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POSITIVE SOLUTIONS TO DISCRETE HARMONIC FUNCTIONS IN UNBOUNDED CYLINDERS

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Abstract. In this paper, we study the positive solutions to a discrete harmonic function for a random walk satisfying finite range and ellipticity conditions, killed at the boundary of an unbounded cylinder in $\mathbb{Z}^d$. We first prove the existence and uniqueness of positive solutions, and then establish that all the positive solutions are generated by two special solutions, which are exponential growth at one end and exponential decay at the other. Our method is based on maximum principle and a Harnack type inequality.

1. Introduction

The positive harmonic functions on Euclidean spaces have been studied extensively, see for examples [3, 9, 10, 12, 18, 22]. It is worth noting that Benedicks [3] proved that the cone of positiveharmonic functions vanishing on the boundary of certain domains in $\mathbb{R}^d$ is generated by twolinearly independent minimal positive harmonic functions, which means that the cone is two-dimensional. Gardiner [9] considered the positive harmonic functions with zero boundary condition in unbounded cylinders and obtained similar result. As is well known, for the equation $\Delta u = 0$ on $\mathbb{R} \times (0, \pi)$ with the zero Dirichlet boundary condition, any positive solution is a linear combination of $e^x \sin y$ and $e^{-x} \sin y$. Later, Gardiner' result was extended to second order elliptic operators, one can see [1, 18, 24–28].

The positive harmonic functions on graphs have also been studied by many authors, e.g., [11, 16, 17, 23]. Recently, random walks conditioned to live in domains $D \subset \mathbb{Z}^d$ are of growing interest because of the range of their applications in enumerative combinatorics, in probability theory and in harmonic analysis, see for examples [2, 5, 7, 8, 19, 20]. The authors in [4] proved the existence and uniqueness of positive harmonic functions for random walks killed at the boundary of an orthant in $\mathbb{Z}^d$. Mustapha and Sifi [21] generalized the results of [4] to a globally Lipschitz unbounded domain in $\mathbb{Z}^d$. For more related

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works, we refer readers to [6, 13–15, 29]. Since it is crucial to identify the set of all positive harmonic functions associated with a killed random walk, in this paper, we study the positive harmonic functions for random walks killed at the boundary of an unbounded cylinder in $\mathbb{Z}^d$. We will generalize Gardiner's result to graph setting.

We recall the setting on graphs. Let Γ be a finite subset of $\mathbb{Z}^d$ containing all unit vectors $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$, where the 1 is the k-th component; and let $\pi : \mathbb{Z}^d \times \Gamma \to [0, 1]$ such that

$$\sum_{e\in\Gamma} \pi(x, e) = 1, \quad x \in \mathbb{Z}^d.$$

Then we let $\{X_n\}_{n\in\mathbb{N}}$ be the Markov chain on $\mathbb{Z}^d$ defined by

$$\mathbb{P}[X_{n+1} = x + e \mid X_n = x] = \pi(x, e), \quad e \in \Gamma,\, x \in \mathbb{Z}^d,\, n = 0, 1, \ldots.$$

$\{X_n\}_{n\in\mathbb{N}}$ is a (spatially) homogeneous random walk with bounded increments if $\pi(x, e) = \pi(e)$; Otherwise, we call $\{X_n\}_{n\in\mathbb{N}}$ spatially inhomogeneous. Let

$$\mathbb{P}_x[X_n = y], \quad n = 1, 2, \ldots, x, y \in \mathbb{Z}^d,$$

be the transition probability corresponding to the chain $\{X_n\}_{n\in\mathbb{N}}$, where $\mathbb{P}_x$ is the law of the chain with $X_0 = x$.

Let A be a set of $\mathbb{Z}^d$. We define the boundary of A as

$$\partial A = \{x \in A^c \mid x = z + e,\, z \in A,\, e \in \Gamma\},$$

where $A^c = \mathbb{Z}^d \backslash A$. We say a function $u : \bar{A} = A \cup \partial A \to \mathbb{R}$ is harmonic in A if $Lu = 0$ in A, where L is the difference operator defined as

$$Lu(x) = \sum_{e\in\Gamma} \pi(x, e)(u(x + e) - u(x)).$$

We assume that the probability $\pi(x, e)$ satisfies the following uniform ellipticity condition:

$$\pi(x, e) \geq \lambda, \quad x \in \mathbb{Z}^d,\, e \in \Gamma,$$

for some $\lambda > 0$. Under the assumption above, such an L is a discrete analogue of a uniformly elliptic, purely second order differential operator with measurable coefficients.

To state our results, we first give some notations. Let $x = (x', y) \in \mathbb{Z}^d \times \mathbb{Z}$. We denote by $\mathcal{C} = \mathcal{D} \times \mathbb{Z}$ an unbounded cylinder in $\mathbb{Z}^d \times \mathbb{Z}$, where $\mathcal{D} \subset \mathbb{Z}^d$ is a bounded domain (i.e., a finite connected set of vertices of $\mathbb{Z}^d$). Moreover, for $E \subset \mathbb{Z}$, let $\mathcal{C}_E = \mathcal{D} \times E = \{(x', y) \in \mathbb{Z}^d \times \mathbb{Z} \mid x' \in \mathcal{D},\ y \in E\}$, $\partial_l \mathcal{C}_E = \partial \mathcal{D} \times E = \{(x', y) \in \mathbb{Z}^d \times \mathbb{Z} \mid x' \in \partial \mathcal{D},\ y \in E\}$ and $\partial_r \mathcal{C}_E = \partial \mathcal{C}_E \backslash \partial_l \mathcal{C}_E$. Without loss of generality, we assume that $0' \in \mathcal{D}$ and the constants c, C may change from line to line.

In this paper, we study the positive functions u which are discrete harmonic for the random walk $\{X_n\}_{n\in\mathbb{N}}$ killed at the boundary of an unbounded cylinder

$\mathcal{C}$, that is,

$$\begin{cases} Lu(x) = 0, & x \in \mathcal{C}, \\ u(x) = 0, & x \in \partial\mathcal{C}, \\ u(x) > 0, & x \in \mathcal{C}. \end{cases} \tag{1.1}$$

We are interested in the existence and uniqueness (up to a multiplicative constant) of a solution to the problem (1.1). In addition, we would like to extend Gardiner's result [9] to the discrete positive harmonic functions for random walks killed at the boundary $\partial\mathcal{C}$.

For any $u \in \mathcal{C}$, let $\hat{u}(y) = \max\limits_{x' \in \mathcal{D}} u^+(x', y)$ and $m(u) = \inf\limits_{y \in \mathbb{Z}} \hat{u}(y)$, where $u^+ = \max\{u, 0\}$. We denote by S the positive solution set of the problem (1.1). Denote

$$\begin{aligned} S^+ &= \{u \in S \mid \lim_{y \to +\infty} u(x', y) = 0\}, \\ S^- &= \{u \in S \mid \lim_{y \to -\infty} u(x', y) = 0\}, \\ S^\vee &= \{u \in S \mid \text{there exists } x^* \in \mathcal{C} \text{ such that } u(x^*) = m(u) > 0\}. \end{aligned}$$

Our results are as follows.

Theorem 1.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a random walk satisfying the above finite support and ellipticity conditions. Assume that $\mathcal{C}$ is an unbounded cylinder in $\mathbb{Z}^d$. Then, up to a multiplicative constant, there exists a unique positive function, harmonic for the random walk killed at the boundary $\partial\mathcal{C}$.*

Theorem 1.2. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a random walk satisfying the above finite support and ellipticity conditions. Assume that $\mathcal{C}$ is an unbounded cylinder in $\mathbb{Z}^d$. Then the solution set S^+ and S^- are well defined. Moreover, S is a linear combination of S^+ and S^-, that is, for any $u \in S^+$ and $v \in S^-$,*

$$S = S^+ + S^- = \{au + bv \mid a, b \geq 0,\ a + b > 0\}.$$

Theorem 1.3. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a random walk satisfying the above finite support and ellipticity conditions. Assume that $\mathcal{C}$ is an unbounded cylinder in $\mathbb{Z}^d$. Then there exist constants $c, C > 0$ depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that, for any $u \in S^+$, $v \in S^-$, $w \in S^\vee$,*

$$-cy \leq \ln\left(\frac{\hat{u}(y)}{\hat{u}(0)}\right) \leq -Cy, \quad y \in \mathbb{Z},$$

$$cy \leq \ln\left(\frac{\hat{v}(y)}{\hat{v}(0)}\right) \leq Cy, \quad y \in \mathbb{Z},$$

$$c|y - y^*| \leq \ln\left(\frac{\hat{w}(y)}{\hat{w}(y^*)}\right) \leq C|y - y^*|, \quad y \in \mathbb{Z},$$

where $w(x^*) = m(w)$.

Remark 1.4. (i) Since the spatially inhomogeneous random walks are considered as the discrete analogues of diffusions generated by second order differential operators in non-divergence form, for the proof of the theorems, we follow the arguments in Bao et al. [1].

(ii) The basic tools used in [1] are maximum principle, boundary Hölder estimate, Harnack inequality, Carleson estimate and boundary Harnack inequality. While in our paper, maximum principle and a Harnack type inequality are enough;

(iii) Our contribution is that we give a natural probabilistic proof for a decay lemma (Lemma 2.1 below) and a Harnack inequality (Lemma 3.1 below), which are the keystones for our theory of this paper.

The paper is organized as follows: in Section 2, we first prove a decay lemma and use it to obtain a maximum principle in unbounded cylinders of $\mathbb{Z}^d$. This yields a decomposition of the positive solution set S. In Section 3, we first prove a Harnack inequality and use it to get the relations among the positive solution sets S^+, S^- and $S^\vee$. In Section 4, we give the proofs of Theorem 1.1-Theorem 1.3.

2. Maximum principle

In this section, we prove a maximum principle in unbounded cylinders. For a finite set $A \subset \mathbb{Z}^d$, we denote by τ_A the exit time from A, i.e., $\tau_A = \inf\{n \geq 0 : X_n \notin A\}$. The harmonic measure in A at the point x is defined by

$$\omega_A^x(E) = \mathbb{P}_x[X_{\tau_A} \in E], \quad E \subset \partial A.$$

It is well known that, for each $\phi : \partial A \to \mathbb{R}$, the solution $u : \bar{A} \to \mathbb{R}$ of the boundary value problem

$$\begin{cases} Lu(x) = 0, & x \in A, \\ u(x) = \phi(x), & x \in \partial A, \end{cases}$$

can be represented by means of ω_A^x, $x \in A$ as follows:

$$u(x) = \sum_{z \in \partial A} \phi(z)\, \mathbb{P}_x[X_{\tau_A} = z]. \tag{2.1}$$

First, we give a discrete version of maximum principle, see [4, Theorem 2.1].

Theorem 2.1 (Maximum principle)**.** *Let $A \subset \mathbb{Z}^d$ be a bounded domain in $\mathbb{Z}^d$ and $u : \bar{A} \to \mathbb{R}$ a harmonic function on A. Assume that $u \geq 0$ on ∂A. Then $u \geq 0$ in A.*

Then we have the following decay lemma, which plays a key role in proving a maximum principle in unbounded cylinders of $\mathbb{Z}^d$.

Lemma 2.2. *Assume that* $u(x)$ *satisfies*

$$\begin{cases} Lu(x) \geq 0, & x \in \mathcal{C}_{(k-1,k+1)}, \\ u(x) \leq 0, & x \in \partial_l \mathcal{C}_{(k-1,k+1)}, \\ u(x) \leq 1, & x \in \partial_r \mathcal{C}_{(k-1,k+1)}. \end{cases}$$

Then for any $k \in \mathbb{Z}$, *there exists a constant* $0 < \delta < 1$ *depending on* d, λ, Γ, $\mathcal{D}$ *such that*

$$u(x) \leq 1 - \delta, \quad x \in \mathcal{C}_{\{k\}}.$$

Proof. Let $w(x)$ be the solution of the Dirichlet boundary problem

$$\begin{cases} Lw(x) = 0, & x \in \mathcal{C}_{(k-1,k+1)}, \\ w(x) = 0, & x \in \partial_l \mathcal{C}_{(k-1,k+1)}, \\ w(x) = 1, & x \in \partial_r \mathcal{C}_{(k-1,k+1)}. \end{cases}$$

Then maximum principle implies that

$$u(x) \leq w(x), \quad x \in \mathcal{C}_{(k-1,k+1)}.$$

Let τ be the exit time from $\mathcal{C}_{(k-1,k+1)}$. We claim that

$$\mathbb{P}_x[X_\tau \in \partial_l \mathcal{C}_{(k-1,k+1)}] \geq \delta > 0,$$

where $0 < \delta < 1$ is a constant depending only on d, λ, Γ, $\mathcal{D}$.

Indeed, we have that

$$\begin{aligned} &\mathbb{P}_x[X_\tau \in \partial_l \mathcal{C}_{(k-1,k+1)}] \\ &\geq \mathbb{P}_x[X_\tau = z \in \partial_l \mathcal{C}_{(k-1,k+1)}] \\ &= \sum_{x_1,\ldots,x_{n-1} \in \mathcal{C}_{(k-1,k+1)}} \mathbb{P}_x[X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_\tau = z] \\ &= \sum_{x_1,\ldots,x_{n-1} \in \mathcal{C}_{(k-1,k+1)}} \mathbb{P}[X_1 = x_1 \mid X_0 = x] \cdots \mathbb{P}[X_\tau = z \mid X_{n-1} = x_{n-1}] \\ &\geq \mathbb{P}[X_1 = x_1 \mid X_0 = x] \cdots \mathbb{P}[X_\tau = z \mid X_{n-1} = x_{n-1}] \\ &\geq \lambda^{\mathrm{diam}(\mathcal{C}_{(k-1,k+1)})} \\ &= \delta. \end{aligned}$$

Clearly, the constant $\delta \in (0,1)$ is dependent of d, λ, Γ, $\mathcal{D}$.

By (2.1), we have that

$$\begin{aligned} w(x) &= \sum_{z \in \partial(\mathcal{C}_{(k-1,k+1)})} w(z)\, \mathbb{P}_x[X_\tau = z] \\ &= \sum_{z \in \partial(\mathcal{C}_{(k-1,k+1)}) \setminus \partial_l \mathcal{C}_{(k-1,k+1)}} \mathbb{P}_x[X_\tau = z] \\ &= 1 - \sum_{z \in \partial_l \mathcal{C}_{(k-1,k+1)}} \mathbb{P}_x[X_\tau = z] \end{aligned}$$

$$\leq 1 - \delta.$$

Consequently, we obtain that

$$u(x) \leq 1 - \delta, \quad x \in \mathcal{C}_{\{k\}}. \qquad \square$$

Lemma 2.3. *Assume that u is a subharmonic function in $\mathcal{C}$, and u is bounded from above. Then*

$$\sup_{x\in\mathcal{C}} u(x) \leq \sup_{x\in\partial\mathcal{C}} u^+(x).$$

Proof. Without loss of generality, we assume that $\sup\limits_{x\in\partial\mathcal{C}} u^+(x) = 0$. Otherwise one can consider $v = u - \sup\limits_{x\in\partial\mathcal{C}} u^+(x)$. Hence we only need to prove that

$$u(x) \leq 0, \quad x \in \mathcal{C}. \tag{2.2}$$

Since u is bounded from above, we let $u \leq M$, where M is a positive constant. In order to apply Lemma 2.2, for any $\varepsilon > 0$ and $k \in \mathbb{Z}$, we consider the function

$$\tilde{u}(x) = \frac{u(x)}{\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \varepsilon}, \quad x \in \mathcal{C}.$$

Obviously, $\tilde{u}(x)$ satisfies

$$\begin{cases} L\tilde{u}(x) \geq 0, & x \in \mathcal{C}_{(k-1,k+1)}, \\ \tilde{u}(x) \leq 0, & x \in \partial_l\mathcal{C}_{(k-1,k+1)}, \\ \tilde{u}(x) \leq 1, & x \in \partial_r\mathcal{C}_{(k-1,k+1)}. \end{cases}$$

By Lemma 2.2, we derive, for $0 < \delta < 1$, that

$$\tilde{u}(x', k) \leq 1 - \delta, \quad x' \in \mathcal{D}.$$

More precisely, we have that

$$u(x', k) \leq (1-\delta)(\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \varepsilon),$$

and hence

$$u^+(x', k) \leq (1-\delta)(\max\{\hat{u}(k-1), \hat{u}(k+1)\} + \varepsilon).$$

Then we get that

$$\hat{u}(k) \leq (1-\delta)\max\{\hat{u}(k-1), \hat{u}(k+1)\}.$$

The previous inequality implies that

$$\begin{aligned} \hat{u}(k-1) &\leq (1-\delta)\max\{\hat{u}(k-2), \hat{u}(k)\} \\ &= (1-\delta)\max_{\mathcal{C}_{[k-2,k]}} u^+ \end{aligned}$$

and

$$\begin{aligned} \hat{u}(k+1) &\leq (1-\delta)\max\{\hat{u}(k), \hat{u}(k+2)\} \\ &= (1-\delta)\max_{\mathcal{C}_{[k,k+2]}} u^+. \end{aligned}$$

Then it follows from the above inequalities that

$$\max_{\mathcal{C}_{[k-1,k+1]}} u^+ \le (1-\delta) \max_{\mathcal{C}_{[k-2,k+2]}} u^+.$$

For any $k \in \mathbb{Z}$, by iteration, we obtain that

$$\begin{aligned}
\hat{u}(k) &\le (1-\delta)\max\{\hat{u}(k-1), \hat{u}(k+1)\} \\
&= (1-\delta) \max_{\mathcal{C}_{[k-1,k+1]}} u^+ \\
&\le (1-\delta)^2 \max_{\mathcal{C}_{[k-2,k+2]}} u^+ \\
&\ \ \vdots \\
&\le (1-\delta)^m \max_{\mathcal{C}_{[k-m,k+m]}} u^+ \\
&\le (1-\delta)^m M \\
&\to 0, \quad m \to +\infty.
\end{aligned}$$

Then the result (2.2) holds. We complete the proof. □

Corollary 2.4. *Assume that u is a subharmonic function in $\mathcal{C}_{(0,+\infty)}$, and u is bounded from above. Then*

$$\sup_{x\in\mathcal{C}_{(0,+\infty)}} u(x) \le \sup_{x\in\partial\mathcal{C}_{(0,+\infty)}} u^+(x).$$

Lemma 2.5. *Assume that u satisfies*

$$\begin{cases} Lu(x) \ge 0, & x \in \mathcal{C}_{(0,+\infty)}, \\ u(x) = 0, & x \in \partial_l\mathcal{C}_{(0,+\infty)}, \end{cases}$$

and u is bounded from above. Then

$$u(x) \le e^{-\beta y}\hat{u}(0), \quad x \in \mathcal{C}_{(0,+\infty)},$$

where $\beta > 0$ is a constant depending on d, λ, Γ, $\mathcal{D}$.

Proof. Corollary 2.4 implies that

$$\hat{u}(y) \le \hat{u}(0), \quad y \in (0,+\infty).$$

By Lemma 2.2, there exists a constant $\delta \in (0,1)$ such that $\hat{u}(1) \le (1-\delta)\hat{u}(0)$. Do the operation repeatedly, we get that

$$\hat{u}(y) \le (1-\delta)^y \hat{u}(0) = e^{y\ln(1-\delta)}\hat{u}(0).$$

Hence the proof ends with $\beta = -\ln(1-\delta) > 0$. □

Now we can prove the following proposition.

Proposition 2.6. *Let $u \in S$. Then $m(u) = \inf\limits_{y\in\mathbb{Z}} \hat{u}(y) \ge 0$.*

(1) *If $m(u) = \inf\limits_{y\in\mathbb{Z}} \hat{u}(y) = 0$, then either of the following alternatives holds*:

(i) *there exists a sequence* $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}$ *such that* $\lim\limits_{j\to\infty} y_j = +\infty$, $\lim\limits_{j\to\infty} u(x'_j, y_j) = 0$, *and* $\hat{u}(y)$ *is a strictly decreasing function in* $\mathbb{Z}$;

(ii) *there exists a sequence* $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}$ *such that* $\lim\limits_{j\to\infty} y_j = -\infty$, $\lim\limits_{j\to\infty} u(x'_j, y_j) = 0$, *and* $\hat{u}(y)$ *is a strictly increasing function in* $\mathbb{Z}$.

(2) *If* $m(u) = \inf\limits_{y\in\mathbb{Z}} \hat{u}(y) > 0$, *then there exists* $x^* = (x'^*, y^*) \in \mathcal{C}$ *such that* $u(x^*) = m(u)$, *and* $\hat{u}(y)$ *is strictly increasing in* $[y^*, +\infty)$ *and strictly decreasing in* $(-\infty, y^*]$.

Proof. Let $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}$ be a minimizing sequence that satisfies $u(x'_j, y_j) = \max\limits_{x'\in\mathcal{D}} u(x', y_j) = \hat{u}(y_j)$ and $\lim\limits_{j\to\infty} u(x_j) = \lim\limits_{j\to\infty} \hat{u}(y_j) = m(u)$.

(1) If $m(u) = 0$, then $\lim\limits_{j\to\infty} u(x_j) = m(u) = 0$.

We claim that there exists a subsequence of $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}$ (still denote itself) such that $\lim\limits_{j\to\infty} y_j = +\infty$ or $\lim\limits_{j\to\infty} y_j = -\infty$.

In fact, if $\{y_j\}$ is bounded in $\mathbb{Z}$, i.e., $|y_j| \le M$ for some $M > 0$, then $\mathcal{C}_{[-M,M]}$ is a bounded domain (a finite vertex set). Hence there exist a subsequence $\{x_j\} = \{(x'_j, y_j)\} \subset \mathcal{C}_{[-M,M]}$ and a point $x_0 \in \mathcal{C}_{[-M,M]}$ such that $\lim\limits_{j\to\infty} u(x_j) = u(x_0) > 0$. We get a contradiction.

(i) If $\lim\limits_{j\to\infty} y_j = +\infty$, then $\hat{u}(y)$ is a strictly decreasing function in $\mathbb{Z}$.

Arguing indirectly, if there exist $y_1, y_2 \in \mathbb{Z}$ with $y_1 < y_2$ such that $\hat{u}(y_2) \ge \hat{u}(y_1) > 0$. Since $\lim\limits_{j\to\infty} y_j = +\infty$ and $\lim\limits_{j\to\infty} \hat{u}(y_j) = 0$, by taking j sufficiently large, one has that

$$y_1 < y_2 < y_j \quad \text{and} \quad \hat{u}(y_j) \le \hat{u}(y_1).$$

Then we get a local maximum point $x_2 = (x'_2, y_2) \in \mathcal{C}_{(y_1, y_j)}$, which contradicts the maximum principle.

(ii) If $\lim\limits_{j\to\infty} y_j = -\infty$, by similar arguments as in (i), we get that $\hat{u}(y)$ is a strictly increasing function in $\mathbb{Z}$.

(2) If $m(u) > 0$, then $\lim\limits_{j\to\infty} \hat{u}(y_j) = \lim\limits_{j\to\infty} u(x_j) = m(u) > 0$. We first prove that there exist a subsequence of $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}$ (still denote itself) and a point $x^* \in \mathcal{C}$ such that

$$\lim_{j\to\infty} x_j = x^* \quad \text{and} \quad \lim_{j\to\infty} u(x_j) = u(x^*).$$

In fact, $|y_j| \le M$ for some $M > 0$. Otherwise $\lim\limits_{j\to\infty} y_j = +\infty$ or $\lim\limits_{j\to\infty} y_j = -\infty$. Without loss of generality, we assume that $\lim\limits_{j\to\infty} y_j = +\infty$. Similar to the proof of (i), we can get that $\hat{u}(y)$ is a strictly decreasing function in $\mathbb{Z}$. Hence $u(x)$ is bounded in $\mathcal{C}_{(0,+\infty)}$. By Lemma 2.5, one gets that $\lim\limits_{j\to\infty} \hat{u}(y_j) = 0$, which is a contradiction. Hence there exist a subsequence $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}_{[-M,M]}$

and a point $x^* \in \mathcal{C}_{[-M,M]}$ such that $\lim_{j\to\infty} x_j = x^*$ and $\lim_{j\to\infty} u(x_j) = u(x^*)$. As a consequence, $u(x^*) = m(u)$.

Next, we show that $\hat{u}(y)$ is strictly increasing in $[y^*, +\infty)$. Suppose it is not true, then there exist $y_1, y_2 \in \mathbb{Z}$ with $y^* < y_1 < y_2$ such that $\hat{u}(y_1) \geq \hat{u}(y_2) \geq \hat{u}(y^*) = m(u) > 0$, we get a local maximum point $x_1 = (x_1', y_1) \in \mathcal{C}_{(y^*, y_2)}$, which contradicts the maximum principle. Similarly, we can prove that $\hat{u}(y)$ is strictly decreasing in $(-\infty, y^*]$. □

Remark 2.7. By Proposition 2.6, one gets that the positive solution set $S = S^+ \cup S^- \cup S^\vee$ and $S^+ \cap S^- = S^- \cap S^\vee = S^+ \cap S^\vee = \emptyset$.

3. Local Harnack inequality

In this section, in order to study the asymptotic behavior of the positive harmonic functions for the random walk $\{X_n\}_{n\in\mathbb{N}}$ killed at the boundary, we first prove a discrete version of Harnack type inequality. We would like to say that this is a local Harnack inequality, which depends on the domain. Hence this Harnack inequality differs from the one that in [21, Theorem 2.1].

Lemma 3.1. *Let $u \in S$. There exists a positive constant C depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that, for any $y \in \mathbb{Z}$,*

$$Cu(z) \leq u(x) \leq C^{-1}u(z), \quad x, z \in \mathcal{C}_{(y-2,y+2)}.$$

Proof. We first claim that if u is harmonic in $\{x, x+e\}$ with $e \in \Gamma$, then

$$\lambda u(x+e) \leq u(x) \leq \lambda^{-1} u(x+e). \tag{3.1}$$

In fact, we have that

$$u(x) = \sum_{e\in\Gamma} \pi(x,e) u(x+e) \geq \lambda u(x+e).$$

Interchanging x and $x+e$ gives the second inequality.

For any $y \in \mathbb{Z}$, since $\mathcal{C}_{(y-2,y+2)} \subset \mathcal{C}$ is connected, by iterating (3.1), we deduce that

$$\lambda^{\mathrm{diam}(\mathcal{C}_{(y-2,y+2)})} u(z) \leq u(x) \leq \lambda^{-\mathrm{diam}(\mathcal{C}_{(y-2,y+2)})} u(z), \quad z \in \mathcal{C}_{(y-2,y+2)}.$$

We get the result by taking $C = \lambda^{\mathrm{diam}(\mathcal{C}_{(y-2,y+2)})}$. □

Remark 3.2. In fact, by similar arguments as in Lemma 3.1, we have the following Harnack inequality: for any given connected finite set $\Omega \subset \mathbb{Z}^d$, if $u \geq 0$ on $\bar{\Omega}$ and harmonic in Ω, then

$$Cu(z) \leq u(x) \leq C^{-1} u(z), \quad x, z \in \Omega, \tag{3.2}$$

where C depends only on d, λ, Γ and Ω.

Corollary 3.3. *Let $u_1, u_2 \in S$. For any $y_0 \in \mathbb{Z}$, if there exists a point $x_0 = (x_0', y_0) \in \mathcal{C}$ such that $u_1(x_0) = u_2(x_0)$, then*

$$Cu_2(x) \le u_1(x) \le \frac{1}{C} u_2(x), \quad x \in \mathcal{C}_{\{y_0\}},$$

where $C > 0$ depends only on $d, \lambda, \Gamma, \mathcal{D}$.

Proof. The result follows from Lemma 3.1. □

Remark 3.4. If the condition $u_1(x_0) = u_2(x_0)$ in Corollary 3.3 is replaced by $u_1(x_0) \le (\ge)\, u_2(x_0)$, then it holds that $u_1(x) \le \frac{1}{C} u_2(x)$ $(u_1(x) \ge Cu_2(x))$. In fact, let $v(x) = \frac{u_1(x_0)}{u_2(x_0)} u_2(x)$. Then $v \in S$ and $v(x_0) = u_1(x_0)$. By Corollary 3.3, there exists a positive constant C depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that

$$Cu_1(x) \le v(x) \le \frac{1}{C} u_1(x), \quad x \in \mathcal{C}_{\{y_0\}}.$$

If $u_1(x_0) \le u_2(x_0)$, then the left inequality implies that

$$u_1(x) \le \frac{1}{C} u_2(x), \quad x \in \mathcal{C}_{\{y_0\}}.$$

If $u_1(x_0) \ge u_2(x_0)$, then the right inequality implies that

$$u_1(x) \ge Cu_2(x), \quad x \in \mathcal{C}_{\{y_0\}}.$$

Proposition 3.5. *For the solution set S^+, S^- and $S^\vee$, we have that*:

(i) *for any $u \in S^+$ and $w \in S^\vee$, there exists a constant $a > 0$ such that*

$$au(x) \le w(x), \quad x \in \mathcal{C};$$

(ii) *for any $v \in S^-$ and $w \in S^\vee$, there exists a constant $b > 0$ such that*

$$bv(x) \le w(x), \quad x \in \mathcal{C}.$$

Proof. For any $w \in S^\vee$, by Proposition 2.6, there exists a point $x^* = (x'^*, y^*) \in \mathcal{C}$ such that $\hat{w}(y)$ is strictly decreasing in $(-\infty, y^*]$ and increasing in $[y^*, +\infty)$. Hence w has no upper bound in $\mathcal{C}$.

(i) For any $u \in S^+$, u is bounded in $\mathcal{C}_{\mathbb{N}}$ since $\hat{u}(y)$ is strictly decreasing in $\mathbb{Z}$. By Corollary 3.3, there exists a positive constant C_0 such that $C_0 u(x) \le w(x)$, $x \in \mathcal{C}_{\{0\}}$. Then it follows from Corollary 2.4 that

$$C_0 u(x) \le w(x), \quad x \in \mathcal{C}_{[0,+\infty)}. \tag{3.3}$$

Now we prove that there exists a constant $a > 0$ such that $au(x) \le w(x)$, $x \in \mathcal{C}$. Suppose it is not true, then there exists a sequence $\{x_j = (x_j', y_j)\} \subset \mathcal{C}$ such that

$$\frac{1}{j} u(x_j) > w(x_j). \tag{3.4}$$

We claim that there exists a subsequence of $\{x_j = (x_j', y_j)\}$ (still denote itself) such that $\lim_{j\to\infty} y_j = -\infty$.

In fact, one gets easily that the sequence $\{y_j\}$ is bounded from above in $\mathbb{Z}$. Otherwise, $x_j = (x'_j, y_j) \in \mathcal{C}_{(0,+\infty)}$ as $j \to \infty$. Then it follows from (3.3) and (3.4) that

$$w(x_j) \geq C_0 u(x_j) > C_0 j w(x_j),$$

which is a contradiction. If the sequence $\{y_j\}$ is bounded from below in $\mathbb{Z}$, then $|y_j| \leq m_0$ for some $m_0 > 0$. By Corollary 3.3 and Lemma 2.3, there exists a constant $C_{m_0} > 0$ such that

$$C_{m_0} u(x) \leq w(x), \quad x \in \mathcal{C}_{[-m_0,+\infty)}. \tag{3.5}$$

Let j be large enough such that $\frac{1}{j} \leq C_{m_0}$. For $x_j \in \mathcal{C}_{[-m_0,+\infty)}$, by (3.4) and (3.5), we get that

$$w(x_j) < \frac{1}{j} u(x_j) \leq C_{m_0} u(x_j) \leq w(x_j).$$

This is a contradiction. We complete the claim.

Without loss of generality, we assume that the sequence $\{y_j\}$ is strictly decreasing, that is, for $1 \leq j_1 < j_2$, $y_{j_2} < y_{j_1} < 0$. For any $j \geq 1$, by (3.4), there exists a point $x_j \in \mathcal{C}_{\{y_j\}}$ such that $w(x_j) < \frac{1}{j} u(x_j)$. Then it follows from Corollary 3.3 that

$$w(x) - \frac{1}{Cj} u(x) \leq 0, \quad x \in \mathcal{C}_{\{y_j\}}.$$

Then by maximum principle, one has that

$$w(x) - \frac{1}{Cj} u(x) \leq \hat{w}(0), \quad x \in \mathcal{C}_{(y_j,0)}.$$

Since $\lim\limits_{j\to\infty} y_j = -\infty$, we get that

$$\lim_{j\to\infty} [w(x) - \frac{1}{Cj} u(x)] = w(x) \leq \hat{w}(0), \quad x \in \mathcal{C}_{(-\infty,0)}.$$

This yields a contradiction since $w(x)$ has no upper bound in $\mathcal{C}$.

(ii) The proof is similar to that of (i), we omit here. □

Proposition 3.6. *For the solution set S^+ and S^-, we have that*:

(i) *for any $u, v \in S^+$, there exists a constant $c > 0$ such that*

$$u(x) \leq cv(x), \quad x \in \mathcal{C};$$

(ii) *for any $u, v \in S^-$, there exists a constant $d > 0$ such that*

$$u(x) \leq dv(x), \quad x \in \mathcal{C}.$$

Proof. For any $y \in \mathbb{Z}$, by Corollary 3.3, there exists a constant $c_y > 0$ such that

$$u(x) \leq c_y v(x), \quad x \in \mathcal{C}_{\{y\}}. \tag{3.6}$$

(i) Suppose it is not true, then for any $j \geq 1$, there exists a sequence $\{x_j = (x'_j, y_j)\} \subset \mathcal{C}$ such that $u(x_j) > jv(x_j)$. By Corollary 3.3, there exists a constant $C > 0$ such that

$$u(x) \geq Cjv(x), \quad x \in \mathcal{C}_{\{y_j\}}.$$

Let $y = 0$ in (3.6), by Corollary 2.4, we get that

$$u(x) \leq c_0 v(x), \quad x \in \mathcal{C}_{[0,+\infty)}.$$

Since $\lim\limits_{j\to\infty} y_j = +\infty$ for j large enough, we have that $\mathcal{C}_{[y_j,+\infty)} \subset \mathcal{C}_{\mathbb{N}}$ and $Cj > c_0$. Then the above inequalities imply that

$$c_0 v(x) \geq u(x) \geq Cjv(x) > c_0 v(x), \quad x \in \mathcal{C}_{[y_j,+\infty)},$$

which is a contradiction.

(ii) The proof is similar to that of (i), we omit here. □

4. Proof of the main theorems

In this section, we are devoted to prove Theorem 1.1–Theorem 1.3.

Proof of Theorem 1.1. For any $k \in \mathbb{N}^+$, consider the equation

$$\begin{cases} Lu(x) = 0, & x \in \mathcal{C}_{(-k,k)}, \\ u(x) = 0, & x \in \partial_l \mathcal{C}_{(-k,k)} \cup \mathcal{C}_{\{-k\}}, \\ u(x) = C\mathrm{dist}(x', \partial\mathcal{D}), & x \in \mathcal{C}_{\{k\}}. \end{cases}$$

By maximum principle, we get a positive solution $u_k(x)$ in $\mathcal{C}_{(-k,k)}$. Without loss of generality, we normalize u_k such that $u_k(0) = 1$.

Let $l \in \mathbb{N}^+ : k > l$. For $x \in \mathcal{C}_{(-l,\,l)}$, by the local Harnack inequality (3.2), we get that

$$\max_{\mathcal{C}_{(-l,\,l)}} u_k(x) \leq C \min_{\mathcal{C}_{(-l,\,l)}} u_k(x) \leq C,$$

where C depends on $d, \Gamma, \lambda, \mathcal{D}$ and l. By taking the diagonal convergent subsequence, we may assume that $\lim\limits_{k\to+\infty} u_k(x) = u(x)$ locally and uniformly in $\mathcal{D}$. Then it is clear that $u(x) \geq 0, u(0) = 1$ and u satisfies the equation (1.1). By maximum principle, we get that $u > 0$ for all $x \in \mathcal{C}$. □

Proof of Theorem 1.2. The proof consists of two steps.

Step 1. The structure of S^+ and S^-.

For any $u, v \in S^+$, define

$$E = \{p > 0 \mid u(x) \leq pv(x),\ x \in \mathcal{C}\} \quad \text{and} \quad P = \inf E.$$

By Proposition 3.6, one gets that $E \neq \emptyset$. Moreover, by the definition of P, we have that, for any $x \in \mathcal{C}$, $Pv(x) - u(x) \geq 0$. This yields that $P > 0$. Now we claim that

$$Pv(x) - u(x) = 0, \quad x \in \mathcal{C}.$$

In fact, if there exists $x_0 \in \mathcal{C}$ such that $Pv(x_0) - u(x_0) > 0$, then $Pv(x_0) - u(x_0) \in S^+$. By Proposition 3.6, there exists a constant $c > 0$ such that

$v(x_0) \le c(Pv(x_0) - u(x_0))$, i.e., $u(x_0) \le (P - \frac{1}{c})v(x_0)$. Then $P - \frac{1}{c} \in E$, which contradicts the definition of P. Hence,

$$u(x) = Pv(x), \quad x \in \mathcal{C}.$$

Therefore, we get the structure of S^+: $S^+ = \{pu \mid p > 0,\ u \in S^+\}$.

Similarly, we can get the structure of S^-: $S^- = \{qv \mid q > 0,\ v \in S^-\}$.

Step 2. The structure of S.

By Proposition 2.6, we have that

$$S = S^+ \cup S^- \cup S^\vee,$$

where $S^+ \cap S^- = S^- \cap S^\vee = S^+ \cap S^\vee = \emptyset$.

For any $u \in S$, if $u \in S^+$, or $u \in S^-$, the result follows from Step 1. Now we assume that $u \in S^\vee$. Define

$$F = \{l > 0 \mid lv \le u,\, v \in S^+\} \quad \text{and} \quad L = \sup F.$$

By Proposition 3.5, one sees that $F \neq \emptyset$, and hence $L > 0$. Clearly, for any $x \in \mathcal{C}$, $u - Lv \ge 0$, which means that $L < +\infty$. Now we claim that

$$u - Lv > 0, \quad x \in \mathcal{C}. \tag{4.1}$$

In fact, if there exists $x_0 \in \mathcal{C}$ such that $u(x_0) - Lv(x_0)$=0, then by Step 1, $u(x_0) = Lv(x_0) \in S^+$, which contradicts $u \in S^\vee$.

The claim (4.1) implies that $u - Lv \in S$. Next, we prove that $u - Lv \in S^-$. By contradiction, if $u - Lv \in S^+$, then there exists $q > 0$ such that $u - Lv = qv$, i.e. $u = (L + q)v \in S^+$, this contradicts $u \in S^\vee$; If $u - Lv \in S^\vee$, then there exists $l > 0$ such that $lv \le (u - Lv)$, i.e., $(L + l)v \le u$. Hence $L + l \in F$, which contradicts the definition of L. Therefore, we get that

$$u - Lv \in S^-.$$

By Step 1, there exist $Q > 0$ and $w \in S^-$ such that $u - Lv = Qw$, i.e., $u = Lv + Qw$, where $v \in S^+, w \in S^-$. Then we get the solution set

$$S = S^+ + S^- = \{pu + qv \mid p, q \ge 0,\ p + q > 0\}. \qquad \square$$

Proof of Theorem 1.3. First, we claim that for any $u \in S$, there exists a constant $\xi > 0$ depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that

$$\hat{u}(y-1) \le (1+\xi)\hat{u}(y), \quad y \in \mathbb{Z}, \tag{4.2}$$

and

$$\hat{u}(y+1) \le (1+\xi)\hat{u}(y), \quad y \in \mathbb{Z}. \tag{4.3}$$

In fact, by Lemma 3.1, there exists a constant $C > 0$ depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that, for any $y \in \mathbb{Z}$,

$$u(x) \le Cu(0', y) \le C\hat{u}(y), \quad x \in \mathcal{C}_{(y-2,y+2)}.$$

Hence, we get that

$$\hat{u}(y-1) \le (1+\xi)\hat{u}(y), \quad \hat{u}(y+1) \le (1+\xi)\hat{u}(y), \quad y \in \mathbb{Z}.$$

Step 1. The asymptotic behavior of positive solutions in S^+ and S^-.

For any $u \in S^+$, $\hat{u}(y)$ is strictly decreasing in $\mathbb{Z}$. We prove that there exists a constant $\zeta > 0$ depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that

$$(1+\zeta)\hat{u}(y+1) \le \hat{u}(y), \quad y \in \mathbb{Z}. \tag{4.4}$$

Suppose it is not true, then for any $k \in \mathbb{Z}^+$, there exists $y_k \in \mathbb{Z}$ such that

$$(1+\frac{1}{k})\hat{u}(y_k+1) > \hat{u}(y_k).$$

Apply Lemma 2.2 to u in $\mathcal{C}_{(y_k, y_k+2)}$, we get that

$$\hat{u}(y_k+1) \le (1-\delta)\hat{u}(y_k) < (1-\delta)(1+\frac{1}{k})\hat{u}(y_k+1).$$

Let k large enough such that $(1-\delta)(1+\frac{1}{k}) < 1$, then

$$\hat{u}(y_k+1) \le (1-\delta)\hat{u}(y_k) < \hat{u}(y_k+1),$$

which is a contradiction. Hence (4.4) holds.

For $y \in (-\infty, 0)$, by (4.2), one has that

$$\hat{u}(y) \le (1+\xi)^{-y}\hat{u}(0) = e^{-\beta_1 y}\hat{u}(0),$$

and by (4.4),

$$\hat{u}(y) \ge (1+\zeta)^{-y}\hat{u}(0) = e^{-\alpha_1 y}\hat{u}(0),$$

where $\beta_1 = \ln(1+\xi) > 0$ and $\alpha_1 = \ln(1+\zeta) > 0$. Hence

$$-\alpha_1 y \le \ln\frac{\hat{u}(y)}{\hat{u}(0)} \le -\beta_1 y, \quad y \in (-\infty, 0). \tag{4.5}$$

For $y \in (0, +\infty)$, by (4.2), one has that

$$\hat{u}(y) \ge (1+\xi)^{-y}\hat{u}(0) = e^{-\beta_1 y}\hat{u}(0),$$

and by (4.4),

$$\hat{u}(y) \le (1+\zeta)^{-y}\hat{u}(0) = e^{-\alpha_1 y}\hat{u}(0).$$

Consequently,

$$-\beta_1 y \le \ln\frac{\hat{u}(y)}{\hat{u}(0)} \le -\alpha_1 y, \quad y \in (0, +\infty). \tag{4.6}$$

Hence, by (4.5) and (4.6), we have that

$$-cy \le \ln\frac{\hat{u}(y)}{\hat{u}(0)} \le -Cy, \quad y \in \mathbb{Z},$$

where c, C are positive constants depending only on $d, \lambda, \Gamma, \mathcal{D}$.

For any $v \in S^-$, since $\hat{v}(y)$ is strictly increasing in $\mathbb{Z}$. By similar arguments as above, we can get that

$$cy \le \ln\frac{\hat{v}(y)}{\hat{v}(0)} \le Cy, \quad y \in \mathbb{Z},$$

where c, C are positive constants depending only on $d, \lambda, \Gamma, \mathcal{D}$.

Step 2. The asymptotic behavior of positive solutions in $S^\vee$.

For any $w \in S^\vee$, since $\hat{w}(y)$ is strictly decreasing in $(-\infty, y^*]$ and increasing in $[y^*, +\infty)$, similar to the proof of (4.4), there exist constants $\eta_1, \eta_2 > 0$ depending only on $d, \lambda, \Gamma, \mathcal{D}$ such that

$$(1+\eta_1)\hat{w}(y^*+y+1) \le \hat{w}(y^*+y), \quad y \in (-\infty, y^*), \tag{4.7}$$

and

$$(1+\eta_2)\hat{w}(y^*+y) \le \hat{w}(y^*+y+1), \quad y \in (y^*, +\infty). \tag{4.8}$$

For $y \in (-\infty, y^*)$, by (4.2),

$$\hat{w}(y) \le (1+\xi)^{y^*-y}\hat{w}(y^*) = e^{\beta_1(y^*-y)}\hat{w}(y^*),$$

and by (4.7),

$$\hat{w}(y) \ge (1+\eta_1)^{y^*-y}\hat{w}(y^*) = e^{\gamma_1(y^*-y)}\hat{w}(y^*),$$

where $\gamma_1 = \ln(1+\eta_1) > 0$. Hence

$$\gamma_1(y^*-y) \le \ln\frac{\hat{w}(y)}{\hat{w}(y^*)} \le \beta_1(y^*-y), \quad y \in (-\infty, y^*). \tag{4.9}$$

For $y \in (y^*, +\infty)$, by (4.3),

$$\hat{w}(y) \le (1+\xi)^{y-y^*}\hat{w}(y^*) = e^{\beta_1(y-y^*)}\hat{w}(y^*),$$

and by (4.8),

$$\hat{w}(y) \ge (1+\eta_2)^{y-y^*}\hat{w}(y^*) = e^{\gamma_2(y-y^*)}\hat{w}(y^*),$$

where $\gamma_2 = \ln(1+\eta_2) > 0$. As a consequence,

$$\gamma_2(y-y^*) \le \ln\frac{\hat{w}(y)}{\hat{w}(y^*)} \le \beta_1(y-y^*), \quad y \in (y^*, +\infty). \tag{4.10}$$

Hence by (4.9) and (4.10), we get the result

$$c|y-y^*| \le \ln\frac{\hat{w}(y)}{\hat{w}(y^*)} \le C|y-y^*|, \quad y \in \mathbb{Z},$$

where c, C are positive constants depending only on $d, \lambda, \Gamma, \mathcal{D}$. □

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References

[1] J. Bao, L. Wang, and C. Zhou, *Positive solutions to elliptic equations in unbounded cylinder*, Discrete Contin. Dyn. Syst. Ser. B **21** (2016), no. 5, 1389–1400. https://doi.org/10.3934/dcdsb.2016001

[2] M. T. Barlow, *Random Walks and Heat Kernels on graphs*, London Mathematical Society Lecture Note Series, 438, Cambridge Univ. Press, Cambridge, 2017. https://doi.org/10.1017/9781107415690

[3] M. Benedicks, *Positive harmonic functions vanishing on the boundary of certain domains in* $\mathbb{R}^n$, Ark. Mat. **18** (1980), no. 1, 53–72. https://doi.org/10.1007/BF02384681

[4] A. Bouaziz, S. Mustapha, and M. Sifi, *Discrete harmonic functions on an orthant in* $\mathbb{Z}^d$, Electron. Commun. Probab. **20** (2015), no. 52, 13 pp. https://doi.org/10.1214/ecp.v20-4249

[5] D. Denisov and V. I. Wachtel, *Random walks in cones*, Ann. Probab. **43** (2015), no. 3, 992–1044. https://doi.org/10.1214/13-AOP867

[6] J. Doob, *Discrete potential theory and boundaries*, J. Math. Mech. **8** (1959), 433–458.

[7] E. Dynkin, *The boundary theory of Markov processes* (*discrete case*), Uspehi Mat. Nauk **24** (1969), no. 2, 3–42.

[8] G. Fayolle, R. Iasnogorodski, and V. Malyshev, *Random walks in the quarter-plane*, Applications of Mathematics (New York), 40, Springer, Berlin, 1999. https://doi.org/10.1007/978-3-642-60001-2

[9] S. J. Gardiner, *The Martin boundary of NTA strips*, Bull. London Math. Soc. **22** (1990), no. 2, 163–166. https://doi.org/10.1112/blms/22.2.163

[10] M. Ghergu and J. Pres, *Positive harmonic functions that vanish on a subset of a cylindrical surface*, Potential Anal. **31** (2009), no. 2, 147–181. https://doi.org/10.1007/s11118-009-9129-5

[11] B. Hua and J. Jost, L^q *harmonic functions on graphs*, Israel J. Math. **202** (2014), no. 1, 475–490. https://doi.org/10.1007/s11856-014-1089-9

[12] R. A. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **147** (1970), 507–527. https://doi.org/10.2307/1995208

[13] H. J. Kuo and N. S. Trudinger, *Positive difference operators on general meshes*, Duke Math. J. **83** (1996), no. 2, 415–433. https://doi.org/10.1215/S0012-7094-96-08314-3

[14] G. F. Lawler, *Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments*, Proc. London Math. Soc. (3) **63** (1991), no. 3, 552–568. https://doi.org/10.1112/plms/s3-63.3.552

[15] G. F. Lawler, *Random walk and the heat equation*, Student Mathematical Library, 55, Amer. Math. Soc., Providence, RI, 2010. https://doi.org/10.1090/stml/055

[16] Y. Lin and H. Song, *Harnack and mean value inequalities on graphs*, Acta Math. Sci. Ser. B (Engl. Ed.) **38** (2018), no. 6, 1751–1758. https://doi.org/10.1016/S0252-9602(18)30843-9

[17] Y. Lin and L. Xi, *Lipschitz property of harmonic function on graphs*, J. Math. Anal. Appl. **366** (2010), no. 2, 673–678. https://doi.org/10.1016/j.jmaa.2009.12.037

[18] R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. **49** (1941), 137–172. https://doi.org/10.2307/1990054

[19] S. Mustapha, *Gaussian estimates for spatially inhomogeneous random walks on* $\mathbb{Z}^d$, Ann. Probab. **34** (2006), no. 1, 264–283. https://doi.org/10.1214/009117905000000440

[20] S. Mustapha, *Gambler's ruin estimates for random walks with symmetric spatially inhomogeneous increments*, Bernoulli **13** (2007), no. 1, 131–147. https://doi.org/10.3150/07-BEJ5135

[21] S. Mustapha and M. Sifi, *Discrete harmonic functions in Lipschitz domains*, Electron. Commun. Probab. **24** (2019), Paper No. 58, 15 pp. https://doi.org/10.1214/19-ecp259

[22] J. Pres, *Positive harmonic functions on comb-like domains*, Ann. Acad. Sci. Fenn. Math. **36** (2011), no. 2, 577–591. https://doi.org/10.5186/aasfm.2011.3630

[23] M. Rigoli, M. Salvatori, and M. Vignati, *Subharmonic functions on graphs*, Israel J. Math. **99** (1997), 1–27. https://doi.org/10.1007/BF02760674

[24] L. Wang, *The exponential property of solutions bounded from below to degenerate equations in unbounded domains*, Acta Math. Sci. Ser. B (Engl. Ed.) **42** (2022), no. 1, 323–348. https://doi.org/10.1007/s10473-022-0118-8

[25] L. Wang, L. Wang, and C. Zhou, *The exponential growth and decay properties for solutions to elliptic equations in unbounded cylinders*, J. Korean Math. Soc. **57** (2020), no. 6, 1573–1590. https://doi.org/10.4134/JKMS.j190836

[26] L. Wang, L. Wang, and C. Zhou, *Classification of positive solutions for fully nonlinear elliptic equations in unbounded cylinders*, Commun. Pure Appl. Anal. **20** (2021), no. 3, 1241–1261. https://doi.org/10.3934/cpaa.2021019

[27] L. Wang, L. Wang, and C. Zhou, *The dimensional estimates of exponential growth solutions to uniformly elliptic equations of non-divergence form*, Discrete Contin. Dyn. Syst. **42** (2022), no. 11, 5223–5238. https://doi.org/10.3934/dcds.2022092

[28] L. Wang, L. Wang, C. Zhou, and Z. Li, *The behavior and classification of solutions bounded from below to degenerate elliptic equations in unbounded cylinders*, J. Math. Anal. Appl. **516** (2022), no. 2, Paper No. 126560, 31 pp. https://doi.org/10.1016/j.jmaa.2022.126560

[29] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, 138, Cambridge Univ. Press, Cambridge, 2000. https://doi.org/10.1017/CBO9780511470967

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