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ON THE LINEAR INDEPENDENCE MEASURES OF LOGARITHMS OF RATIONAL NUMBERS. II

Abderraouf Bouchelaghem, Yuxin He, Yuanhang Li, and Qiang Wu

Abstract. In this paper, we give a general method to compute the linear independence measure of $1, \log(1-1/r), \log(1+1/s)$ for infinitely many integers r and s. We also give improvements for the special cases when $r = s$, for example, $\nu(1, \log 3/4, \log 5/4) \leq 9.197$.

1. Introduction

For an irrational real number α, the real number $\mu > 0$ is said to be an irrationality measure of α if, for any $\varepsilon > 0$, there exists $q_0 = q_0(\varepsilon) > 0$, such that

$$\left|\alpha - \frac{p}{q}\right| \geq q^{-\mu-\varepsilon}$$

for all integers p and q with $q \geq q_0$. If $q_0(\varepsilon)$ is effectively computable, we say that it is an effective irrationality measure of α.

If $\alpha_0, \alpha_1, \ldots, \alpha_n$ are real numbers linearly independent over $\mathbb{Q}$, we say that $\nu > 0$ is a linearly independence measure of $\alpha_0, \alpha_1, \ldots, \alpha_n$ if, for any $\varepsilon > 0$, there exists $H_0(\varepsilon) > 0$, such that

$$|p\alpha_0 + q_1\alpha_1 + \cdots + q_n\alpha_n| \geq H^{-\nu-\varepsilon}$$

for all integers $p, q_1, \ldots, q_n$, with $H = \max\left(|q_1|, |q_2|, \ldots, |q_n|\right) \geq H_0(\varepsilon)$.

The minimum of those numbers μ is denoted by $\mu(\alpha)$, the minimum of those ν is denoted by $\nu(\alpha_0, \alpha_1, \ldots, \alpha_n)$. We have $\mu(\alpha) = \nu(1, \alpha) + 1$.

A classical problem is to study the irrationality measure of logarithm of rational number and the linear independence measure of logarithms of rational numbers. Baker [4] gave effective lower bounds of nonvanishing linear forms of logarithms

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

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where α_i and β_i are algebraic numbers. In the particular case, where α_i are rationals and β_i are integers, we obtain linear independence measures of logarithms of rational numbers (see [5] or [19]). However, the linear independence (or irrationality) measures are very large, for example, we have $\mu(\log 2) \leq 10^{22}$. In 1964, Baker [3] gave the first effective irrationality measure of $\log 2$ which is 12.5. After that, many improvements appeared as follows: in 1979, van der Poorten [11] showed that the measure of $\log 2$ can be reduced to 4.622, in 1982, Chudnovskys [7] improved it to 4.269, in 1987, Rhin [13] established that $\mu(\log 2) \leq 4.0765$, in 1993, Amoroso [2] obtained $\mu(\log 2) \leq 3.991$, in 1987, Rukhadze [16] found that $\mu(\log 2) \leq 3.891399$. At present, the best known irrationality measure of $\log 2$ is 3.57455391 which is obtained by Marcovecchio [10] in 2009.

In 1987, with an "arithmetical method", Rhin [13] obtained $\mu(\log 3) \leq 8.616$ with the help of $\nu(1, \log 3/2, \log 4/3) \leq 7.616$, i.e., $\nu(1, \log 2, \log 3) \leq 7.616$. In 2007, Salikhov [17] improved it to 5.125 with an "analytical method" by considering two integrals of symmetric rational function. In 2014, the fourth author and Wang [21] improved it to 5.1163 with the "arithmetical method" applied to the Salikhov's integrals.

In 2003, the fourth author [20] obtained $\nu\left(1, \log 16/15, \log 6/5, \log 4/3\right) \leq 15.27049$ and $\nu\left(1, \log 36/35, \log 8/7, \log 6/5, \log 9/7\right) \leq 256.865$, that is to say $\nu(1, \log 2, \log 3, \log 5) \leq 15.27049$ and $\nu(1, \log 2, \log 3, \log 5, \log 7) \leq 256.865$, and then $\mu(\log 5) \leq 16.27049$ and $\mu(\log 7) \leq 257.865$. In 2020, Bondareva et al. [6] improved the irrationality measure of $\log 7$ to 36.0099 with the "arithmetical method" using the integrals of symmetric rational function.

In 1980, Alladi and Robinson [1] gave a general method to compute the irrational measure of $\log(r/s)$, where r/s is a rational number close to 1. In 1989, Rhin [14] proved that $\mu(\log 5/3) \leq 7.224$. In 1993, Amoroso [2] improved the measure to $\mu(\log 5/3) \leq 6.851$ and obtained the following results $\mu(\log 2/3) \leq 3.402$, $\mu(\log 3/4) \leq 3.154$, $\mu(\log 4/5) \leq 3.017$ and $\mu(\log 7/5) \leq 5.456$. In 2010, Salnikova [18] improved the irrationality measure of $\log 5/3$ to 5.6514 and obtained $\mu(\log 8/5) \leq 7.2173$.

The fourth author [20] gave a general method to compute the linear independence measure of $1, \log(1 - 1/a), \log(1 + 1/a)$ for all integers $a \geq 4$. This method replaced the measure ($\nu(1, \log 3/4, \log 5/4) \leq 88$) of Rhin and Toffin [15] by 36.86, and then improved it to 20.515 with the "arithmetical method".

In this paper, we focus on the linear independence measure of logarithms of rational numbers. We give a general method to compute the linear independence measure of $1, \log(1 - 1/r), \log(1 + 1/s)$ for infinitely many integers r and s. In particular, we obtain the linear independence measure of $1, \log(1 - 1/a), \log(1+1/a)$ for all integers $a \geq 2$ and, for example, $\nu(1, \log 3/4, \log 5/4) \leq 10.789$, which is better than the bound (20.515) in [20]. And we give some improvements in the special case, for example, $\nu(1, \log 3/4, \log 5/4) \leq 9.197$.

In Section 2, we will give some lemmas. In Section 3, we will give the Theorems that provide the methods to computer the linear independence measure of $1, \log(1-1/r), \log(1+1/s)$ for $r = a$, $s = ma$ or $r = m'a$, $s = ma$, where a, m, m' are integers and prove them. In Section 4, we will show some numerical results and some improvements in the special case.

2. Some lemmas

We first recall Lemma 1 in [20], which is a generalization of Hata's lemma [9].

Lemma 2.1. *Let $m \in \mathbb{Z}^+$, and $\theta_1, \theta_2, \ldots, \theta_m$ be m real numbers. Suppose that for any $n \geq 1$, there exist integers $r_n > 0$, $P_n^{(1)}, \ldots, P_n^{(m)}$, such that if $\varepsilon_n^{(i)} = r_n\theta_i - P_n^{(i)}$, then $\varepsilon_n^{(i)} \neq 0$ for $1 \leq i \leq m$ and*

$$\lim_{n\to\infty} \frac{1}{n} \log |r_n| \leq \sigma, \quad \lim_{n\to\infty} \frac{1}{n} \log |\varepsilon_n^{(i)}| = -\tau^{(i)}, \ 1 \leq i \leq m,$$

where $\sigma, \tau^{(i)}(1 \leq i \leq m)$ are positive numbers.

Let $\tau = \min_{1\leq i\leq m}(\tau^{(i)})$, if for any $i \neq j$, $\tau^{(i)} \neq \tau^{(j)}$, then $1, \theta_1, \theta_2, \ldots, \theta_m$ are linearly independent over $\mathbb{Q}$, and for any $\varepsilon > 0$, there exists a positive integer $H_0(\varepsilon)$ such that

$$|p + q_1\theta_1 + q_1\theta_2 + \cdots + q_m\theta_m| \geq H^{-\frac{\sigma}{\tau}-\varepsilon}$$

for all integers $p, q_i(1 \leq i \leq m)$ with $H = \max_{1\leq i\leq m}(|q_i|) \geq H_0(\varepsilon)$.

Let r and s be two positive integers with $s \geq r$. Let $B = (2r-1)(2s+1)$, $C_1 = 2(s+1)(2r-1)$, $C_2 = 2r(2s+1)$ with $C_2 > C_1$ because $s \geq r$. Let $H_0(x) = 2B - x$, $H_1(x) = x-(2B-C_2)$, $H_2(x) = x-(2B-C_1)$, $H_3(x) = x-B$, $H_4(x) = x - C_1$, $H_5(x) = x - C_2$, and

$$F(x) = \frac{H_1^{\alpha_1 n}(x)H_2^{\alpha_2 n}(x)H_3^{\alpha_3 n}(x)H_4^{\alpha_2 n}(x)H_5^{\alpha_1 n}(x)}{x^{n+1}H_0(x)^{n+1}} = \frac{(f(x))^n}{xH_0(x)},$$

where α_i are the rational numbers and n is an even integer large enough such that $\alpha_i n \in \mathbb{Z}$ for $i = 1, 2, 3$. Let $\xi_1, \xi_2 \in (B, C_2)$, $\xi_3 \notin (B, C_2)$ be the extremum points of $f(x)$ with $|f(\xi_2)| \geq |f(\xi_1)|$.

As the function $F(x)$ is invariant by the transformation $x \to 2B - x$, i.e., $F(x) = F(2(2r-1)(2s+1) - x)$, we can write

$$F(x) = P(x) + \sum_{l=1}^{n+1} \left(\frac{A_l}{x^l} + \frac{A_l}{H_0(x)^l} \right), \tag{1}$$

where $P(x) \in \mathbb{Z}[x]$. If we take $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = 1$, then

$$\deg P(x) = (2\alpha_1 + 2\alpha_2 + \alpha_3)n - 2n - 2 = n - 2.$$

For A_l defined in (1), we have the following result.

Lemma 2.2. *Let* $d_1=\gcd(r,s+1)$, $d_2=\gcd(r-1,s)$, $d_3=\gcd(r,s)$, $d_4=\gcd(s+1,r-1)$. *Then* A_l *can be written as*

$$\begin{aligned}A_l &= 2^{l-2}d_1^{l-1}d_2^{l-1}d_3^{l-1}d_4^{l-1}\left(\tfrac{r}{d_1d_3}\right)^{-\frac{n}{2}+l-1}\left(\tfrac{s+1}{d_1d_4}\right)^{-\frac{n}{2}+l-1}\left(\tfrac{r-1}{d_2d_4}\right)^{-\frac{n}{2}+l-1}\\ &\quad\times\left(\tfrac{s}{d_2d_3}\right)^{-\frac{n}{2}+l-1}(2r-1)^{l-2}(2s+1)^{l-2}B_l,\end{aligned}$$

where $B_l\in\mathbb{Z}$ *for* $l=1,2,\ldots,n+1$.

Proof. We denote $\mathbf{D}_k(f(x))=\frac{f^{(k)}(0)}{k!}$ for $k\geq 0$, then

$$\begin{aligned}A_l &= \mathbf{D}_{n+1-l}(F(x)x^{n+1})\\ &= \sum_{\sum_{0\leq i\leq 5}k_i=n+1-l}\mathbf{D}_{k_0}\left(H_0(x)^{-n-1}\right)\mathbf{D}_{k_1}\left(H_1(x)^{\frac{n}{2}}\right)\mathbf{D}_{k_2}\left(H_2(x)^{\frac{n}{2}}\right)\\ &\quad\times\mathbf{D}_{k_3}\left(H_3(x)^{n}\right)\mathbf{D}_{k_4}\left(H_4(x)^{\frac{n}{2}}\right)\mathbf{D}_{k_5}\left(H_5(x)^{\frac{n}{2}}\right)\\ &= \sum_{\overline{k}}\gamma_{\overline{k}}(2(2r-1)(2s+1))^{-n-1-k_0}(2(r-1)(2s+1))^{\frac{n}{2}-k_1}(2s(2r-1))^{\frac{n}{2}-k_2}\\ &\quad\times((2r-1)(2s+1))^{n-k_3}(2(s+1)(2r-1))^{\frac{n}{2}-k_4}(2r(2s+1))^{\frac{n}{2}-k_5}\\ &= \sum_{\overline{k}}\gamma_{\overline{k}}2^{n-1-k_0-k_1-k_2-k_4-k_5}r^{\frac{n}{2}-k_5}(s+1)^{\frac{n}{2}-k_4}(r-1)^{\frac{n}{2}-k_1}s^{\frac{n}{2}-k_2}\\ &\quad\times(2r-1)^{n-1-k_0-k_2-k_3-k_4}(2s+1)^{n-1-k_0-k_1-k_3-k_5}\\ &= \sum_{\overline{k}}\gamma_{\overline{k}}2^{n-1-k_0-k_1-k_2-k_4-k_5}d_1^{n-k_4-k_5}d_2^{n-k_1-k_2}d_3^{n-k_2-k_5}d_4^{n-k_1-k_4}\\ &\quad\times\left(\tfrac{r}{d_1d_3}\right)^{\frac{n}{2}-k_5}\left(\tfrac{s+1}{d_1d_4}\right)^{\frac{n}{2}-k_4}\left(\tfrac{r-1}{d_2d_4}\right)^{\frac{n}{2}-k_1}\left(\tfrac{s}{d_2d_3}\right)^{\frac{n}{2}-k_2}\\ &\quad\times(2r-1)^{n-1-k_0-k_2-k_3-k_4}(2s+1)^{n-1-k_0-k_1-k_3-k_5},\end{aligned}$$

where $0\leq k_i\leq n/2$ for $i=1,2,4,5$ and $0\leq k_3\leq n$, $0\leq k_0\leq n+1$, the summation is over the sextuple $\overline{k}=(k_0,k_1,k_2,k_3,k_4,k_5)$ such that $\sum_{0\leq i\leq 5}k_i=n+1-l$ and $\gamma_{\overline{k}}\in\mathbb{Z}$.

As $k_0+k_1+k_2+k_3+k_4+k_5=n+1-l$, then

$$\begin{aligned}A_l &= 2^{l-2}d_1^{l-1}d_2^{l-1}d_3^{l-1}d_4^{l-1}\left(\tfrac{r}{d_1d_3}\right)^{-\frac{n}{2}+l-1}\left(\tfrac{s+1}{d_1d_4}\right)^{-\frac{n}{2}+l-1}\left(\tfrac{r-1}{d_2d_4}\right)^{-\frac{n}{2}+l-1}\\ &\quad\times\left(\tfrac{s}{d_2d_3}\right)^{-\frac{n}{2}+l-1}(2r-1)^{l-2}(2s+1)^{l-2}B_l,\end{aligned}$$

where $B_l\in\mathbb{Z}$. □

We consider the integrals

$$I_n(B,C_j)=\int_B^{C_j}F(x)\mathrm{d}x$$

for $j=1,2$. Then we have:

Lemma 2.3. *Suppose $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = 1$ if we take $D_n = \mathrm{lcm}(1, 2, \dots, n)$ and*

$$Q_n = 2\left(\frac{r}{d_1 d_3}\right)^{\frac{n}{2}} \left(\frac{s+1}{d_1 d_4}\right)^{\frac{n}{2}} \left(\frac{r-1}{d_2 d_4}\right)^{\frac{n}{2}} \left(\frac{s}{d_2 d_3}\right)^{\frac{n}{2}} (2r-1)(2s+1).$$

Then we have

$$Q_n D_n \int_B^{C_1} F(x)\mathrm{d}x \in \mathbb{Z} + \mathbb{Z}\log\left(1 + \frac{1}{s}\right),$$

and

$$Q_n D_n \int_B^{C_2} F(x)\mathrm{d}x \in \mathbb{Z} + \mathbb{Z}\log\left(1 - \frac{1}{r}\right).$$

Proof. We have

$$\begin{aligned}
&\int_B^{C_j} F(x)\mathrm{d}x \\
&= \int_B^{C_j} \left(P(x) + \sum_{l=1}^{n+1}\left(\frac{A_l}{x^l} + \frac{A_l}{H_0(x)^l}\right)\right)\mathrm{d}x \\
&= \int_B^{C_j} P(x)\mathrm{d}x + \int_B^{C_j}\left(\frac{A_1}{x} + \frac{A_1}{H_0(x)}\right)\mathrm{d}x + \sum_{l=2}^{n+1}\int_B^{C_j}\left(\frac{A_l}{x^l} + \frac{A_l}{H_0(x)^l}\right)\mathrm{d}x \\
&= \Lambda(B, C_j) + \Lambda_1(B, C_j) + \sum_{l=2}^{n+1}\Lambda_l(B, C_j),
\end{aligned}$$

where

$$\begin{aligned}
\Lambda(B, C_j) &= \int_B^{C_j} P(x)\mathrm{d}x, \\
\Lambda_1(B, C_j) &= \int_B^{C_j}\left(\frac{A_1}{x} + \frac{A_1}{H_0(x)}\right)\mathrm{d}x, \\
\Lambda_l(B, C_j) &= \int_B^{C_j}\left(\frac{A_l}{x^l} + \frac{A_l}{H_0(x)^l}\right)\mathrm{d}x.
\end{aligned}$$

(1) Obviously, we have $D_n Q_n \Lambda(B, C_j) \in \mathbb{Z}$ for $j = 1, 2$.
(2) For $l > 1$ and $j = 1, 2$, we have

$$\Lambda_l(B, C_j) = -\frac{A_l}{l-1}\left(\frac{1}{C_j^{l-1}} - \frac{1}{(2B - C_j)^{l-1}}\right).$$

As $l = 1, 2, \dots, n+1$, then $D_n/(l-1) \in \mathbb{Z}$. By Lemma 2.2 and the definitions of C_1, C_2 and B, $Q_n A_l/C_j$ and $Q_n A_l/(2B - C_j)$ are integers, where $2B - C_1 = 2s(2r-1)$ and $2B - C_2 = 2(r-1)(2s+1)$. Then for $l > 1$, $D_n Q_n \Lambda_l(B, C_j) \in \mathbb{Z}$ for $j = 1, 2$.

(3) For $l=1$, we have

$$\Lambda_1(B,C_j)=A_1\int_B^{C_j}\left(\frac{1}{x}+\frac{1}{(2B-x)}\right)\mathrm{d}x=A_1\log\frac{x}{2B-x}\bigg|_B^{C_j},$$

i.e., $\Lambda_1(B,C_1)=A_1\log\frac{s+1}{s}$ and $\Lambda_1(B,C_2)=A_1\log\frac{r}{r-1}$. By Lemma 2.2, $D_nQ_nA_1\in\mathbb{Z}$. Finally, we have Lemma 2.3. □

For proving that τ in Lemma 2.1 is a positive number in our cases, we need the following lemmas.

Lemma 2.4. *Let α,β be integers with $\beta>\alpha>0$ and $h(y)=y(y-\alpha^2)(y-\beta^2)$ for any $y\in[0,\beta^2]$. Then*

$$\frac{\max_{y\in[0,\beta^2]}|h(y)|}{(\alpha^2\beta^2-\beta^2)^2}\le\frac{\sqrt{2}\sqrt[4]{35\alpha^8-70\alpha^6\beta^2+59\alpha^4\beta^4-24\alpha^2\beta^6+4\beta^8}}{(\alpha^2-1)^2\sqrt[4]{3150}}.$$

Proof. Noting Agmon's inequality, we have

$$\begin{aligned}\Big(\max_{y\in[0,\beta^2]}|h(y)|\Big)^2&\le 2\int_0^{\beta^2}|h(y)||h'(y)|dy\\&\le 2\left(\int_0^{\beta^2}(h(y))^2dy\int_0^{\beta^2}(h'(y))^2dy\right)^{\frac{1}{2}},\end{aligned}$$

i.e.,

$$\max_{y\in[0,\beta^2]}|h(y)|\le\sqrt{2}\left(\int_0^{\beta^2}(h(y))^2dy\int_0^{\beta^2}(h'(y))^2dy\right)^{\frac{1}{4}}.$$

As

$$\int_0^{\beta^2}(h(y))^2dy=\frac{\alpha^4\beta^{10}}{30}-\frac{\alpha^2\beta^{12}}{30}+\frac{\beta^{14}}{105},$$
$$\int_0^{\beta^2}(h'(y))^2dy=\frac{\alpha^4\beta^6}{3}-\frac{\alpha^2\beta^8}{3}+\frac{2\beta^{10}}{15},$$

then

$$\int_0^{\beta^2}(h(y))^2dy\int_0^{\beta^2}(h'(y))^2dy=\frac{\beta^{16}(35\alpha^8-70\alpha^6\beta^2+59\alpha^4\beta^4-24\alpha^2\beta^6+4\beta^8)}{3150},$$

i.e.,

$$\max_{y\in[0,\beta^2]}|h(y)|\le\frac{\beta^4\sqrt{2}\sqrt[4]{35\alpha^8-70\alpha^6\beta^2+59\alpha^4\beta^4-24\alpha^2\beta^6+4\beta^8}}{\sqrt[4]{3150}},$$

consequently, Lemma 2.4 is proved. □

Considering Theorem 1.1.2 in [12], we have:

Lemma 2.5. *Let*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

and

$$\overline{a_0} = \max(|a_0|, |a_1|, \ldots, |a_{n-1}|).$$

Then all the positive zeros of $p(x)$ *are in the interval*

$$\left[0,\ 1 + \frac{\overline{a_0}}{|a_n|}\right).$$

By Lemma 2.5, we obtain the following lemma.

Lemma 2.6. *Let* $\alpha = 2x - 1$, $\beta = 2mx + 1$, *where* m *is a positive integer and* $x \geq m(m+1) + 1$. *Then we have*

$$35\alpha^8 - 70\alpha^6\beta^2 + 59\alpha^4\beta^4 - 24\alpha^2\beta^6 + 4\beta^8 < 4(\beta - 1)^8.$$

Proof. Let

$$H(x) = \sum_{i=0}^{8} A_i(m)x^i = 4(\beta-1)^8 - (35\alpha^8 - 70\alpha^6\beta^2 + 59\alpha^4\beta^4 - 24\alpha^2\beta^6 + 4\beta^8),$$

where $A_8(m) = 6144m^6 - 15104m^4 + 17920m^2 - 8960$, $A_7(m) = -4096m^7 - 6144m^6 + 18432m^5 + 30208m^4 - 30208m^3 - 53760m^2 + 17920m + 35840$, and $A_i(m) \in \mathbb{Z}[x]$ for $0 \leq i \leq 6$.

If $m = 1, 2$, by the numerical computation, the equation $H(x) = 0$ has no real solution in $[3, +\infty)$ and $H(3) > 0$, as the function $H(x)$ is continuous on $[3, +\infty)$, then for all $x \geq 3$ we have $H(x) > 0$.

If $m \geq 3$, then $A_8(m) > 0$, and $\overline{a_0} = \max\limits_{0 \leq i \leq 7}(|A_i(m)|) = A_7(m)$. As $m(m+1) + 1 > 1 + \left|\frac{A_7(m)}{A_8(m)}\right|$, it implies that $x > 1 + \left|\frac{A_7(m)}{A_8(m)}\right|$. By Lemma 2.5 the equation $H(x) = 0$ has no real solution in interval $\left[1 + \left|\frac{A_7(m)}{A_8(m)}\right|, +\infty\right)$, as $H(x)$ is continuous and $A_8(m) > 0$ we obtain $H(x) > 0$ for all $x \geq m(m+1) + 1$, then we have Lemma 2.6. □

3. The linear independence measure of 1, $\log(1 - 1/r)$, $\log(1 + 1/s)$

In the case $r = a$, $s = ma$, where $m \geq 1$ and $a \geq 2$ are integers, we take $\alpha_1 = \alpha_2 = 1/2$, $\alpha_3 = 1$, and $d = \lim\limits_{n\to\infty} \frac{1}{n}\log D_n = 1$, where $D_n = \operatorname{lcm}(1, 2, \ldots, n)$, then we have:

Theorem 3.1. *If* $\frac{2(a-1)}{m(m+1)}$ *is even, and* $q = \lim\limits_{n\to\infty} \frac{1}{n}\log Q_n$, *where*

$$Q_n = 2(2a-1)(2ma+1)\left(\frac{a-1}{m(m+1)}\right)^{\frac{n}{2}}\left(\frac{ma+1}{m+1}\right)^{\frac{n}{2}},$$

then for any $\varepsilon > 0$, there exists a positive integer $q_0(\varepsilon)$, such that, for all integers p, q_1, q_2 with $\max\{|q_1|, |q_2|\} \geq q_0(\varepsilon)$,

$$\left|p + q_1 \log\left(1 - \frac{1}{a}\right) + q_2 \log\left(1 + \frac{1}{ma}\right)\right| \geq q_0^{-\nu-\varepsilon},$$

where $q_0(\varepsilon)$ is effectively computable, and

$$\nu = -\frac{d + q + \log|f(\xi_3)|}{d + q + \log|f(\xi_2)|}$$

is a positive number.

Proof. Let $m \geq 1$ and $a \geq 2$ be two integers with $\frac{2(a-1)}{m(m+1)}$ is even, i.e., $m(m+1) \mid (a-1)$. Then $(m+1) \mid (ma+1)$. If we substitute r by a and s by ma, with the definitions in Lemma 2.2, then we have $d_1 = \gcd(a, ma+1) = 1$, $d_2 = \gcd(a-1, ma) = m$, $d_3 = \gcd(a, ma) = a$, $d_4 = \gcd(ma+1, a-1) = m+1$. Therefore using Lemma 2.3, we get

$$Q_n = 2(2a-1)(2ma+1)\left(\frac{a-1}{m(m+1)}\right)^{\frac{n}{2}}\left(\frac{ma+1}{m+1}\right)^{\frac{n}{2}}.$$

If we take $D_n = \mathrm{lcm}(1, 2, \ldots, n)$, then,

$$D_nQ_nI_n(B, C_1) = D_nQ_n\Lambda(B, C_1) + \sum_{l=2}^{n} D_nQ_n\Lambda_l(B, C_1) + D_nQ_nA_1 \log\frac{ma+1}{ma},$$

$$D_nQ_nI_n(B, C_2) = D_nQ_n\Lambda(B, C_2) + \sum_{l=2}^{n} D_nQ_n\Lambda_l(B, C_2) + D_nQ_nA_1 \log\frac{a}{a-1}.$$

We denote

$$D_nQ_n\Lambda(B, C_1) + \sum_{l=2}^{n} D_nQ_n\Lambda_l(B, C_1) = P_n^{(1)},$$

$$D_nQ_n\Lambda(B, C_2) + \sum_{l=2}^{n} D_nQ_n\Lambda_l(B, C_2) = P_n^{(2)},$$

$$D_nQ_nI_n(B, C_1) = \varepsilon_n^{(1)},\ D_nQ_nI_n(B, C_2) = \varepsilon_n^{(2)},$$

$$D_nQ_nA_1 = r_n,\ \log\frac{ma+1}{ma} = \theta_1,\ \log\frac{a}{a-1} = \theta_2.$$

Hence using Lemma 2.3, we have $P_n^{(1)} \in \mathbb{Z}$, $P_n^{(2)} \in \mathbb{Z}$ and $r_n \in \mathbb{Z}$.

In light of the result in Chapter IX of Dieudonné's book [8] (see also Lemma 2.4 in [9]), we have

$$\lim_{n\to\infty} \frac{1}{n} \log I_n(B, C_j) = \lim_{n\to\infty} \frac{1}{n} \log \int_B^{C_j} F(x)\mathrm{d}x = \log f(\xi_j)$$

for $j = 1, 2$, where B, C_j, $F(x)$, $f(x)$ and ξ_j are defined in Section 2. That is to say

$$-\tau^{(1)} = \lim_{n\to\infty} \frac{1}{n} \log|\varepsilon_n^{(1)}| = \log f(\xi_1) + d + q,$$

$$-\tau^{(2)} = \lim_{n\to\infty} \frac{1}{n} \log |\varepsilon_n^{(2)}| = \log f(\xi_2) + d + q,$$

and

$$\sigma = \lim_{n\to\infty} \frac{1}{n} \log |r_n| = \log f(\xi_3) + d + q,$$

where $d = \lim\limits_{n\to\infty} \frac{1}{n} \log D_n = 1$, $q = \lim\limits_{n\to\infty} \frac{1}{n} \log Q_n$. As $|f(\xi_2)| \geq |f(\xi_1)|$ then

$$\tau = \min\{\tau^{(1)}, \tau^{(2)}\} = -(\log |f(\xi_2)| + d + q).$$

Now we will prove that $\log |f(\xi_2)| + d + q < 0$ for all $m \geq 1$ and $a \geq 2$ with $\frac{2(a-1)}{m(m+1)}$ is even.

Let $\alpha = 2a - 1$, $\beta = 2ma + 1$ and

$$g(y) = (f(x))^2 = \frac{y(y-\alpha^2)(y-\beta^2)}{(\alpha^2\beta^2 - y)^2},$$

where $y = (x - \alpha\beta)^2$. We have

$$|f(\xi_2)|^2 \leq \max_{y\in[0,\beta^2]} |g(y)| \leq \frac{\max_{y\in[0,\beta^2]} |y(y-\alpha^2)(y-\beta^2)|}{(\alpha^2\beta^2 - \beta^2)^2}.$$

With Lemma 2.4 and Lemma 2.6, we have

$$|f(\xi_2)|^2 \leq \frac{\sqrt{2}\sqrt[4]{35\alpha^8 - 70\alpha^6\beta^2 + 59\alpha^4\beta^4 - 24\alpha^2\beta^6 + 4\beta^8}}{(\alpha^2-1)^2\sqrt[4]{3150}} < \frac{2(\beta-1)^2}{(\alpha^2-1)^2\sqrt[4]{3150}},$$

i.e.,

$$|f(\xi_2)| < \frac{m}{(a-1)\sqrt{2}\sqrt[8]{3150}}.$$

As $q = \frac{1}{2} \log\left(\frac{(a-1)(ma+1)}{m(m+1)^2}\right)$ and $d = 1$, then $\log |f(\xi_2)| + d + q < l(a)$, where

$$l(x) = \frac{1}{2} \log\left(\frac{m(mx+1)}{(m+1)^2(x-1)}\right) + 1 - \frac{\log(2)}{2} - \frac{\log(3150)}{8}.$$

It is very easy to proof for all $m \geq 1$ and $x \in [2, +\infty)$ we have $l(x) < 0$. Then for all $m \geq 1$ and all $a \geq 2$ with $m(m+1) \mid a-1$, we have $\log |f(\xi_2)| + d + q < 0$, i.e., $\tau = \min\{\tau^{(1)}, \tau^{(2)}\} > 0$.

Then by Lemma 2.1, Theorem 3.1 is proved. □

Theorem 3.2. *If* $\frac{2(a-1)}{m(m+1)}$ *is odd,* $q = \lim\limits_{n\to\infty} \frac{1}{n} \log Q_n$*, where*

$$Q_n = 2^{n+1}(2a-1)(2ma+1)\left(\frac{a-1}{m(m+1)}\right)^{\frac{n}{2}} \left(\frac{ma+1}{m+1}\right)^{\frac{n}{2}},$$

and if $d + q + \log |f(\xi_2)| < 0$*, then for any* $\varepsilon > 0$*, there exists a positive integer* $q_0(\varepsilon)$*, such that, for all integers* p*,* q_1*,* q_2 *with* $\max\{|q_1|, |q_2|\} \geq q_0(\varepsilon)$*,*

$$\left| p + q_1 \log\left(1 - \frac{1}{a}\right) + q_2 \log\left(1 + \frac{1}{ma}\right) \right| \geq q_0^{-\nu-\varepsilon},$$

where $q_0(\varepsilon)$ *is effectively computable, and*

$$\nu = -\frac{d+q+\log|f(\xi_3)|}{d+q+\log|f(\xi_2)|}$$

is a positive number.

In fact, by Lemma 2.3,

$$Q_n = 2^{n+1}(2a-1)(2ma+1)\left(\frac{a-1}{m(m+1)}\right)^{\frac{n}{2}}\left(\frac{ma+1}{m+1}\right)^{\frac{n}{2}}.$$

If $\log|f(\xi_2)|+d+q<0$, with the same argument, Theorem 3.2 is proved.

In particular, if $m=1$, then we have:

Corollary 3.3. *Let* $q=\lim\limits_{n\to\infty}\frac{1}{n}\log Q_n$, *where*

$$Q_n = 2(2a-1)(2a+1)\left(\frac{a-1}{2}\right)^{\frac{n}{2}}\left(\frac{a+1}{2}\right)^{\frac{n}{2}}$$

if $2\nmid a$, *and*

$$Q_n = 2(2a-1)(2a+1)(a-1)^{\frac{n}{2}}(a+1)^{\frac{n}{2}}$$

if $2\mid a$. *Then for any* $\varepsilon>0$, *there exists a positive integer* $q_0(\varepsilon)$, *such that, for all integers* p, q_1, q_2 *with* $\max\{|q_1|,|q_2|\}\geq q_0(\varepsilon)$,

$$\left|p+q_1\log\left(1-\frac{1}{a}\right)+q_2\log\left(1+\frac{1}{a}\right)\right|\geq q_0^{-\nu-\varepsilon},$$

where $q_0(\varepsilon)$ *is effectively computable, and*

$$\nu = -\frac{d+q+\log|f(\xi_3)|}{d+q+\log|f(\xi_2)|}$$

is a positive number.

Corollary 3.3 is a consequence of Theorem 3.1 and Theorem 3.2 when $m=1$, for $2\nmid a$ and $2\mid a$, respectively.

In the case $r=m'a$, $s=ma$, where $m>m'\geq 1$ are integers, we take also $\alpha_1=\alpha_2=1/2$, $\alpha_3=1$, and $d=\lim\limits_{n\to\infty}\frac{1}{n}\log D_n=1$, where $D_n=\operatorname{lcm}(1,2,\ldots,n)$, then we have:

Theorem 3.4. *If* $m(m+m')\mid m'a-1$, $m'(m+m')\mid ma+1$ *with* $\gcd(m',m)=1$, $q=\lim\limits_{n\to\infty}\frac{1}{n}\log Q_n$, *where*

$$Q_n = 2(2m'a-1)(2ma+1)\left(\frac{m'a-1}{m(m+m')}\right)^{\frac{n}{2}}\left(\frac{ma+1}{m'(m+m')}\right)^{\frac{n}{2}},$$

and if $d+q+\log|f(\xi_2)|<0$, *then for any* $\varepsilon>0$, *there exists a positive integer* $q_0(\varepsilon)$, *such that for all integers* p, q_1, q_2 *with* $\max\{|q_1|,|q_2|\}\geq q_0(\varepsilon)$,

$$\left|p+q_1\log\left(1-\frac{1}{m'a}\right)+q_2\log\left(1+\frac{1}{ma}\right)\right|\geq q_0^{-\nu-\varepsilon},$$

where $q_0(\varepsilon)$ *is effectively computable, and*

$$\nu = -\frac{d+q+\log|f(\xi_3)|}{d+q+\log|f(\xi_2)|}$$

is a positive number.

By Lemma 2.3 we get:

$$Q_n = 2(2m'a-1)(2ma+1)\left(\frac{m'a-1}{m(m+m')}\right)^{\frac{n}{2}}\left(\frac{ma+1}{m'(m+m')}\right)^{\frac{n}{2}}.$$

With the same method, we obtain Theorem 3.4.

Remark 3.5. By numerical computation, for all integers $1 \leq m < 22$ and $a \geq 2$ in Theorem 3.2, and for infinitely many integers m, m', a in Theorem 3.4, we have $\log|f(\xi_2)| + d + q < 0$.

If we replace a by $-a$ in Theorems 3.1, 3.2 and 3.4, we can also obtain the measure with different value.

4. Numerical results and two specials cases

With the method above we get, for example, $\nu(1, \log 3/4, \log 5/4) \leq 10.789$ replaces 20.515 in [20]. We give some numerical results for $\nu(1, \log(1-1/a), \log(1+1/ma))$ in Table 1 and Table 2.

Table 1. The numerical results of Theorem 3.1

$m=2$		$m=3$		$m=4$		$m=5$		$m=25$		$m=102$	
a	ν	a	ν	a	ν	a	ν	a	ν	a	ν
7	8.037	13	11.247	21	13.868	31	16.050	651	33.070	10507	47.303
13	9.047	25	12.855	41	15.824	61	18.234	1301	36.071	21013	50.462
19	9.754	37	13.879	61	17.031	91	19.561	1951	37.831	31519	52.311
25	10.295	49	14.634	81	17.908	121	20.517	2601	39.081	42025	53.623
151	14.189	373	20.356	341	22.466	721	26.628	18201	47.551	178603	60.222

Table 2. The numerical results of Theorem 3.2

$m=2$		$m=3$		$m=4$		$m=5$		$m=21$	
a	ν	a	ν	a	ν	a	ν	a	ν
4	31.041	7	49.058	11	72.054	16	99.991	232	32614.364
10	23.499	19	44.592	31	70.321	46	100.952	694	10721.976
16	23.965	31	46.600	51	73.995	76	106.452	1156	9982.251
22	24.741	43	48.483	71	77.074	106	110.849	1618	9851.788
388	37.084	571	68.969	751	104.928	916	145.524	48742	12192.241

Remark 4.1. The numerical results of case $m = 1$ in Theorem 3.1 can be found in Table 4.

Some numerical results for $\nu(1, \log(1-1/m'a), \log(1+1/ma))$ can be found in Table 3.

Table 3. The numerical results of Theorem 3.4

$m=3$			$m=4$			$m=5$			$m=6$		
m'	a	ν	m'	a	ν	m'	a	ν	m'	a	ν
2	23	4.355	3	47	3.990	2	53	6.257	5	119	3.926
2	53	5.152	3	131	4.786	3	187	5.366	5	449	4.763
2	83	5.587	3	215	5.171	3	307	5.770	5	779	5.111
2	113	5.889	3	803	6.194	4	79	3.955	5	1109	5.333
2	263	6.719	3	887	6.271	4	259	4.772	5	1439	5.498
$m=7$			$m=17$			$m=23$			$m=43$		
m'	a	ν	m'	a	ν	m'	a	ν	m'	a	ν
2	95	7.436	2	485	10.428	5	773	6.026	2	6773	14.631
3	47	4.667	5	1197	6.190	14	6261	4.689	3	1319	8.891
4	289	5.088	13	2707	4.289	17	3193	4.056	34	37687	4.379
5	17	2.606	14	2221	4.172	19	13219	4.591	41	881	3.032
6	167	3.903	16	1087	3.803	22	2023	3.779	42	7223	3.739

On the other hand, in Corollary 3.3, if we optimize the α_i in some special cases, we can improve the measures.

Special case 1. For $2a = (2h-1)3^k - 1$ with integers $h \geq 1$, $k \geq 2$, we take

$$\alpha_1 = \frac{2k+1}{4k+1},\ \alpha_2 = \frac{2k}{4k+1},\ \alpha_3 = \frac{4k+2}{4k+1},$$
$$D'_n = \operatorname{lcm}(1, 2, \ldots, \alpha_3 n),$$

and

$$Q'_n = 2(2a-1)(2a+1)\left(\frac{a-1}{3}\right)^{\alpha_2 n}(a+1)^{\alpha_1 n}$$

if a is even, or

$$Q'_n = 2(2a-1)(2a+1)\left(\frac{a-1}{6}\right)^{\alpha_2 n}\left(\frac{a+1}{2}\right)^{\alpha_1 n}$$

if a is odd, then we have

$$Q'_n D'_n \int_{4a^2-1}^{4a^2+2a} F(x)dx \in \mathbb{Z} + \mathbb{Z}\log\left(1 - \frac{1}{a}\right),$$

and

$$Q'_n D'_n \int_{4a^2-1}^{4a^2+2a-2} F(x)dx \in \mathbb{Z} + \mathbb{Z}\log\left(1 + \frac{1}{a}\right).$$

Hence we have:

Corollary 4.2. *If* $2a = (2h-1)3^k - 1$ *with integers* $h \geq 1$, $k \geq 2$, *then*

$$\nu'\left(1, \log\left(1 - \frac{1}{a}\right), \log\left(1 + \frac{1}{a}\right)\right) = -\frac{d' + q' + \log|f(\xi_3)|}{\max_{1\leq i\leq 2}(d' + q' + \log|f(\xi_i)|)},$$

where $d' = \lim_{n\to\infty} \frac{1}{n}\log D'_n$ *and* $q' = \lim_{n\to\infty} \frac{1}{n}\log Q'_n$, *and* $\xi_1, \xi_2 \in (4a^2-1, 4a^2+2a)$, $\xi_3 \notin (4a^2-1, 4a^2+2a)$ *are the extremum points of* $f(x)$.

Special case 2. For $2a = (2h-1)3^k + 1$ with integers $h \geq 1$, $k \geq 2$, we take

$$\alpha_1 = \frac{2k}{4k+1},\ \alpha_2 = \frac{2k+1}{4k+1},\ \alpha_3 = \frac{4k+2}{4k+1}$$

and

$$Q''_n = 2(2a-1)(2a+1)(a-1)^{\alpha_2 n}\left(\frac{a+1}{3}\right)^{\alpha_1 n}$$

if a is even, or

$$Q''_n = 2(2a-1)(2a+1)\left(\frac{a-1}{2}\right)^{\alpha_2 n}\left(\frac{a+1}{6}\right)^{\alpha_1 n}$$

if a is odd, we have

$$Q''_n D'_n \int_{4a^2-1}^{4a^2+2a} F(x)dx \in \mathbb{Z} + \mathbb{Z}\log\left(1-\frac{1}{a}\right),$$

or

$$Q''_n D'_n \int_{4a^2-1}^{4a^2+2a-2} F(x)dx \in \mathbb{Z} + \mathbb{Z}\log\left(1+\frac{1}{a}\right).$$

In a similar way, we have:

Corollary 4.3. *If* $2a = (2h-1)3^k + 1$ *with integers* $h \geq 1$, $k \geq 2$, *then*

$$\nu''\left(1, \log\left(1-\frac{1}{a}\right), \log\left(1+\frac{1}{a}\right)\right) = -\frac{d' + q'' + \log|f(\xi_3)|}{\max_{1\leq i\leq 2}(d' + q'' + \log|f(\xi_i)|)},$$

where $q'' = \lim_{n\to\infty}\frac{1}{n}\log Q''_n$, *and* d', ξ_i *are defined in Corollary 4.2.*

We then obtain, for example, $\nu'(1, \log 3/4, \log 5/4) \leq 9.197$ replaces 10.789 given by Theorem 3.2 for $m = 1$ and $a = 4$. We give in Table 4, some numerical results of the linear independence measure of $1, \log(1-1/a), \log(1+1/a)$ for $a \geq 2$, where ν', ν'' are given by Corollary 4.2 and Corollary 4.3 respectively.

Table 4. The linear independence measure of $1, \log(1-1/a), \log(1+1/a)$

a	ν	ν'	a	ν	ν''	a	ν	ν'	a	ν	ν''
2	20.019	-	3	4.125	-	4	10.789	9.197	5	5.441	4.736
6	13.042	-	7	6.251	-	8	14.530	-	9	6.840	-
10	15.644	-	11	7.304	-	13	7.686	6.741	14	17.279	12.915
40	22.222	16.535	41	10.279	8.808	67	11.380	10.906	68	24.688	21.983
121	12.703	10.828	122	27.399	22.033	148	28.294	27.662	149	13.169	13.191
202	29.735	24.928	203	13.860	12.762	283	14.604	13,655	284	31,313	26.927
364	32.462	25.399	365	15.172	13.426	850	36.389	29.986	851	17.067	15.553

Remark 4.4. For an integer a in the special cases, there are maybe different m and k, we may have the different numerical results. In this case, we take the best.

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Abderraouf Bouchelaghem
Department of Mathematics
Southwest University of China
Tiansheng Road Beibei
400715 Chongqing, P. R. China
Email address: raouf1994@163.com

Yuxin He
Department of Mathematics
Southwest University of China
Tiansheng Road Beibei
400715 Chongqing, P. R. China
Email address: hyx3491@163.com

Yuanhang Li
Department of Mathematics
Southwest University of China
Tiansheng Road Beibei
400715 Chongqing, P. R. China
Email address: 1059221470@qq.com

Qiang Wu
Department of Mathematics
Southwest University of China
Tiansheng Road Beibei
400715 Chongqing, P. R. China
Email address: qiangwu@swu.edu.cn