_2 with $t_1 < t_2$.

By taking $t_1 = -1 - \epsilon$, regular value of f for sufficiently small ϵ with $\epsilon > 0$, we may conclude that $C = 0$ on the set $\{x : f(x) \geq -1\}$. Similarly, by taking $t_2 = -1 + \epsilon$, regular value of f for sufficiently small ϵ with $\epsilon > 0$, we may conclude that $C = 0$ on $f = t_1$ for all regular values t_1 of f with $t_1 < -1$. Hence, we have $C = 0$ on the whole M by continuity. In other words, M has harmonic curvature. The conclusion follows from a result in [10] and [11]. $\square$

Proposition 3.15. *Let* (g, f) *be a nontrivial solution of the CPE on a compact smooth* n*-manifold* M. *Assume that* $\operatorname{div}^3 C \leq 0$ *and* $\tilde{i}_{\nabla f}\mathcal{W} = 0$. *Then* $\operatorname{div}^2 Q \geq 0$.

Proof. Since the scalar curvature is constant, from Lemma 2.4 and Proposition 2.11, we have

$$\operatorname{div}^2 C = \frac{1}{n-2}\delta Q$$

and so $\delta\delta Q = -(n-2)\operatorname{div}^3 C$. $\square$

From Theorem 3.14 and Proposition 3.15, we can see that the exactly same property as Theorem 3.5 holds for the critical point equation. This result is an extension of a result in [6].

Corollary 3.16. *Let* (g, f) *be a non-trivial solution of the critical point equation on an* n*-dimensional compact manifold* M. *If* $\operatorname{div}^3 C \leq 0$ *and* $\tilde{i}_{\nabla f}\mathcal{W} = 0$, *then* (M, g) *is isometric to a standard sphere* $\mathbb{S}^n$.

4. Einstein-type equation

In this section, we consider the harmonicity of Weyl tensor and conservation law of the stress-energy tensor of the traceless Ricci tensor for Riemannian manifolds satisfying the Einstein-type equation (1.2). Note that the scalar curvature of a Riemannian manifold satisfying the Einstein-type equation is not necessarily constant. By letting

$$\lambda = h - \frac{sf}{n-1},$$

and using $z = \mathrm{Ric} - \frac{s}{n}g$, the Einstein-type equation (1.2) can be rewritten in the following form

$$fz = Ddf + \left(\frac{sf}{n(n-1)} + \lambda\right) g. \tag{4.1}$$

Taking the trace of (4.1), we have

$$\Delta f = -\frac{s}{n-1}f - n\lambda. \tag{4.2}$$

Also by taking the divergence in (4.1), and using $\delta z = -\frac{n-2}{2n}ds$ and (4.2), we obtain

$$\frac{f}{2}ds + (n-1)d\lambda = 0. \tag{4.3}$$

This shows that an Einstein-type manifold does not necessarily have constant scalar curvature which is different from both CPE case and vacuum static equation. In particular, if the scalar curvature is constant, then λ must be constant and vice versa.

If $f = 0$, then λ should also be zero, and if f is a nonzero constant, from (4.1) and (4.2), we have $\lambda = -\frac{s}{n(n-1)}f$ and $fz = 0$, which mean (M, g) is Einstein. So if $n \geq 3$, λ must be constant since the scalar curvature is constant. From now on, we assume that (g, f, λ) is a nontrivial solution of an Einstein-type equation which means f is not a constant function.

By taking the exterior derivative d of (4.3), we have

$$df \wedge ds = 0, \tag{4.4}$$

which means 1-forms df and ds are parallel. In fact, we have

$$ds(\nabla f)df = |\nabla f|^2 ds, \quad \langle \nabla s, \nabla f\rangle^2 = |\nabla s|^2|\nabla f|^2 \tag{4.5}$$

and

$$ds = \frac{\langle \nabla f, \nabla s\rangle}{|\nabla f|^2}df \quad \text{on} \quad M \setminus \mathrm{Crit}(f),$$

where $\mathrm{Crit}(f)$ denotes the set of all critical points of f on M. So,

$$i_{\nabla s}z = \frac{\langle \nabla f, \nabla s\rangle}{|\nabla f|^2} i_{\nabla f}z. \tag{4.6}$$

Lemma 4.1. *Let* (g, f, λ) *be a solution of an Einstein-type equation* (4.1) *on an* n*-dimensional manifold* M. *Then*

$$\delta i_{\nabla f} z = -f|z|^2 - \frac{n-2}{2n}\langle \nabla s, \nabla f\rangle. \tag{4.7}$$

Proof. Introducing a geodesic orthonormal frame $\{e_i\}$ which is normal at a point, it follows from definition that, at the point,

$$\begin{aligned}\delta i_{\nabla f} z &= -e_i(z(\nabla f, e_i))\\ &= -D_{e_i} z(\nabla f, e_i) - z(D_{e_i}\nabla f, e_i)\\ &= \delta z(\nabla f) - f z(e_i, e_j) z(e_j, e_i)\\ &= -\frac{n-2}{2n}\langle \nabla s, \nabla f\rangle - f|z|^2.\end{aligned}$$

□

For Einstein-type manifolds, using (4.3), we can show that the exactly same property as Lemma 3.1 and Lemma 3.11 holds.

Lemma 4.2 (cf. [12])**.** *Let* (g, f, λ) *be a solution of the Einstein-type equation* (4.1)*. Then*

$$f\,C = \tilde{i}_{\nabla f}\mathcal{W} - (n-1)\,T. \tag{4.8}$$

Recall that the tensor T is defined by

$$T = \frac{1}{n-2} df \wedge z_g + \frac{1}{(n-1)(n-2)} i_{\nabla f} z_g \wedge g.$$

Lemma 4.3. *On an Einstein-type equation, we have*

$$\mathrm{div}^2 C(\nabla f) = \frac{1}{2} f|C|^2 + \langle i_{\nabla f} C, z\rangle.$$

Proof. Note that

$$\langle T, C\rangle = \frac{1}{n-2}\langle df \wedge z, C\rangle = \frac{2}{n-2}\langle i_{\nabla f} C, z\rangle.$$

Thus, by Proposition 2.11 and Lemma 4.2, we have

$$\begin{aligned}\mathrm{div}^2 C(\nabla f) &= \frac{1}{2} f|C|^2 + \frac{n-1}{2}\langle T, C\rangle - \frac{1}{n-2}\langle i_{\nabla f} C, z\rangle\\ &= \frac{1}{2} f|C|^2 + \langle i_{\nabla f} C, z\rangle.\end{aligned}$$

□

Lemma 4.4. *Let* (g, f, λ) *be a nontrivial solution of an Einstein-type equation* (4.1) *on an* n*-dimensional manifold* M. *If* $C = 0$*, then for any vector* X *which is orthogonal to* ∇f,

$$z(\nabla f, X) = 0.$$

Proof. Since $C = 0$, by Lemma 4.2,

$$\tilde{i}_{\nabla f}\mathcal{W} = (n-1)\,T.$$

Substituting a triple $(X, Y, \nabla f)$ into this, we obtain

$$\begin{aligned} 0 &= (n-1)T(X, Y, \nabla f) \\ &= df(X)z(Y, \nabla f) - df(Y)z(X, \nabla f). \end{aligned}$$

In the second equality, we used the identity (4.3): $\frac{f}{2}ds + (n-1)d\lambda = 0$. Since X and Y are arbitrary, by choosing $X \perp \nabla f$ and $Y = \nabla f$, we have

$$|df|^2 z(X, \nabla f) = 0. \qquad \square$$

If (g, f, λ) is a nontrivial solution of an Einstein-type equation (4.1) on an n-dimensional manifold M satisfying $z(\nabla f, X) = 0$ for any vector field X orthogonal to ∇f, then we can write

$$i_{\nabla f} z = \alpha df, \quad \alpha = z(N, N), \;\; N = \frac{\nabla f}{|\nabla f|},$$

as a 1-form. Moreover, since $|\alpha| \leq |z|$, the function α can be extended on the whole manifold M to a C^0-function. On the other hand, the orthogonality $z(\nabla f, X) = 0$ for $X \perp \nabla f$ implies that

$$D_N N = 0 \quad \text{and} \quad D_{e_i} N = \frac{1}{|\nabla f|} Ddf(e_i, X) = 0$$

for a local frame $\{e_1 = N, e_2, \ldots, e_n\}$ (cf. [5]).

Theorem 4.5. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M. Assume that (M, g) has harmonic Weyl tensor. Then Q is conservative if and only if the scalar curvature s is constant.*

Proof. Since $C = 0$, by Lemma 2.4, we have

$$\delta Q = \frac{(n-2)^2}{2n(n-1)} i_{\nabla s} z.$$

If s is constant, then $\delta Q = 0$ trivially. Conversely, assume that Q is conservative. Since ∇s and ∇f are parallel, by Lemma 4.4,

$$z(\nabla s, X) = 0$$

for any vector X with $X \perp \nabla f$. This together with (4.6) implies that

$$i_{\nabla s} z = \alpha ds.$$

Since $\delta Q = 0$, either $\alpha = 0$ or s is constant. In the last of the proof, we will show that $\alpha = 0$ on the whole M or $ds = 0$ on the whole M. First we are going to show the following Assertion.

Assertion. If $\alpha = 0$, then s is constant.

If $\alpha = 0$, then $i_{\nabla f} z = \alpha df = 0$ and so, by definition of T, we have

$$i_{\nabla f} T = \frac{1}{n-2} |\nabla f|^2 z.$$

Since $C = 0$, we have $\tilde{i}_{\nabla f} \mathcal{W} = (n-1)T$. So,

$$\frac{n-1}{n-2} |\nabla f|^2 z = (n-1) i_{\nabla f} T = -|\nabla f|^2 \mathcal{W}_N,$$

where $\mathcal{W}_N$ is defined by $\mathcal{W}_N(X,Y) = \mathcal{W}(X,N,Y,N)$ with $N = \frac{\nabla f}{|\nabla f|} = \pm \frac{\nabla s}{|\nabla s|}$. In other words, we obtain

$$z = -\frac{n-2}{n-1} \mathcal{W}_N. \tag{4.9}$$

Taking the divergence of (4.9), we have

$$-\frac{n-2}{2n} ds = \delta z = -\frac{n-2}{n-1} \delta \mathcal{W}_N = 0,$$

which completes our assertion. In fact, we have

$$\begin{aligned} \delta \mathcal{W}_N(X) &= -D_{e_i} \mathcal{W}_N(e_i, X) = -e_i(\mathcal{W}(e_i, N, X, N)) \\ &= \delta \mathcal{W}(N, X, N) = -\frac{n-3}{n-2} C(N, X, N) = 0. \end{aligned}$$

Finally, we will prove that if Q is conservative, then the set, $\mathrm{Crit}(s)$, of all critical points of s is closed and open. It is obvious that $\mathrm{Crit}(s)$ is closed. To show $\mathrm{Crit}(s)$ is open, let $p \in \mathrm{Crit}(s)$ be an isolated critical point. Then $\alpha = 0$ on a neighborhood of p. The proof of Assertion shows that $ds = 0$ around p, which is a contradiction. □

Lemma 4.6. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M. If $\tilde{i}_{\nabla f} \mathcal{W} = 0$, then*

$$\mathrm{div}^2 C(\nabla f) = -\frac{1}{n-2} \langle i_{\nabla f} C, z \rangle = \frac{1}{2(n-1)} f |C|^2$$

and

$$f|C|^2 = \frac{2(n-1)}{n-2} \delta Q(\nabla f) - \frac{n-2}{n} z(\nabla s, \nabla f).$$

Proof. If $\tilde{i}_{\nabla f} \mathcal{W} = 0$, by Proposition 2.11,

$$\mathrm{div}^2 C(\nabla f) = -\frac{1}{n-2} \langle i_{\nabla f} C, z \rangle.$$

Also, by Lemma 4.2, we have

$$fC = -(n-1)T$$

and so

$$f|C|^2 = -(n-1)\langle T, C \rangle = -\frac{2(n-1)}{n-2} \langle i_{\nabla f} C, z \rangle.$$

Hence,

$$\mathrm{div}^2 C(\nabla f) = \frac{1}{2(n-1)} f|C|^2.$$

Finally, by Lemma 2.4,

$$\begin{aligned}\delta Q(\nabla f) &= -\langle i_{\nabla f} C, z\rangle + \frac{(n-2)^2}{2n(n-1)} z(\nabla s, \nabla f) \\ &= \frac{n-2}{2(n-1)} f|C|^2 + \frac{(n-2)^2}{2n(n-1)} z(\nabla s, \nabla f).\end{aligned}$$

□

Lemma 4.7. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. If* $\mathrm{div}^3 C \le 0$ *and* $\tilde{i}_{\nabla f}\mathcal{W} = 0$, *then Then*

$$\Lambda(t) := t \int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma$$

is monotone decreasing with respect to regular values t's of f.

Proof. Since $\tilde{i}_{\nabla f}\mathcal{W} = 0$, it follows from Lemma 4.6 together with divergence theorem that

$$\begin{aligned}\int_{t_1 \le f \le t_2} \mathrm{div}^3 C dv_g &= \int_{f=t_2} \mathrm{div}^2 C(N) d\sigma - \int_{f=t_1} \mathrm{div}^2 C(N) d\sigma \\ &= \frac{t_2}{2(n-1)} \int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma - \frac{t_1}{2(n-1)} \int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma.\end{aligned}$$

Since $\mathrm{div}^3 C \le 0$, this shows that

$$\frac{t_2}{2(n-1)} \int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma \le \frac{t_1}{2(n-1)} \int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma.$$

□

The following result is a generalization of a result in [6].

Theorem 4.8. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. If* $\mathrm{div}^3 C \le 0$ *and* $\tilde{i}_{\nabla f}\mathcal{W} = 0$, *then (M, g) has harmonic Weyl tensor, or equivalently $C = 0$.*

Proof. Since $\mathrm{div}^3 C \le 0$ and $\tilde{i}_{\nabla f}\mathcal{W} = 0$, it follows from Lemma 4.6 together with divergence theorem that

$$0 \ge \int_{f \le t} \mathrm{div}^3 C dv_g = \int_{f=t} \mathrm{div}^2 C(N) d\sigma = \frac{t}{2(n-1)} \int_{f=t} \frac{|C|^2}{|\nabla f|} d\sigma.$$

This implies that $C = 0$ on the set $\{f \ge 0\}$ and so $\Lambda(t) = 0$ for $t \ge 0$.

On the other hand, by Lemma 4.7, we can see that $C = 0$ on the set $f < 0$ by letting $t_2 = 0$. □

Corollary 4.9 ([6]). *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. If (M, g) has zero radial Weyl curvature with* $\operatorname{div}^4 \mathcal{W} = 0$, *then (M, g) has harmonic Weyl tensor.*

Proof. It is obvious since $\operatorname{div}^4 \mathcal{W} = 0$ implies $\operatorname{div}^3 C \leq 0$. □

Combining Theorem 4.5 to Theorem 4.8, we obtain the following result.

Corollary 4.10. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. Assume that* $\operatorname{div}^3 C \leq 0$ *and $\tilde{i}_{\nabla f} \mathcal{W} = 0$. Then Q is conservative if and only if the scalar curvature s is constant.*

Theorem 4.11. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M. Assume that* $\operatorname{div}^3 C \leq 0$ *and $\tilde{i}_{\nabla f} \mathcal{W} = 0$, then (M, g) is Bach-flat.*

Proof. By Lemma 4.2 and Theorem 4.8, we have $C = 0$ and $T = 0$. It follows from the definition of T that

$$z_g + \frac{\alpha}{n-1} g = 0$$

on each level hypersurface $f^{-1}(t)$ for a regular value. Furthermore, by Lemma 4.4, $z(\nabla f, X) = 0$ for any vector field X such that $X \perp \nabla f$. Consequently, by choosing a local frame $\{e_i\}$ with $e_1 = N = \frac{\nabla f}{|\nabla f|}$, we have

$$z(e_i, e_j) = -\frac{\alpha}{n-1} \delta_{ij} \quad (2 \leq i, j \leq n) \quad \text{and} \quad z(N, N) = \alpha.$$

So,

$$\begin{aligned} \mathring{\mathcal{W}}(z)(X, Y) &= \mathcal{W}(X, e_i, Y, e_j) z(e_i, e_j) \\ &= -\frac{\alpha}{n-1} \sum_{i=2}^{n} \mathcal{W}(X, e_i, Y, e_i) + \alpha \mathcal{W}(X, N, Y, N) \\ &= \frac{n\alpha}{n-1} \mathcal{W}(X, N, Y, N) = 0. \end{aligned}$$

Hence we have $(n-2)B = -\delta C + \mathring{\mathcal{W}}(z) = 0$. □

Now we consider Einstein-type manifolds with nonnegative complete divergence of stress-energy tensor of the traceless Ricci tensor. In this case, we can prove the same property as Einstein-type manifolds with zero radial Weyl curvature hold when $\delta(i_{\nabla s} z) \leq 0$. Note that this condition is trivial if the scalar curvature is constant.

Lemma 4.12. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. Assume that*

$\mathrm{div}^2 Q \geq 0$ *and* $\mathrm{div}(i_{\nabla s} z) \geq 0$. *Then*

$$\int_{f=t} \frac{1}{|\nabla f|} \langle i_{\nabla f} C, z \rangle$$

is nonnegative and monotone increasing with respect to regular values t's of f.

Proof. From Lemma 2.10 together with (2.15), we have

$$\begin{aligned} \delta\delta Q &= \frac{1}{2}|C|^2 - \langle \delta C, z \rangle + \frac{(n-2)^2}{2n(n-1)} \delta i_{\nabla s} z \\ &\leq \frac{1}{2}|C|^2 - \langle \delta C, z \rangle. \end{aligned}$$

So from assumption, we have

$$0 \leq \frac{1}{2}|C|^2 - \langle \delta C, z \rangle.$$

The rest of proof is the same as that of Lemma 3.6. □

Theorem 4.13. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. Assume that* $\mathrm{div}^2 Q \geq 0$ *and* $\mathrm{div}(i_{\nabla s} z) \geq 0$. *If* $\mathrm{div}^3 C \leq 0$, *then* (M, g) *has harmonic Weyl tensor.*

Proof. As in the proof of Theorem 3.7, by Lemma 4.3 and Lemma 4.12 together with the assumption $\mathrm{div}^3 C \leq 0$, we have

$$\frac{t_2}{2} \int_{f=t_2} \frac{|C|^2}{|\nabla f|} \leq \frac{t_1}{2} \int_{f=t_1} \frac{|C|^2}{|\nabla f|}$$

for any two regular values t_1, t_2 with $t_1 < t_2$.

By taking $t_1 = 0$ we may conclude that $C = 0$ on the set $\{x : f(x) \geq 0\}$. Similarly, by taking $t_2 = 0$ we may conclude that $C = 0$ on $f = t$ for all regular values t of f with $t < t_2 = 0$. Hence, we have $C = 0$ on the whole M by continuity. In other words, M has harmonic curvature. □

Combining Theorem 4.5 to Theorem 4.13, we obtain the following result.

Corollary 4.14. *Let (g, f, λ) be a solution of an Einstein-type equation* (4.1) *on an n-dimensional manifold M with compact level sets of f. Assume that* $\mathrm{div}^3 C \leq 0$ *and* $\mathrm{div}^2 Q \geq 0$ *with* $\mathrm{div}(i_{\nabla s} z) \geq 0$. *Then Q is conservative if and only if the scalar curvature s is constant.*

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