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REMARKS ON ULRICH BUNDLES OF SMALL RANKS OVER QUARTIC FOURFOLDS

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Abstract. In this paper, we investigate a few strategies to construct Ulrich bundles of small ranks over smooth fourfolds in $\mathbb{P}^5$, with a focus on the case of special quartic fourfolds. First, we give a necessary condition for Ulrich bundles over a very general quartic fourfold in terms of the rank and the Chern classes. Second, we give an equivalent condition for Pfaffian fourfolds in every degree in terms of arithmetically Gorenstein surfaces therein. Finally, we design a computer-based experiment to look for Ulrich bundles of small rank over special quartic fourfolds via deformation theory. This experiment gives a construction of numerically Ulrich sheaf of rank 4 over a random quartic fourfold containing a del Pezzo surface of degree 5.

1. Introduction

A vector bundle $\mathcal{E}$ on a projective variety $X \subseteq \mathbb{P}^N$ of dimension n is called *Ulrich* if it satisfies

$$H^*(X, \mathcal{E}(-i)) = 0$$

for every $1 \le i \le n$. It is well-known that the pushforward $\pi_*\mathcal{E}$ of an Ulrich bundle by a general linear projection $\pi : X \to \mathbb{P}^n$ is trivial, that is, $\pi_*\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^n}^t$ for some integer t. In the case, we also have $t = dr$, where $d = \deg(X) = [\mathcal{O}_X(1)]^n$ and $r = \operatorname{rank} \mathcal{E}$. Hence, the Hilbert polynomial $H_\mathcal{E}(t) = \chi(\mathcal{E}(t))$ of an Ulrich bundle $\mathcal{E}$ is

$$H_\mathcal{E}(t) = dr\binom{n+t}{n}.$$

Even in the 1980s and 1990s, Ulrich bundles received some highlights but had a different name, MGMCM (maximally generated maximal Cohen-Macaulay) modules. Indeed, many people asked about the existence of maximal Cohen-Macaulay modules over various rings. Among these maximal Cohen-Macaulay modules, the most interesting one is achieved when its number of generators

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is maximal. In this case, such an MCM module has a minimal free resolution which is linear. Ulrich bundles are geometric analogues of these MGMCM modules, by twisting them (so that they are generated in degree 0), and then sheafifying them. Hence, the existence of Ulrich bundles implies several questions arose from commutative algebra such as Lech's conjecture (cf. [19,20,26]), and also some geometric properties such as determinantal representations of Cayley-Chow forms, the cone of cohomology tables, etc. [9, 10].

In the case when X is a (smooth) hypersurface of degree d in $\mathbb{P}^{n+1}$, the existence of Ulrich bundles on X is well-known [1]. A standard way to attack these objects is using matrix factorizations [8]. When X is reduced and irreducible, matrix factorizations of X contain the same amount of information as set-theoretic determinantal representations of X. In particular, an Ulrich bundle on $X = V(F)$ is equivalent to its presentation matrix, which is a linear matrix M whose determinant is a nonzero constant multiple of some powers F^r. In other words, we ask whether X (or some powers of it) is linearly determinantal or not. In terms of the generalized Clifford algebra, any hypersurface X carries an Ulrich sheaf of rank d^τ for some τ [1, 3]. On the other hand, the *Ulrich complexity* $uc(X)$, the smallest possible rank of an Ulrich bundle on X, is very mysterious and it is widely open for most cases. For instance, when $d = 1$ (in other words, $X = \mathbb{P}^n$ is the projective n-space), it is clear that $uc(X) = 1$ since the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ is an Ulrich line bundle. Even the case $d = 2$ looks not very trivial: it is known that the Ulrich complexity of a smooth hyperquadric of dimension n is $2^{\lfloor (n-1)/2 \rfloor}$ which comes from the spinor bundles [5, 17]. There are only a few more results on Ulrich complexity of hypersurfaces of degree $d = 3$: see [11] and references therein.

Let us move our focus into quartic hypersurfaces. When $n = \dim X = 1$, a classical result of Dixon tells us that X is linearly determinantal [7]. When $n = \dim X = 2$, a quartic surface in $\mathbb{P}^3$, there is no Ulrich line bundle in most cases. In fact, if it has an Ulrich line bundle $\mathcal{L}$, then $\mathcal{L}^2 = 4$ and $\mathcal{L} \cdot \mathcal{O}_X(1) = 6$. The existence of such a line bundle is characterized by the Noether-Lefschetz divisor, in particular, a general quartic surface has no Ulrich line bundle. It is known that the Ulrich complexity of a general quartic surface and of a general quartic threefold are 2, in other words, they are linearly Pfaffian (but not linearly determinantal). When X is a sufficiently general quartic hypersurface of dimension $n \geq 4$, a study on Horrocks-type splitting theorem implies that there is no Ulrich bundle of rank ≤ 3 (cf. [22]).

The main objective of this paper is to describe conditions for Ulrich bundles on quartic fourfolds and to suggest some ideas on explicit constructions of them together with Ulrich bundles of small ranks. We characterize Pfaffian fourfolds, which is a generalization of Beauville's description of Pfaffian cubic fourfolds. A big difference between the cubic case is that they do not form a divisor in the moduli of hypersurfaces of a given degree. And then, we exhibit a computer-aided construction of numerically Ulrich sheaves of rank 4 over a certain class of special quartic fourfolds. These numerically Ulrich sheaves are not locally free, but they have the same Chern classes as Ulrich bundles of rank 4 (if exist). Hence, numerically Ulrich sheaves have the potential that can be deformed into stable Ulrich bundles.

The structure of the paper is as follows. In Section 2, we recall definitions for ACM and Ulrich bundles and describe a necessary condition for Ulrich bundles on a very general quartic fourfold. In Section 3, we analyze Pfaffian fourfolds in $\mathbb{P}^5$ together with surfaces contained in them. In particular, a smooth Pfaffian fourfold in $\mathbb{P}^5$ must contain an arithmetically Gorenstein (AG for short) surface of a certain degree, and vice versa. We give the description of the family of such AG surfaces, and thus it provides a generalization of Beauville's description of Pfaffian cubic fourfolds [2]. In Section 4, we address a computer-based experiment to look for an Ulrich bundle on a specific class of quartic fourfolds, namely, quartic fourfolds containing a del Pezzo surface of degree 5. Unfortunately, the experiment is not very successful to construct an Ulrich bundle explicitly, however, we find that there are plenty of numerically Ulrich sheaves of rank 4. As we can find in the case of cubic threefolds and fourfolds, a nontrivial extension of two numerically Ulrich sheaves has a high chance to deform into an Ulrich bundle.

2. Preliminaries

Throughout this paper, we work over an algebraically closed field of characteristic 0 for simplicity, even though most of the notions can be generalized over non-algebraically closed fields or fields of positive characteristics. Recall first the definition of ACM and Ulrich bundles on a (smooth) projective variety $X \subset \mathbb{P}^N$. Note that it depends on an embedding, or the choice of a polarization $(X, \mathcal{O}_X(1))$, where $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$ is a very ample line bundle on X.

Definition. Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension n, and let $\mathcal{E}$ be a coherent sheaf on X.

(i) $\mathcal{E}$ is called *arithmetically Cohen-Macaulay* (ACM for short) if $\mathcal{E}$ is locally Cohen-Macaulay, and $\mathcal{E}$ has no intermediate cohomology, that is,
$$H^i(X, \mathcal{E}(j)) = 0$$
for every $0 < i < n$ and $j \in \mathbb{Z}$.

(ii) $\mathcal{E}$ is called *Ulrich* if $\mathcal{E}$ has the same cohomological behavior as the structure sheaf of $\mathbb{P}^n$, that is,
$$H^i(X, \mathcal{E}(-j)) = 0$$
for every $i \in \mathbb{Z}$ and $1 \leq j \leq n$.

Remark 2.1. It is easy to check that an Ulrich sheaf is ACM. It is also easy to check that an ACM sheaf over a smooth projective variety is locally free, which is the reason why we often call these objects ACM or Ulrich bundles. We refer to [6, 10] for more basic properties of Ulrich bundles.

Now let us turn our focus to the main object of this paper. Let X be a very general quartic fourfold in $\mathbb{P}^5$. We first describe a necessary condition for Ulrich bundles of rank r on X.

Proposition 2.2. *Let $\mathcal{E}$ be an Ulrich bundle of rank r on a very general quartic fourfold $X \subset \mathbb{P}^5$. Then r is divisible by 4.*

Proof. Let $\mathcal{E}$ be an Ulrich bundle of rank r on X, and let $\mathcal{T}_X$ be the tangent bundle of X. Let $c_i = c_i(\mathcal{E})$ be the i-th Chern class in the Chow ring $A(X)$ of X. Note that the intersection theory on X is determined by multiples of codimension i cycles H^i, where $H \subset X$ denotes the general hyperplane section of X so that $H^4 = 4$. Together with the Poincaré duality on the cohomology ring $H^\bullet(X, \mathbb{Z})$, we may write c_i as a multiple of the class H^i by a rational number q_i so that $4q_i \in \mathbb{Z}$.

To find a necessary condition that r should satisfy, we use the Hirzebruch-Riemann-Roch formula

$$\chi(\mathcal{E}) = \deg\left(ch(\mathcal{E}).td(\mathcal{T}_X)\right)_{\dim X},$$

where $ch(\mathcal{E})$ denotes the Chern character of $\mathcal{E}$ and $td(\mathcal{T}_X)$ denotes the Todd class of $\mathcal{T}_X$. Note that the tangent bundle $\mathcal{T}_X$ fits into the short exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^5}|_X \to \mathcal{N}_{X/\mathbb{P}^5} = \mathcal{O}_X(4) \to 0$$

which helps to compute the Todd class $td(\mathcal{T}_X)$. Indeed, we have identities

$$\begin{aligned} ch(\mathcal{E}) &= r + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\ &\quad + \tfrac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4), \\ td(\mathcal{T}_X) &= 1 + H + \tfrac{11}{12}H^2 + \tfrac{7}{12}H^3 + \tfrac{1}{4}H^4, \end{aligned}$$

see for instance [13, Appendix A].

Since $\mathcal{E}$ is an Ulrich bundle of rank r on a fourfold X, it satisfies $\chi(\mathcal{E}(-1)) = \chi(\mathcal{E}(-2)) = \chi(\mathcal{E}(-3)) = \chi(\mathcal{E}(-4)) = 0$ which induces 4 equations with indeterminates $r, c_1, \ldots, c_4$. To solve this system of equations, the Chern classes $c_1, \ldots, c_4$ can be written in terms of r, namely,

$$\begin{aligned} c_1 &= \tfrac{3r}{2}H, \\ c_2 &= \left(\tfrac{9}{8}r^2 - \tfrac{1}{2}r\right)H^2, \\ c_3 &= \left(\tfrac{9}{16}r^3 - \tfrac{3}{4}r^2 - \tfrac{3}{4}r\right)H^3 = \left(\tfrac{9}{4}r^3 - 3r^2 - 3r\right)\cdot[\text{line}], \\ c_4 &= \left(\tfrac{27}{128}r^4 - \tfrac{9}{16}r^3 - r^2 + \tfrac{5}{4}r\right)H^4 = \left(\tfrac{27}{32}r^4 - \tfrac{9}{4}r^3 - 4r^2 + 5r\right)\cdot[\text{point}]. \end{aligned}$$

The statement follows from the fact that $c_4(\mathcal{E})$ is an integer multiple of the class of the point. $\square$

3. Pfaffian fourfolds

Let us consider a problem to look for smooth quartic fourfolds X endowed with an Ulrich bundle of small ranks. Notice that X has an Ulrich line bundle if and only if it is linearly determinantal, which is impossible since a determinantal hypersurface is singular along a codimension 3 locus. Hence, the smallest possible Ulrich bundle that X can have is of rank 2. Such an Ulrich bundle of rank 2 admits a linearly Pfaffian representation of the given hypersurface. Note also that a Pfaffian hypersurface is singular along a codimension 5 locus, so the question makes sense for hypersurfaces of dimension ≤ 4. We made several computations with a help of the computer algebra system Macaulay2 [12]: see also [15] for the implemented code for these computations.

Proposition 3.1. *A smooth quartic fourfold $X \subset \mathbb{P}^5$ has an Ulrich bundle of rank 2 if and only if it contains an AG surface $S \subset \mathbb{P}^5$ of degree 14 which is defined as 6-Pfaffians of a skew-symmetric 7×7 linear matrix M. In particular, S has a minimal free resolution of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \mathcal{O}_{\mathbb{P}^5}(-4)^7 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^5}(-3)^7 \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_S \to 0.$$

Proof. First, assume that X has an Ulrich bundle $\mathcal{E}$ of rank 2. Since $\mathcal{E}$ is globally generated, Serre's correspondence implies that a general global section $s \in H^0(\mathcal{E})$ degenerates along a surface S which fits into the following short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{E} \to \mathcal{I}_{S/X}(3) \to 0.$$

By considering their minimal free resolutions over $\mathbb{P}^5$ and the mapping cone, we see that $\mathcal{I}_{S/X}(3)$ has a minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{O}_{\mathbb{P}^5}(-1)^8 \to \mathcal{O}_{\mathbb{P}^5}^7 \to \mathcal{I}_{S/X}(3) \to 0.$$

We also have the short exact sequence of ideal sheaves

$$0 \to \mathcal{I}_{X/\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^5}(-4) \to \mathcal{I}_{S/\mathbb{P}^5} \to \mathcal{I}_{S/X} \to 0,$$

and a similar argument yields a minimal free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \mathcal{O}_{\mathbb{P}^5}(-4)^7 \overset{M}{\to} \mathcal{O}_{\mathbb{P}^5}(-3)^7 \to \mathcal{I}_{S/\mathbb{P}^5} \to 0.$$

In particular, S is an AG surface in $\mathbb{P}^5$. The Buchsbaum-Eisenbud structure theorem [4] implies that S is indeed defined as Pfaffians of a skew-symmetric matrix M. In this case, such a matrix M must be linear, and we also have $\deg S = 14$ by reading off its Hilbert function.

Conversely, suppose that X contains such a surface S. Applying Shamash's construction [23] for $S \subset X$, we have the following 2-periodic free resolution over X

$$\cdots \to \mathcal{O}_X(-8)^8 \xrightarrow{A} \mathcal{O}_X(-7)^8 \to \mathcal{O}_X(-4)^8 \to \mathcal{O}_X(-3)^7 \to \mathcal{O}_X \to \mathcal{O}_S \to 0.$$

Since X is an arithmetically Cohen-Macaulay subvariety in X of codimension 3, hence, the 3rd syzygy sheaf $\operatorname{coker} A$ is locally free on X. Thanks to Eisenbud [8], A induces a linear matrix factorization of X, and thus the sheaf $\mathcal{E} := (\operatorname{coker} A) \otimes \mathcal{O}_X(7)$ is indeed an Ulrich bundle on X, whose rank is 2 since $\det A$ vanishes along X of multiplicity 2. $\square$

Corollary 3.2. *A smooth fourfold $X_d \subset \mathbb{P}^5$ of degree $d \geq 2$ has an Ulrich bundle of rank 2 if and only if it contains an AG surface S_d of degree $\frac{d(d-1)(2d-1)}{6}$ which is defined as $(2d-2)$-Pfaffians of a skew-symmetric $(2d-1) \times (2d-1)$ linear matrix M_d. In particular, S_d has a minimal free resolution of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-2d+1) \to \mathcal{O}_{\mathbb{P}^5}(-d)^{2d-1} \xrightarrow{M_d} \mathcal{O}_{\mathbb{P}^5}(-d+1)^{2d-1} \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{S_d} \to 0.$$

Proof. The construction is exactly the same, and the degree of S_d can be read off from the Hilbert function of $\mathcal{O}_{S_d}$. □

Note that $\deg S_2 = 1$, $\deg S_3 = 5 = 1^2 + 2^2$, $\deg S_4 = 14 = 1^2 + 2^2 + 3^2$ and so on. Thanks to Beauville, such a hypersurface equipped with Ulrich bundles of rank 2 (and no Ulrich line bundle) is linearly Pfaffian [2]. He also showed that a cubic fourfold containing a del Pezzo surface S of degree 5 is linearly Pfaffian, and such Pfaffian cubic fourfolds form a hypersurface in the space of all smooth cubic fourfolds. We compare both ideas and give a slightly further analysis on the families of Pfaffian fourfolds as follows.

Theorem 3.3. *Let $d \geq 2$. There is a $(8d^2 - 6d)$-dimensional family of smooth Pfaffian fourfolds $X_d \in |\mathcal{O}_{\mathbb{P}^5}(d)|$ of degree d. A very general Pfaffian fourfold X contains a $(2d-1)$-dimensional family of AG surfaces of degree $\frac{d(d-1)(2d-1)}{6}$ and carries a discrete family of stable Ulrich bundle of rank 2.*

Proof. As in [2, Theorem B, Corollary 2.4], the dimension of the family of Pfaffian hypersurfaces can be computed as $\dim \mathcal{M}_d/GL(2d)$, where $\mathcal{M}_d$ is the space of $2d \times 2d$ skew-symmetric matrices whose entries are linear forms over $\mathbb{P}^5$. Hence, its dimension is nothing but

$$\frac{1}{2}(2d)(2d-1) \cdot h^0(\mathcal{O}_{\mathbb{P}^5}(1)) - (2d)^2 = 8d^2 - 6d.$$

Suppose that X_d is a general Pfaffian fourfold of degree d so that it has an Ulrich bundle of rank 2. Thanks to Corollary 3.2, it is equivalent to say that X_d contains an AG surface S_d. Let us consider the incidence families of S_d's in $\mathbb{P}^5$ and also in X_d. Since S_d is defined as $(2d-2)$-Pfaffians of a $(2d-1) \times (2d-1)$ skew-symmetric linear matrix, hence, the same dimension count works: the family of such surfaces

$$\mathcal{S}_d := \left\{ S_d \subset \mathbb{P}^5 \;\middle|\; S_d \text{ is AG surface of degree } \frac{d(d-1)(2d-1)}{6} \right\}$$

has dimension $8d^2 - 14d + 5$. Note that the number is also the same as the local dimension of the 1st order deformations, that is, $h^0(\mathcal{N}_{S_d/\mathbb{P}^5})$.

Each S_d is defined by $(2d-1)$ equations of degree $(d-1)$ having $(2d-1)$ linear syzygies, hence, the space of degree d fourfolds containing a single S_d is characterized as $\mathbb{P}H^0(\mathcal{I}_{S_d/\mathbb{P}^5}(d)) = 6(2d-1) - (2d-1) - 1 = 10d - 6$. Hence, the incidence locus

$$\mathfrak{I}_d := \{(S_d, X_d) \mid S_d \in \mathcal{S}_d,\ S_d \subset X_d \in |\mathcal{O}_{\mathbb{P}^5}(d)|\}$$

has dimension $8d^2 - 4d - 1$.

The dimension of Pfaffian fourfolds of degree d is nothing but the dimension of the image under the natural projection $p : \mathfrak{I}_d \to |\mathcal{O}_{\mathbb{P}^5}(d)|$, which must be $8d^2 - 6d$ as computed above. Hence, by shrinking the codomain into $p(\mathfrak{I}_d)$, a general Pfaffian fourfold X_d of degree d contains $(8d^2 - 4d - 1) - (8d^2 - 6d) = 2d - 1$ dimensional family of AG surfaces S_d, which is also the same as $h^0(\mathcal{N}_{S_d/X_d})$ as well.

Now consider a very general Pfaffian fourfold X_d of degree d, equipped with an Ulrich bundle $\mathcal{E}$ of rank 2. As discussed above, we have a $(2d-1)$-dimensional family of AG surfaces $S_d \subset X_d \subset \mathbb{P}^5$ of degree $\frac{d(d-1)(2d-1)}{6}$ parametrized by $\mathbb{P}H^0(\mathcal{E}) \simeq \mathbb{P}^{2d-1}$. We claim that such an Ulrich bundle is uniquely determined by the choice of S_d. In what happens (we may use either Serre's correspondence or Shamash's construction) there is an Ulrich bundle $\mathcal{E}$ of rank 2 which fits into the short exact sequence

$$0 \to \mathcal{O}_{X_d} \to \mathcal{E} \to \mathcal{I}_{S_d/X_d}(d-1) \to 0,$$

in other words, $\mathcal{E}$ is a locally free extension of $\mathcal{I}_{S_d/X_d}(d-1)$ by $\mathcal{O}_{X_d}$. Such extensions are parametrized by $\mathbb{P}\operatorname{Ext}^1_{X_d}(\mathcal{I}_{S_d/X_d}(d-1), \mathcal{O}_{X_d})$.

Thanks to Grothendieck-Serre duality, we have

$$\begin{aligned}\operatorname{Ext}^1_{X_d}(\mathcal{I}_{S_d/X_d}(d-1), \mathcal{O}_{X_d})^\vee &\simeq \operatorname{Ext}^3_{X_d}(\mathcal{O}_{X_d}, \mathcal{I}_{S_d/X_d}(d-1) \otimes \omega_{X_d}) \\ &= H^3(\mathcal{I}_{S_d/X_d}(2d-7)).\end{aligned}$$

On the other hand, from the short exact sequence of ideal sheaves, we have

$$0 \to \mathcal{I}_{X_d/\mathbb{P}^5}(2d-7) = \mathcal{O}_{\mathbb{P}^5}(d-7) \to \mathcal{I}_{S_d/\mathbb{P}^5}(2d-7) \to \mathcal{I}_{S_d/X_d}(2d-7) \to 0,$$

in particular, $H^3(\mathcal{I}_{S_d/X_d}(2d-7)) \simeq H^3(\mathcal{I}_{S_d/\mathbb{P}^5}(2d-7))$. Since S_d is AG, we have $\omega_{S_d} \simeq \omega_{\mathbb{P}^5} \otimes \mathcal{O}_{S_d}(2d-1) = \mathcal{O}_{S_d}(2d-7)$ which yields

$$H^3(\mathcal{I}_{S_d/\mathbb{P}^5}(2d-7)) \simeq H^2(\omega_{S_d})$$

which is 1-dimensional. Indeed, $\mathbb{P}\operatorname{Ext}^1_{X_d}(\mathcal{I}_{S_d/X_d}(d-1), \mathcal{O}_{X_d})$ is a single point, that is, such an Ulrich bundle $\mathcal{E}$ of rank 2 is locally rigid. The stability of Ulrich bundles of rank 2 (either in the sense of Gieseker or in the sense of Mumford) is provided by the fact that there is no Ulrich line bundle, see [6]. $\square$

Remark 3.4. When $d = 2$, the above theorem fits with the well-known fact that a general hyperquadric $Q \subset \mathbb{P}^5$ is Pfaffian, and contains a 3-dimensional family of planes. In this case, there are exactly two stable Ulrich bundles of rank 2 over Q, namely, spinor bundles parametrizing these planes (there are two families of planes contained in Q). We may ask then what will be the number of stable Ulrich bundles of rank 2 over a general Pfaffian fourfold X_d of degree d, which is the same question as the number of Pfaffian representations. A similar question can be found in several works, for instance, counting the number of Cartan representations considered by Iliev and Manivel [14], or the generalized Casson invariant considered by Thomas [24].

4. Experiments: quartic fourfolds having small rank Ulrich bundles

In this section, we address experimental observations and ideas on how to look for quartic fourfolds with Ulrich bundles of small rank. For cubic fourfolds, there are a few attempts to characterize, or to find examples of cubic fourfolds having Ulrich bundles of small rank: see [2, 21, 25] for instance. However, there is almost no attempt to find a single example of smooth quartic fourfolds which has an Ulrich bundle of rather smaller rank.

A folklore expectation says that the projected Segre variety $Sym_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ and its (higher) secant varieties seem to be non-defective except only for a few reported cases. If we trust this expectation, we guess that a general quartic form F in 6 variables can be written as a sum of 6 products of linear forms, namely,

$$F = \sum_{i=1}^{6} \left(\prod_{j=1}^{4} L_{i,j} \right)$$

for some linear forms $L_{i,j}$ in 6 variables. In other words, a general quartic form has Chow rank 6. Such a decomposition of F yields a linear matrix factorization of F, in particular, one can construct an Ulrich sheaf supported on $V(F)$ of rank $\leq 4^{6-2} = 256$, cf. [3, Section 2]. The upper bound obtained from these kinds of decompositions is sharp in the case of hyperquadrics, however, we strongly expect that this upper bound is pretty far away from the exact answer when the degree is > 2. In most cases, it is extremely hard to determine the Ulrich complexity of a given hypersurface explicitly.

In most cases, it is very hard to construct an Ulrich bundle of small rank on a given variety X. It is still meaningful to imitate a construction of [11] using deformation theory. We summarize their main strategy.

(i) Instead of finding an Ulrich bundle $\mathcal{F}$ of given rank r on X, we first find a numerically Ulrich sheaf (= coherent sheaf whose Hilbert polynomial is the same as the one of an Ulrich sheaf of the same rank r). We hopefully expect that $\mathcal{F}$ is stable and $\mathcal{F}$ satisfies a lot of cohomology vanishing conditions to be Ulrich.

(ii) If $\mathcal{F}$ is stable (at least we need its simpleness) but not an Ulrich bundle, then we study its deformations. If $\mathcal{F}$ belongs to a "good" moduli space of stable sheaves (or of simple sheaves) which contains enough vector bundles, and if $\mathcal{F}$ is unobstructed, then we have a chance that $\mathcal{F}$ deforms to an Ulrich bundle.

There are a few difficult problems to apply these arguments.

(1) Finding a numerically Ulrich sheaf itself is heuristic and difficult. Since X is a (quartic) hypersurface, Shamash's construction is a nice tool to pick up an ACM bundle, and certain elementary modifications (supported on a codimension 2 subscheme in practice) could produce a candidate sheaf. We need to carefully choose the sheaves for each step. Even though we found

a numerically Ulrich sheaf, it is even harder to check whether a candidate sheaf is stable/simple or not.

(2) Smoothing such a numerically Ulrich sheaf into a locally free sheaf is also very difficult. In practice, a numerically Ulrich sheaf $\mathcal{G}$ we found as in the first step, it is almost never locally free since the cohomology groups $H^3(X, \mathcal{G}(-j))$ will not vanish for $j \gg 0$. We need to check that $\mathcal{G}$ "deforms" to a locally free sheaf, which is almost the same as saying that these cohomology groups H^3 cancel out with H^4. Unfortunately, we do not understand enough about infinitesimal deformations of sheaves on quartic fourfolds.

We report some observations with special quartic fourfolds X which seem to have Ulrich bundles of rather smaller ranks, in particular, $2 < uc(X) \ll 256$. Motivated by Beauville's description of cubic fourfolds containing a del Pezzo surface of degree 5, we try with quartic fourfolds containing a del Pezzo surface of degree 5.

Definition. Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension n and degree d, and let $\mathcal{F}$ be a coherent sheaf of rank r. We say $\mathcal{F}$ is *numerically Ulrich* if it has the same Hilbert polynomial as an Ulrich sheaf of rank r on X, that is,

$$p_t(\mathcal{F}) = dr\binom{t+n}{n}.$$

Notice that the existence of numerically Ulrich sheaf does not imply the existence of an Ulrich sheaf of the same rank: for instance, if we take a line $\ell \subset X \subset \mathbb{P}^4$ inside a smooth cubic threefold X, then the twisted ideal sheaf $\mathcal{I}_{\ell/X}(1)$ is numerically Ulrich. However, there is no Ulrich line bundle on X. It is also mysterious whether a given projective variety $X \subset \mathbb{P}^N$ has a numerically Ulrich sheaf or not, and if yes, what should be its minimal possible rank.

Lemma 4.1. *A smooth quartic fourfold X containing a del Pezzo surface $Y \subset \mathbb{P}^5$ of degree 5 carries an ACM bundle of rank 4 with $c_1 = 2H$.*

Proof. Applying Shamash's construction, we have a free resolution of an $\mathcal{O}_X$-module $\mathcal{O}_Y$

$$\begin{aligned} \cdots \to \mathcal{O}_X(-7)^5 \oplus \mathcal{O}_X(-8) &\xrightarrow{d_4} \mathcal{O}_X(-5) \oplus \mathcal{O}_X(-6)^5 \\ &\xrightarrow{d_3} \mathcal{O}_X(-3)^5 \oplus \mathcal{O}_X(-4) \xrightarrow{d_2} \mathcal{O}_X(-2)^5 \xrightarrow{d_1} \mathcal{O}_X \to \mathcal{O}_Y \to 0. \end{aligned}$$

Since Y is an ACM subscheme in X of codimension 2, the above free resolution of $\mathcal{O}_Y$ becomes 2-periodic after the 2nd term. In particular, each of the differentials d_i ($i \geq 3$) induces a matrix factorization of the hypersurface X, and hence $\operatorname{coker} d_i$ is an ACM sheaf supported on X [8, Corollary 6.3]. In particular, we have an ACM bundle $\mathcal{E} = im(d_4) \otimes \mathcal{O}_X(7)$ which fits into an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X(2) \oplus \mathcal{O}_X(1)^{\oplus 5} \to \mathcal{O}_X(4)^{\oplus 5} \oplus \mathcal{O}_X(3) \to \mathcal{O}_X(5)^{\oplus 5} \to \mathcal{I}_{Y/X}(7) \to 0,$$

and hence $c_1(\mathcal{E}) = 2H$. $\square$

Notice that the Hilbert polynomial of $\mathcal{E}$ is

$$p_t(\mathcal{E}) = \frac{1}{6}(t+1)(t+2)(4t^2+12t+15),$$

and $h^0(\mathcal{E}) = h^4(\mathcal{E}(-3)) = 5$.

Observation. There is an elementary modification $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{L} \to 0$ so that the induced map $H^0(\mathcal{E}) \to H^0(\mathcal{L})$ is an isomorphism, and $\mathcal{G}$ is a numerically Ulrich sheaf on X.

If there is such an elementary modification, one can check that $\mathcal{L}$ is a numerically Ulrich sheaf on Y of rank 1 by comparing their Hilbert polynomials. Since Y is a del Pezzo surface of degree 5, a good candidate will be an Ulrich line bundle on Y, which is of the form $\mathcal{O}_Y(C)$, where C is a rational normal quintic curve contained in Y. We made a Macaulay2 experiment:

- Y: a del Pezzo surface of degree 5 obtained by blowing up 4 generic points (given by the projective frame $\{[1:0:0], [0:1:0], [0:0:1], [1:1:1]\}$),
- X: a randomly chosen quartic fourfold containing Y,
- $\mathcal{E}$: an ACM bundle obtained as above,
- $\mathcal{L}$: an Ulrich line bundle on Y (there are exactly 20 such line bundles),

and check whether there is a surjection $\mathcal{E} \to \mathcal{L}$ which induces an isomorphism on their global sections, see [15]. We observed that $\mathcal{E}$ admits an elementary modification by each of these 20 Ulrich line bundles on Y.

Note that the existence of an elementary modification is not quite obvious. It is clear that an Ulrich line bundle $\mathcal{L}$ on Y is globally generated by its 5 sections, and thus we have surjections $\mathcal{O}_X^5 \twoheadrightarrow \mathcal{O}_Y^5 \twoheadrightarrow \mathcal{L}$. However, since $\operatorname{rank} \mathcal{E} = 4$ is not big enough, those 5 global sections of $\mathcal{E}$ are not general enough. In particular, we cannot say that either $\mathcal{E}$ contains $\mathcal{O}_X^5$ as subsheaves or $\mathcal{E}$ surjects to $\mathcal{O}_X^5$. In any case, we have a surjection $\mathcal{O}_X^5 \oplus \mathcal{O}_X(-1) \twoheadrightarrow \mathcal{E}$, and thus a surjection $\xi : \mathcal{O}_Y^5 \oplus \mathcal{O}_Y(-1) \twoheadrightarrow \mathcal{E}|_Y$. Hence, the lifting of $\mathcal{O}_Y^5 \twoheadrightarrow \mathcal{L}$ is provided under a cohomology vanishing condition, e.g., $\hom_Y(\ker \xi, \mathcal{L}) = 0$, which seems to make sense thanks to Riemann-Roch computations. However, we do not know an explicit description of $\ker \xi$ as rank 2 vector bundles on Y.

As a result, we obtain a numerically Ulrich sheaf $\mathcal{G}$ on X. We compute its cohomology groups:

$$H^*(\mathcal{G}) = H^*(\mathcal{G}(-1)) = H^*(\mathcal{G}(-2)) = 0, \quad H^i(\mathcal{G}(-3)) = 0 \text{ for } i \leq 2.$$

From the cohomology groups of $\mathcal{F}$ and $\mathcal{L}$, we conclude that $h^3(\mathcal{G}(-3)) = h^4(\mathcal{G}(-4)) = 5$. One can also check by Macaulay2 computations that $\dim \operatorname{Ext}^1(\mathcal{G}, \mathcal{G}) = 17$. Hence, if we are able to deform $\mathcal{G}$ into $\widetilde{\mathcal{G}}$ so that $h^3(\widetilde{\mathcal{G}}(-3)) = h^4(\widetilde{\mathcal{G}}(-3)) = 0$, then $\widetilde{\mathcal{G}}$ enjoys the Ulrich condition, and thus $\widetilde{\mathcal{G}}(1)$ will be an Ulrich bundle we want to find.

Notice that $\mathcal{G}$ fits into a free resolution over X of the form

$$\cdots \to \mathcal{O}_X(-3)^5 \oplus \mathcal{O}_X(-5)^{16} \xrightarrow{\psi} \mathcal{O}_X(-2)^{16} \oplus \mathcal{O}_X(-3)^5 \xrightarrow{\varphi} \mathcal{O}_X(-1)^{16} \to \mathcal{G} \to 0.$$

Hence, a 1st order infinitesimal deformation of $\mathcal{G}$ can be described as a pair of homomorphisms (U, V) so that $(\varphi + \epsilon U)(\psi + \epsilon V) = 0$, $\epsilon^2 = 0$. Equivalently, we need to find (U, V) such that $U\psi + \varphi V = 0$. If there is a solution (U, V) so that the certain restriction V onto $\mathcal{O}_X(-3)^5 \to \mathcal{O}_X(-3)^5$ is of full rank, then these terms of the same degree cancel out, and hence the free resolution can be reduced as

$$\cdots \to \mathcal{O}_X(-5)^{16} \to \mathcal{O}_X(-2)^{16} \to \mathcal{O}_X(-1)^{16} \to \widetilde{\mathcal{G}} \to 0,$$

so that $\widetilde{\mathcal{G}}$ will be the one we want to find. Unfortunately, we did not yet find such a 1st order infinitesimal deformation (U, V) for $\mathcal{G}$.

Remark 4.2. A numerically Ulrich sheaf needs not to have such a smoothing. For instance, consider a smooth twisted cubic curve C lying on a (very general) cubic fourfold $X \subset \mathbb{P}^5$. The Lehn-Lehn-Sorger-van Straten sheaf $\mathcal{G}_C$ is defined as the kernel of the evaluation map

$$0 \to \mathcal{G}_C \to \mathcal{O}_X^3 \to \mathcal{O}_Y(C^t) \to 0,$$

where $Y = \langle C \rangle \cap X$ is the linear section of X determined by the linear span of C [11,18]. It is easy to check that $\mathcal{G}_C$ is a stable sheaf of rank 3 and numerically Ulrich, however, it is not smoothable. If it is, then it implies the existence of Ulrich bundles of rank 3 on a very general cubic fourfold whose non-existence is reported in [16]. See also [11] for more detailed explanations.

Unfortunately, we found no reason that either $\mathcal{G}$ is smoothable, or not. We expect that $\mathcal{G}$ does not allow such a deformation, but some extension of $\mathcal{G}$ by $\mathcal{G}'$ (here, $\mathcal{G}, \mathcal{G}'$ are numerically Ulrich sheaves we obtained) admits such a deformation. For instance, a smooth cubic threefold has plenty of numerically Ulrich sheaves of the form $\mathcal{I}_{\ell/X}(1)$. Under a certain condition, there is a nontrivial extension of them – as a result, one has a simple, strictly semistable numerically Ulrich sheaf $\mathcal{F}$ of rank 2. Unlikely as "minimal" numerically Ulrich sheaves $\mathcal{I}_{\ell/X}(1)$, this $\mathcal{F}$ allows an infinitesimal deformation to a stable vector bundle $\widetilde{F}$. One can easily check that $\widetilde{F}$ is indeed Ulrich. The same story also holds for cubic fourfolds: see [11].

If the same story also makes sense for these special quartic fourfolds, then we expect that X will have an Ulrich sheaf of rank 8, however, we do not know much about deformation theory of simple sheaves on quartic fourfolds. Furthermore, the computational cost for calculating the space of 1st order deformations is too high, so it seems to be very difficult to make a computer-based observation.

We finish this paper by addressing two questions about Ulrich bundles on quartic fourfolds.

Question 1. Describe more classes of special quartic fourfolds which may carry (numerically) Ulrich sheaves of small ranks.

Question 2. What is the Ulrich complexity of a very general quartic fourfold?

References

[1] J. Backelin and J. Herzog, *On Ulrich-modules over hypersurface rings*, in Commutative algebra (Berkeley, CA, 1987), 63–68, Math. Sci. Res. Inst. Publ., 15, Springer, New York, 1989. https://doi.org/10.1007/978-1-4612-3660-3_4

[2] A. Beauville, *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64. https://doi.org/10.1307/mmj/1030132707

[3] M. Bläser, D. Eisenbud, and F.-O. Schreyer, *Ulrich complexity*, Differential Geom. Appl. **55** (2017), 128–145. https://doi.org/10.1016/j.difgeo.2017.06.001

[4] D. A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485. https://doi.org/10.2307/2373926

[5] R.-O. Buchweitz, D. Eisenbud, and J. Herzog, *Cohen-Macaulay modules on quadrics*, in Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 58–116, Lecture Notes in Math., 1273, Springer, Berlin, 1987. https://doi.org/10.1007/BFb0078838

[6] M. Casanellas, R. Hartshorne, F. Geiss, and F.-O. Schreyer, *Stable Ulrich bundles*, Internat. J. Math. **23** (2012), no. 8, 1250083, 50 pp. https://doi.org/10.1142/S0129167X12500838

[7] A. C. Dixon, *Note on the reduction of a ternary quantic to a symmetrical determinant*, Proc. Cambridge Philos. Soc. **11** (1902), 350–351.

[8] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64. https://doi.org/10.2307/1999875

[9] D. Eisenbud and F.-O. Schreyer, *Boij-Söderberg theory*, in Combinatorial aspects of commutative algebra and algebraic geometry, 35–48, Abel Symp., 6, Springer, Berlin, 2011. https://doi.org/10.1007/978-3-642-19492-4_3

[10] D. Eisenbud, F.-O. Schreyer, and J. Weyman, *Resultants and Chow forms via exterior syzygies*, J. Amer. Math. Soc. **16** (2003), no. 3, 537–579. https://doi.org/10.1090/S0894-0347-03-00423-5

[11] D. Faenzi and Y. Kim, *Ulrich bundles on cubic fourfolds*, Comment. Math. Helv. **97** (2022), no. 4, 691–728. https://doi.org/10.4171/cmh/546

[12] D. Grayson and M. Stillman, *Macaulay2 – a software system for algebraic geometry and commutative algebra*, available at: https://macaulay2.com

[13] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer, New York, 1977. https://doi.org/10.1007/978-1-4757-3849-0

[14] A. Iliev and L. Manivel, *On cubic hypersurfaces of dimensions* 7 *and* 8, Proc. Lond. Math. Soc. (3) **108** (2014), no. 2, 517–540. https://doi.org/10.1112/plms/pdt042

[15] Y. Kim, *Macaulay2 scripts*: *"Remarks on Ulrich bundles on Pfaffian fourfolds*, available at: https://sites.google.com/view/yeongrak/publications

[16] Y. Kim and F.-O. Schreyer, *An explicit matrix factorization of cubic hypersurfaces of small dimension*, J. Pure Appl. Algebra **224** (2020), no. 8, 106346, 13 pp. https://doi.org/10.1016/j.jpaa.2020.106346

[17] H. Knörrer, *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. **88** (1987), no. 1, 153–164. https://doi.org/10.1007/BF01405095

[18] C. Lehn, M. Lehn, C. Sorger, and D. van Straten, *Twisted cubics on cubic fourfolds*, J. Reine Angew. Math. **731** (2017), 87–128. https://doi.org/10.1515/crelle-2014-0144

[19] L. Ma, *Lech's conjecture in dimension three*, Adv. Math. **322** (2017), 940–970. https://doi.org/10.1016/j.aim.2017.10.032

[20] L. Ma, *Lim Ulrich sequences and Lech's conjecture*, Invent. Math. **231** (2023), no. 1, 407–429. https://doi.org/10.1007/s00222-022-01149-2

[21] L. Manivel, *Ulrich and aCM bundles from invariant theory*, Comm. Algebra **47** (2019), no. 2, 706–718. https://doi.org/10.1080/00927872.2018.1495222

[22] G. V. Ravindra and A. Tripathi, *Remarks on higher-rank ACM bundles on hypersurfaces*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 11-12, 1215–1221. https://doi.org/10.1016/j.crma.2018.10.004

[23] J. Shamash, *The Poincaré series of a local ring*, J. Algebra **12** (1969), 453–470. https://doi.org/10.1016/0021-8693(69)90023-4

[24] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau* 3*-folds, and bundles on* $K3$ *fibrations*, J. Differential Geom. **54** (2000), no. 2, 367–438. https://doi.org/10.4310/jdg/1214341649

[25] H. L. Truong and H. N. Yen, *Stable Ulrich bundles on cubic fourfolds*, preprint, arXiv:2206.05285.

[26] B. Ulrich, *Gorenstein rings and modules with high numbers of generators*, Math. Z. **188** (1984), no. 1, 23–32. https://doi.org/10.1007/BF01163869

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