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ON $\chi \otimes \eta$-STRONG CONNES AMENABILITY OF CERTAIN DUAL BANACH ALGEBRAS

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Abstract. In this paper, the notions of strong Connes amenability for certain products of Banach algebras and module extension of dual Banach algebras is investigated. We characterize $\chi \otimes \eta$-strong Connes amenability of projective tensor product $\mathbb{K}\widehat{\otimes}\mathbb{H}$ via $\chi \otimes \eta$-σwc virtual diagonals, where $\chi \in \mathbb{K}_*$ and $\eta \in \mathbb{H}_*$ are linear functionals on dual Banach algebras $\mathbb{K}$ and $\mathbb{H}$, respectively. Also, we present some conditions for the existence of (χ, θ)-σwc virtual diagonals in the θ-Lau product of $\mathbb{K} \times_\theta \mathbb{H}$. Finally, we characterize the notion of $(\chi, 0)$-strong Connes amenability for module extension of dual Banach algebras $\mathbb{K} \oplus \mathbb{X}$, where $\mathbb{X}$ is a normal Banach $\mathbb{K}$-bimodule.

1. Introduction

Let $\mathbb{K}$ be a Banach algebra, and let $\mathbb{E}$ be a Banach $\mathbb{K}$-bimodule. A bounded linear map $\mathcal{D} : \mathbb{K} \to \mathbb{E}$ is called a derivation if $\mathcal{D}(\mathbf{k}\mathbf{k}') = \mathcal{D}(\mathbf{k}).\mathbf{k}' + \mathbf{k}.\mathcal{D}(\mathbf{k}')$ for all $\mathbf{k}, \mathbf{k}' \in \mathbb{K}$. Take $\mathbf{x} \in \mathbb{E}$, and set

$$ad_{\mathbf{x}} : \mathbb{K} \to \mathbb{E}; \qquad ad_{\mathbf{x}}(\mathbf{k}) = \mathbf{x}.\mathbf{k} - \mathbf{k}.\mathbf{x} \ \ (\mathbf{k} \in \mathbb{K}).$$

In this case, $ad_{\mathbf{x}}$ is called inner derivation. A Banach $\mathbb{K}$-bimodule $\mathbb{E}$ is called dual if there exists a closed submodule $\mathbb{E}_* \subseteq \mathbb{E}^*$ such that $\mathbb{E} = (\mathbb{E}_*)^*$, where $\mathbb{E}_*$ is predual of $\mathbb{E}$. The Banach algebra $\mathbb{K}$ is called dual if it is dual as a Banach $\mathbb{K}$-bimodule. One can see that a Banach algebra which is also a dual space is a dual Banach algebra if the multiplication maps are separately w^*-continuous and vice versa [15]. For instance, $\mathbb{K}^{**}$ is dual if $\mathbb{K}$ is an Arens regular Banach algebra. In this paper, we write $\mathbb{K} = (\mathbb{K}_*)^*$ if $\mathbb{K}$ is a dual Banach algebra, where $\mathbb{K}_*$ is predual of $\mathbb{K}$. The

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suitable concept of amenability for dual Banach algebras is Connes amenability. This notion under different name, for the first time has been introduced by Johnson et al. for von Neumann algebras [8]. Also, Connes amenability for the larger class of dual Banach algebras, was introduced and later extended by Runde, see [15].

A dual Banach $\mathbb{K}$-bimodule $\mathbb{E}$ is called normal, if the module maps

$$\mathbb{K} \to \mathbb{E}, \qquad \mathbf{k} \mapsto \begin{cases} \mathbf{x}.\mathbf{k}, \\ \mathbf{k}.\mathbf{x} \end{cases} \tag{1.1}$$

are w^*-w continuous for every $\mathbf{k} \in \mathbb{K}$ and $\mathbf{x} \in \mathbb{E}$. Indeed, a dual Banach algebra $\mathbb{K}$ is Connes amenable if every w^*-continuous derivation from $\mathbb{K}$ onto a normal, dual Banach $\mathbb{K}$-bimodule $\mathbb{E}$ is inner, see [15, Definition 1.8]. Let $\mathbb{K}$ be a dual Banach algebra, and let $\mathbb{E}$ be a Banach $\mathbb{K}$-bimodule, then $\sigma wc(\mathbb{E})$ denotes the collection of all elements $\mathbf{x} \in \mathbb{E}$ that the module maps in (1.1), are w^*-w continuous. In [16, Proposition 4.4], Runde showed that every $\mathbb{K}$-bimodule $\mathbb{E}$ is equals to $\sigma wc(\mathbb{E})$ if and only if $\mathbb{E}^*$ is a normal dual Banach $\mathbb{K}$-bimodule. The concept of Connes amenability of a dual Banach algebra $\mathbb{K}$ has been characterized through the existence of a σwc-virtual diagonal for $\mathbb{K}$ [16, Theorem 4.8]. Connes amenability of both the l^1-Munn algebras and certain product of Banach algebras were investigated by Ghaffari, Javadi and Tamimi in [6, 7]. Also, for a linear functional χ on an arbitrary Banach algebra, χ-Connes amenability that appear to be normal type of Connes amenability for dual Banach algebras, was investigated by Ghaffari and Javadi in [4]. Ideal Connes amenability of certain dual Banach algebras and Lau product of Banach algebras were studied by Minapoor et al. in [12]. In [5], χ-Connes module amenability of dual Banach algebras studied by Ghaffari, Javadi and Tamimi.

Strong amenability have studied recently in some of papers [4, 14]. Much later in [8] Johnson, Kadison and Ringrose have defined strong amenable unital C^*-algebra. Also, it is shown that the strong amenability implies symmetric amenability [9].

In [14, Definition 4.6], Runde introduced another version of Connes amenability, called strong Connes amenability and he developed two notions of amenability-Connes amenability and strong Connes amenability on dual Banach algebras. Also, Runde characterized strong Connes amenability through the virtual w^*-diagonals. The χ-strong Connes amenability for dual Banach algebras was defined by Ghaffari and Javadi in [4]. They showed that for mentioned algebras, the existence of χ-normal virtual diagonals and χ-strong Connes amenability are equivalent [4, Theorem 2.7].

The notion of weak amenability of module extension Banach algebra $\mathbb{K}\oplus\mathbb{X}$, where $\mathbb{K}$ is a dual Banach algebra and $\mathbb{X}$ is a normal Banach $\mathbb{K}$-bimodule was investigated by Zhang, see [17, Proposition 5.1].

Let $\mathbb{K}$ be a dual Banach algebra. Throughout this paper, the collection of all non-zero multiplicative linear functionals on $\mathbb{K}$ that are w^*-continuous is denoted by $\Delta_{w^*}(\mathbb{K})$.

In this paper, we are going to investigate the new version of strong Connes amenability for certain Banach algebras, such as $\mathbb{K}\widehat{\otimes}\mathbb{H}$, $\mathbb{K}\times_\theta \mathbb{H}$ and $\mathbb{K}\oplus\mathbb{X}$.

2. strong Connes Amenability of Projective Tensor Product and θ-Lau Product of Banach Algebras

This section consists of two subsections. We investigate the notion of $\chi\otimes\eta$-strong Connes amenability of projective tensor product of dual Banach algebras and the notion of (χ, η)-strong Connes amenability of θ-Lau product of Banach algebras. For this purpose first, we remark some standard notations and definitions that we shall need in the sequel.

Definition 2.1. Let $\mathbb{K}$ be a dual Banach algebra, and let $\chi \in \Delta_{\omega^*}(\mathbb{K}) \cap \mathbb{K}_*$. We say that $\mathbb{K}$ is *χ-Connes amenable* if there exists a bounded linear functional μ on $\sigma wc(\mathbb{K})^*$ such that $\mu(\chi) = 1$ and $\mu(\mathbf{f}.\mathbf{k}) = \chi(\mathbf{k})\mu(\mathbf{f})$ for all $\mathbf{k} \in \mathbb{K}$ and $\mathbf{f} \in \sigma wc(\mathbb{K})^*$.

We apply the Banach $\mathbb{K}$-bimodule $\mathbb{E}$, whose left action is of the form $\mathbf{k}.\mathbf{x} = \chi(\mathbf{k})\mathbf{x}$, where $\chi \in \Delta_{\omega^*}(\mathbb{K}) \cap \mathbb{K}_*$, $\mathbf{k} \in \mathbb{K}$ and $\mathbf{x} \in \mathbb{E}$. For the sake of brevity, such $\mathbb{E}$ will often be named a Banach χ-bimodule.

Let $\mathbb{K}$ be a dual Banach algebra, and let $\mathbb{E}$ be a Banach χ-bimodule. Then an element $\mathbf{y} \in \mathbb{E}^*$ is called a w^*-element if the maps $\mathbf{k} \to \mathbf{k}.\mathbf{y}$ and $\mathbf{k} \to \mathbf{y}.\mathbf{k}$ are w^*-continuous.

Definition 2.2. A dual Banach algebra with identity $\mathbb{K}$ is called *strong Connes amenable* if, for every unital Banach $\mathbb{K}$-bimodule $\mathbb{E}$, every w^*-continuous derivation $\mathcal{D} : \mathbb{K} \to \mathbb{E}^*$, whose range consists of w^*-elements is inner [14, Definition 4.6].

Definition 2.3. Let $\mathbb{K}$ be a dual Banach algebra, and let $\mathbb{E}^*$ be an arbitrary Banach χ-bimodule, where $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$. Then $\mathbb{K}$ is called *χ-strong Connes amenable* if every w^*-continuous derivation $\mathcal{D} : \mathbb{K} \to \mathbb{E}^*$, whose range $\mathcal{D}(\mathbb{K})$ consists of w^*-elements, is inner [4, Definition 2.6].

Example 2.4. Let $\mathbb{F}$ be a reflexive Banach space, and let $\mathcal{L}(\mathbb{F}) := \mathcal{L}(\mathbb{F}, \mathbb{F})$ be the set of all bounded linear operators from $\mathbb{F}$ to itself and $\chi \in \Delta_{w^*}(\mathbb{F}) \cap \mathbb{F}_*$. The only Banach spaces $\mathbb{F}$ for which, $\mathcal{L}(\mathbb{F})$ is amenable and therefore χ-strong Connes amenable, are the finite dimensional ones.

Example 2.5. Let $\mathbb{G}$ be a compact group. Then $M(\mathbb{G})$, measure algebra of $\mathbb{G}$, is id-strong Connes amenable (for more details see the proof of [14, Proposition 5.2]).

Example 2.6. Every Connes amenable von Neumann algebra is id-strong Connes amenable, see [14, Theorem 4.7] and [2].

Remark 2.7. Note that if S is a weakly cancellative semigroup, then semigroup algebra $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$. We denote by $\widehat{S}$ the set of all nonzero bounded continuous characters of S. More precisely $\widehat{S} = \{\chi \in c_b(S); \chi \neq 0, \chi(xy) = \chi(x)\chi(y)\}$ for all $x, y \in S$. For $\chi \in \widehat{S}$ and $\mu \in l^1(S)$, define $\widehat{\chi}(\mu) = \int \overline{\chi(x)} d\mu$. Then semigroup S is said to be χ-amenable if there exists a bounded linear functional ρ on $l^1(S)^*$ satisfying $\rho(\widehat{\chi}) = 1$ and $\rho(\delta_x f) = \widehat{\chi}(\delta_x)\rho(f)$ for all $x \in S$ and $f \in l^1(S)^*$, see [3].

Example 2.8. Let S be a weakly cancellative semigroup that is χ-amenable, where $\chi \in \widehat{S}$. By Remark 2.7 and [3], $l^1(S)$ is χ-amenable and so, it follows that $l^1(S)$ is $\widehat{\chi}$-strong Connes amenable.

The next definition is analogous to [11, Definition 3.1].

Definition 2.9. Suppose that $\mathbb{K}$ is a dual Banach algebra, and $\chi \in \Delta_{\omega^*}(\mathbb{K}) \cap \mathbb{K}_*$. An element $\Gamma \in \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ is called *χ-σwc virtual diagonal* for $\mathbb{K}$ if $\mathbf{k}.\Gamma = \chi(\mathbf{k}).\Gamma$ for all $\mathbf{k} \in \mathbb{K}$ and $\langle \chi \otimes \chi, \Gamma \rangle = 1$, where $\chi \otimes \chi(k_1 \otimes k_2) = \chi(k_1)\chi(k_2)$ for every $k_1, k_2 \in \mathbb{K}$.

Remark 2.10. By [11, Theorem 3.2], it is clear to see that χ-Connes amenability of dual Banach algebra $\mathbb{K}$ is equivalent to the existence of the χ-σwc virtual diagonal for $\mathbb{K}$. In [14, Theorem 4.7], Runde showed that a dual Banach algebra $\mathbb{K}$, has a w^*-virtual diagonal if $\mathbb{K}$ is strong Connes amenable and vice versa.

With above-arrangement, we characterize χ-strong Connes amenability of dual Banach algebras via the existence of χ-σwc virtual diagonal. First, we give a technical lemma to reach our purpose.

Lemma 2.11. *Let $\mathbb{K}$ be a dual Banach algebra, and let $\chi \in \Delta_{\omega^*}(\mathbb{K}) \cap \mathbb{K}_*$. If $\mathbb{K}$ has a χ-σwc virtual diagonal, then $\mathbb{K}$ is χ-strong Connes amenable and vice versa.*

Proof. Let $\mathbb{K}$ has a χ-σwc virtual diagonal. Let $\mathbb{K}\widehat{\otimes}\mathbb{K}$ be a Banach $\mathbb{K}$-bimodule with the following module actions

$$(\mathbf{k}_1 \otimes \mathbf{k}_2).\mathbf{k}_3 = \mathbf{k}_1 \otimes \mathbf{k}_2\mathbf{k}_3, \qquad \mathbf{k}_3.(\mathbf{k}_1 \otimes \mathbf{k}_2) = \chi(\mathbf{k}_3)\mathbf{k}_1 \otimes \mathbf{k}_2 \tag{2.1}$$

for every $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \in \mathbb{K}$. Then $\sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ is a Banach χ-bimodule. By hypothesis, choose an element $\Gamma_0 \in \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ such that $\langle \chi \otimes \chi, \Gamma_0 \rangle = 1$. Define $\mathcal{D} : \mathbb{K} \to \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ by $\mathcal{D}(\mathbf{k}) = \Gamma_0.\mathbf{k} - \chi(\mathbf{k})\Gamma_0$. This will allow us to show that $\mathcal{D}(\mathbb{K})$ is a subset of ω^*-closed submodule $\ker(\chi \otimes \chi)$. By assumption, we find $\Gamma_1 \in \ker(\chi \otimes \chi)$ such that

$$\Gamma_0.\mathbf{k} - \chi(\mathbf{k})\Gamma_0 = \Gamma_1.\mathbf{k} - \chi(\mathbf{k})\Gamma_1. \tag{2.2}$$

We claim that $\Gamma := \Gamma_0 - \Gamma_1$ is a χ-σwc virtual diagonal for $\mathbb{K}$. This follows that $\mathbb{K}$ is χ-strong Connes amenable.

The converse follows directly from this real that if consider $\mathbb{K}\widehat{\otimes}\mathbb{K}$, which the module actions are specified by the formulae (2.1) in above argument, then it is clear that $\sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ is a Banach χ-bimodule. Now, we can choose $\mathbf{v} \in \mathbb{K}$ such that $\chi(\mathbf{v}) = 1$. We consider the following inner derivation

$$\mathcal{D} : \mathbb{K} \to \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*, \qquad \mathbf{k} \mapsto (\mathbf{v} \otimes \mathbf{v})\mathbf{k} - \chi(\mathbf{k})(\mathbf{v} \otimes \mathbf{v}).$$

One can see that $\mathcal{D}$ is ω^*-continuous, and attains its values in the kernel of $\chi \otimes \chi$. Therefore, there exists $\Gamma' \in \ker(\chi \otimes \chi) \subseteq (\mathbb{K}\widehat{\otimes}\mathbb{K})^*$ such that $\mathcal{D}(\mathbf{k}) = \Gamma'.\mathbf{k} - \mathbf{k}.\Gamma'$ for all $\mathbf{k} \in \mathbb{K}$. Put $\Gamma := \mathbf{v} \otimes \mathbf{v} - \Gamma'$. We show that Γ has the desired properties of mentioned diagonal for $\mathbb{K}$. But we obtain

$$\mathbf{k}.\Gamma = \mathbf{k}.(\mathbf{v} \otimes \mathbf{v}) - \mathbf{k}.\Gamma' = \chi(\mathbf{k})\mathbf{v} \otimes \mathbf{v} - \chi(\mathbf{k})\Gamma' = \chi(\mathbf{k}).\Gamma.$$

and

$$\langle \chi \otimes \chi, \Gamma \rangle = \langle \chi \otimes \chi, \mathbf{v} \otimes \mathbf{v} - \Gamma' \rangle = \langle \chi \otimes \chi, \mathbf{v} \otimes \mathbf{v} \rangle - \langle \chi \otimes \chi, \Gamma' \rangle = 1.$$

Therefore, Γ is a χ-σwc virtual diagonal and the proof is complete. □

Example 2.12. Set $\mathbb{K}=\begin{pmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$, that $\mathbb{C}$ denotes the set of all complex numbers. Consider the usual matrix multiplication and l^1-norm, $\mathbb{K}$ is a dual Banach algebra. We define

$$\chi : \mathbb{K} \to \mathbb{C}; \qquad \chi\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix} = t,$$

for all $z, t \in \mathbb{C}$. Let $\alpha \in \mathbb{C}$. We have

$$\chi\Big[\alpha.\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}\Big] = \chi\begin{pmatrix} 0 & 0 \\ \alpha z & \alpha t \end{pmatrix} = \alpha t = \alpha.\chi\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}.$$

Thus χ is linear. We claim that χ is w^*-continuous. For this purpose, suppose that $A = \begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}$, $A_n = \begin{pmatrix} 0 & 0 \\ z_n & t_n \end{pmatrix} \in \mathbb{K}$ and $A_n \to A$, in w^*-topology$(n \in \mathbb{N})$. Since range of χ has finite dimensional and we known that in such spaces all topologies are coincide. It is clear that $t_n \xrightarrow{w^*} t$ and $z_n \xrightarrow{w^*} z$. Therefore χ is w^*-continuous. We set

$$\Gamma = \begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} \in \mathbb{K}\widehat{\otimes}\mathbb{K}.$$

It is clear that $\Gamma \in \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$. We show that Γ has the properties of χ-σwc virtual diagonal for $\mathbb{K}$. For this, take $A = \begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix} \in \mathbb{K}$. Then by (2.1)

$$A.\Gamma = \begin{pmatrix} 0 & 0 \\ t & -it \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} = \chi\Big[\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}\Big]\Gamma$$

and

$$\langle \chi \otimes \chi, \Gamma \rangle = \chi\begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix}\chi\begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} = -i.i = 1.$$

This shows that $\Gamma \in \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ is a χ-σwc virtual diagonal for Banach algebra $\mathbb{K}$. Now, suppose that w^*-continuous derivation $\mathcal{D} : \mathbb{K} \to \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ is given by $A \mapsto \Gamma.A - \chi(A)\Gamma$. Since $\sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ is Banach χ-bimodule. It is sufficient to show that $\mathcal{D}$ is inner. Therefore, $\mathbb{K}$ is χ-strong Connes amenable.

Example 2.13. Suppose that $\mathbb{G}$ is a locally compact group, $C_0(\mathbb{G})$ denotes the set of all continuous functions on $\mathbb{G}$ that vanishing at infinity and $\chi \in \Delta_{\omega^*}(M(\mathbb{G})) \cap C_0(\mathbb{G})$. Define $\chi : M(\mathbb{G}) \to \mathbb{C}$ by $\chi(\nu) = \int \chi d\nu$. Then one can see that $M(\mathbb{G}) = (C_0(\mathbb{G}))^*$ is a χ-strong Connes amenable dual Banach algebra. Using Lemma 2.11, $M(\mathbb{G})$ has a χ-σwc virtual diagonal.

2.1. $\chi \otimes \eta$-strong Connes amenability of projective tensor product of dual Banach algebras

In this subsection, we characterize the $\chi \otimes \eta$-strong Connes amenability of projective tensor product of dual Banach algebras through the existence of χ-σwc virtual diagonals and η-σwc virtual diagonals. Also, the $\chi \otimes \eta$-normal virtual diagonals for projective tensor product of dual Banach algebras is defined.

In the next theorem, we modify the argument of Lemma 2.11 to complete the proof.

Theorem 2.14. *Let $\mathbb{K}$ and $\mathbb{H}$ be unital dual Banach algebras. Let $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\eta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. If $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi\otimes\eta$-strong Connes amenable dual Banach algebra, then $\mathbb{K}$ has a χ-σwc virtual diagonal and $\mathbb{H}$ has a η-σwc virtual diagonal and vice versa.*

Proof. First, suppose that $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi \otimes \eta$-strong Connes amenable. Consider the Banach $\mathbb{K}$-bimodule $\mathbb{K}\widehat{\otimes}\mathbb{K}$ and the Banach $\mathbb{H}$-bimodule $\mathbb{H}\widehat{\otimes}\mathbb{H}$ with the module actions given by

$$(\mathbf{k}_1 \otimes \mathbf{k}_2).\mathbf{k}_3 = \mathbf{k}_1 \otimes \mathbf{k}_2\mathbf{k}_3, \quad \mathbf{k}_3.(\mathbf{k}_1 \otimes \mathbf{k}_2) = \chi(\mathbf{k}_3)\mathbf{k}_1 \otimes \mathbf{k}_2; \quad (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \in \mathbb{K})$$

and

$$(\mathbf{h}_1 \otimes \mathbf{h}_2).\mathbf{h}_3 = \mathbf{h}_1 \otimes \mathbf{h}_2\mathbf{h}_3, \quad \mathbf{h}_3.(\mathbf{h}_1 \otimes \mathbf{h}_2) = \eta(\mathbf{h}_3)\mathbf{h}_1 \otimes \mathbf{h}_2; \quad (\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in \mathbb{H}).$$

Put $X^* := \sigma wc((\mathbb{K}\widehat{\otimes}\mathbb{K})^*)^*$ and $Y^* := \sigma wc((\mathbb{H}\widehat{\otimes}\mathbb{H})^*)^*$. Then X^* and Y^* are Banach χ-bimodule and Banach η-bimodule, respectively. Choose $M_0 \in X^*$ and $N_0 \in Y^*$ such that $\langle \chi \otimes \chi, M_0 \rangle = 1$ and $\langle \eta \otimes \eta, N_0 \rangle = 1$. We define two derivations as

$$\mathcal{D}_{M_0} : \mathbb{K} \to X^*, \qquad \mathbf{k} \longmapsto M_0.\mathbf{k} - \chi(\mathbf{k})M_0$$

and

$$\mathcal{D}_{N_0} : \mathbb{H} \to Y^*, \qquad \mathbf{h} \longmapsto N_0.\mathbf{h} - \eta(\mathbf{h})N_0.$$

Then we see that the images of the inner derivations namely, $ad_{M_0} : \mathbb{K} \to X^*$ and $ad_{N_0} : \mathbb{H} \to Y^*$ are subsets of w^*-closed submodules $\ker(\chi \otimes \chi)$ and $\ker(\eta \otimes \eta)$, respectively. By hypothesis, there exists $M_1 \in \ker(\chi \otimes \chi)$ and $N_1 \in \ker(\eta \otimes \eta)$ such that

$$ad_{M_0} = ad_{M_1}, \qquad ad_{N_0} = ad_{N_1}. \tag{2.3}$$

Put $M := M_0 - M_1$, and $N := N_0 - N_1$. Then

$$\langle \chi \otimes \chi, M \rangle = \langle \chi \otimes \chi, M_0 - M_1 \rangle = 1 \tag{2.4}$$

and

$$\langle \eta \otimes \eta, N \rangle = \langle \eta \otimes \eta, N_0 - N_1 \rangle = 1. \tag{2.5}$$

Then by (2.3), we have

$$M_0\mathbf{k} - \mathbf{k}M_0 = M_1\mathbf{k} - \mathbf{k}M_1, \qquad N_0\mathbf{h} - \mathbf{h}N_0 = N_1\mathbf{h} - \mathbf{h}N_1. \tag{2.6}$$

for every $\mathbf{k} \in \mathbb{K}$ and $\mathbf{h} \in \mathbb{H}$. So, from (2.6),

$$M.\mathbf{k} = \chi(\mathbf{k})M, \qquad N.\mathbf{h} = \eta(\mathbf{h})N. \tag{2.7}$$

Then from the relations (2.4), (2.5) and (2.7), it follows that M has desirable property for $\mathbb{K}$, and also N is for $\mathbb{H}$.

For the converse, let $\mathbb{K}\widehat{\otimes}\mathbb{H}$ be a dual Banach algebra and define $\mathcal{D}:\mathbb{K}\widehat{\otimes}\mathbb{H}\to\mathbb{E}^*$ such that $\mathbb{E}^*$ be a Banach $\chi\otimes\eta$-bimodule and consists of w^*-elements. By using Lemma 2.11, it is sufficient to show that $\mathcal{D}$ is inner. □

Indeed, in Theorem 2.14, we characterize the $\chi\otimes\eta$-strong Connes amenability of projective tensor product dual Banach algebras through the existence of χ-σwc virtual diagonals and η-σwc virtual diagonals.

Example 2.15. Set $\mathbb{K}=\begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{H}=\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{pmatrix}$. By considering usual matrix multiplication and l^1-norm, one can see that $\mathbb{K}$ and $\mathbb{H}$ are unital dual Banach algebras. Also, $e_{\mathbb{K}}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_{\mathbb{H}}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are units of $\mathbb{K}$ and $\mathbb{H}$, respectively. We define

$$(K\otimes H)(K'\otimes H')=KK'\otimes HH' \qquad (K,K'\in\mathbb{K},H,H'\in\mathbb{H}).$$

Thus, with the defined multiplication, $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is a Banach algebra. It is easy to see that $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is unital dual Banach algebra with unit $e_{\mathbb{K}\widehat{\otimes}\mathbb{H}}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For this reason, consider $\chi:\mathbb{K}\to\mathbb{C}$ by $\chi\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}=z$ and $\eta:\mathbb{H}\to\mathbb{C}$ by $\eta\begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}=w$ for all $z,w\in\mathbb{C}$. By using similar argument in Example 2.12, it is easy to see that χ and η are w^*-continuous bounded linear functionals and so $\chi\otimes\eta$. Obtain

$$\chi\otimes\eta\Big[\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}\Big]=\chi\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\eta\begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}=zw,$$

for every $z,w\in\mathbb{C}$. Hence, $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is unital dual Banach algebra and $\chi\otimes\eta$ is w^*-continuous bounded linear functional. We show that $\Gamma_{\mathbb{K}}=\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix}$ is a χ-σwc virtual diagonal for $\mathbb{K}$. Indeed, for $K=\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}\in\mathbb{K}$ by (2.1), we have

$$K.\Gamma_{\mathbb{K}}=\begin{pmatrix} 5z & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix}=\chi(K).\Gamma_{\mathbb{K}}.$$

Also, $\langle\chi\otimes\chi,\Gamma_{\mathbb{K}}\rangle=5.\frac{1}{5}=1$. By similar argument, $\Gamma_{\mathbb{H}}=\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is a η-σwc virtual diagonal for $\mathbb{H}$. Therefore by Lemma 2.11, $\mathbb{K}$ is χ-strong Connes amenable and $\mathbb{H}$ is η-strong Connes amenable. Also, by putting $(\chi\otimes\eta)\otimes(\chi\otimes\eta):=$

$(\chi \otimes \chi) \otimes (\eta \otimes \eta)$, it is clear that $\Gamma_{\mathbb{K}} \otimes \Gamma_{\mathbb{H}}$ is a $\chi \otimes \eta$-σwc virtual diagonal for $\mathbb{K}\widehat{\otimes}\mathbb{H}$. Thus, $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi \otimes \eta$-strong Connes amenable.

Theorem 2.14, let us to prove the next corollary.

Corollary 2.16. *Let $\mathbb{K}$ and $\mathbb{H}$ be unital dual Banach algebras. Let $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\eta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. Then $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi \otimes \eta$-strong Connes amenable dual Banach algebra if $\mathbb{K}$ is χ-strong Connes amenable, and $\mathbb{H}$ is η-strong Connes amenable and vice versa.*

Proof. Suppose that Banach algebras $\mathbb{K}$ and $\mathbb{H}$ are χ-strong Connes amenable and η-strong Connes amenable, respectively. Using Lemma 2.11, $\mathbb{K}$ and $\mathbb{H}$ have χ-σwc virtual diagonal and η-σwc virtual diagonal, respectively. By Theorem 2.14, $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi \otimes \eta$-strong Connes amenable. The converse is routine. □

Let $\mathbb{K}$ be a dual Banach algebra, and $\Omega_{\mathbb{K}}$ from $\mathbb{K}\widehat{\otimes}\mathbb{K}$ onto $\mathbb{K}$ be a multiplication operator, i.e., $\Omega_{\mathbb{K}}(\mathbf{k} \otimes \mathbf{k}') = \mathbf{k}\mathbf{k}'$, for all $\mathbf{k}, \mathbf{k}' \in \mathbb{K}$. Clearly, $\Omega^*_{\mathbb{K}}(\mathbb{K}_*) \subseteq (\mathbb{K}\widehat{\otimes}\mathbb{K})^*$. We can easily see that $\Omega^*_{\mathbb{K}}(\mathbb{K}^*) \subseteq (\mathbb{K}\widehat{\otimes}\mathbb{K})^*$; therefore, if $\chi \in \Delta_{w^*}(\mathbb{K})$ then $\chi \otimes \chi = \Omega^*_{\mathbb{K}}(\chi) \in (\mathbb{K}\widehat{\otimes}\mathbb{K})^*$, where $\chi \otimes \chi(\mathbf{k}_1 \otimes \mathbf{k}_2) = \chi(\mathbf{k}_1)\chi(\mathbf{k}_2)$ for all $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{K}$.

Suppose that $\mathbb{K}$ is a dual Banach algebra and $\mathcal{L}^2(\mathbb{K}, \mathbb{C}) \cong (\mathbb{K}\widehat{\otimes}\mathbb{K})^*$ is the collection of all bounded linear functionals from $\mathbb{K}\widehat{\otimes}\mathbb{K}$ onto $\mathbb{C}$. Then the collection of separately w^*-continuous elements of $\mathcal{L}^2(\mathbb{K}, \mathbb{C})$ is denoted by $\mathcal{L}^2_{\omega^*}(\mathbb{K}, \mathbb{C})$. Since $\Omega^*_{\mathbb{K}}$ maps $\mathbb{K}_*$ into $\mathcal{L}^2_{w^*}(\mathbb{K}, \mathbb{C})$, it follows that $\Omega^{**}_{\mathbb{K}}$ drops to an $\mathbb{K}$-bimodule homomorphism $\Omega^{**}_{\mathbb{K}} : \mathcal{L}^2_{w^*}(\mathbb{K}, \mathbb{C})^* \longrightarrow \mathbb{K}$, see [4].

We know that in [2], the meaning of Connes amenability for some Banach algebras characterized through the normal virtual diagonals. Now, we characterize the $\chi \otimes \eta$-strong Connes amenability through the χ-normal virtual diagonals and η-normal virtual diagonals. For this purpose, in the sequel we present definition of the mentioned diagonals.

Definition 2.17. Let $\mathbb{K}$ be a dual Banach algebra, and $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$. $\Gamma \in \mathcal{L}^2_{w^*}(\mathbb{K}, \mathbb{C})^*$ is called *χ-normal virtual diagonal* for $\mathbb{K}$ if $\langle \Omega^{**}_{\mathbb{K}}(\Gamma), \chi \rangle$=1 and $\mathbf{k}.\Gamma = \chi(\mathbf{k}).\Gamma$ for every $\mathbf{k} \in \mathbb{K}$.

Example 2.18. Let G be a discrete group, then $M(G) = l^1(G)$ is id-Connes amenable if and only if it has an id-normal virtual diagonal. Also, if G is compact, then $M(G)$ has an id-normal virtual diagonal.

Example 2.19. Let G be a discrete amenable group. Then, we claim that there is an id-normal virtual diagonal for $l^1(G)$. Indeed, if G is amenable then $l^1(G)$ is amenable, so that there is a virtual diagonal $\Gamma \in (l^1(G)\widehat{\otimes}l^1(G))^{**}$ for $l^1(G)$. Let $\rho : (l^1(G)\widehat{\otimes}l^1(G))^{**} \to \mathcal{L}^2_{w^*}(l^1(G), \mathbb{C})^*$, denotes the restriction map. Then $\rho(\Gamma)$ is an id-normal virtual diagonal for $l^1(G)$.

We remind that the collection of all non-zero multiplicative linear functionals on Banach algebra $\mathbb{K}$ is denoted by $\Delta(\mathbb{K})$. In [4, Theorem 2.7], the authors proved that if Banach algebra $\mathbb{K}$ has a χ-normal virtual diagonal, where $\chi \in \Delta(\mathbb{K}) \cap \mathbb{K}_*$ then $\mathbb{K}$ is χ-strong Connes amenable. Is this statement true for $\mathbb{K}\widehat{\otimes}\mathbb{H}$, where $\mathbb{K}$, $\mathbb{H}$ and $\mathbb{K}\widehat{\otimes}\mathbb{H}$ are dual Banach algebras? We answer to this question in the next theorem.

Theorem 2.20. *Let $\mathbb{K}$ and $\mathbb{H}$ be dual Banach algebras with preduals $\mathbb{K}_*$ and $\mathbb{H}_*$, respectively, and let $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\eta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. Then dual Banach algebra $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi \otimes \eta$-strong Connes amenable if $\mathbb{K}$ has a χ-normal virtual diagonal and $\mathbb{H}$ has a η-normal virtual diagonal and vice versa.*

Proof. First, suppose that $\mathbb{K}$ and $\mathbb{H}$ have χ-normal and η-normal, virtual diagonals, respectively. Suppose that $X^* = \mathbb{K}\widehat{\otimes}\mathbb{K}$ and $Y^* = \mathbb{H}\widehat{\otimes}\mathbb{H}$ are Banach $\mathbb{K}$-bimodule and Banach $\mathbb{H}$-bimodule with the module actions are those where in Theorem 2.14. Then by similar argument in the first of Lemma 2.11, X^* and Y^* are Banach χ-bimodule and Banach η-bimodule, respectively. We take $\mathbf{p} \in \mathbb{K}$ and $\mathbf{q} \in \mathbb{H}$ such that $\chi(\mathbf{p}) = 1$ and $\eta(\mathbf{q}) = 1$. Define the following derivations

$$\mathcal{D}_1 : \mathbb{K} \to X^*, \quad \mathcal{D}_1(\mathbf{k}) = (\mathbf{p} \otimes \mathbf{p}).\mathbf{k} - \chi(\mathbf{k})(\mathbf{p} \otimes \mathbf{p}), \qquad \mathbf{k} \in \mathbb{K}$$

$$\mathcal{D}_2 : \mathbb{H} \to Y^*, \quad \mathcal{D}_2(\mathbf{h}) = (\mathbf{q} \otimes \mathbf{q}).\mathbf{h} - \eta(\mathbf{h})(\mathbf{q} \otimes \mathbf{q}), \qquad \mathbf{h} \in \mathbb{H}.$$

One can see that $\mathcal{D}_1$ and $\mathcal{D}_2$ are inner derivations. It is easy to see that $\mathcal{D}_1$ and $\mathcal{D}_2$ are ω^*-continuous and attain thier values in the ω^*-closed submodules $\ker(\chi \otimes \chi)$ and $\ker(\eta \otimes \eta)$, respectively. Hence, there exists $\Gamma_1 \in \ker(\chi \otimes \chi)$ and $\Gamma_2 \in \ker(\eta \otimes \eta)$ such that

$$\mathcal{D}_1(\mathbf{k}) = \mathbf{k}.\Gamma_1 - \Gamma_1.\mathbf{k}, \qquad \mathcal{D}_2(\mathbf{h}) = \mathbf{h}.\Gamma_2 - \Gamma_2.\mathbf{h}, \qquad (\mathbf{k} \in \mathbb{K}, \mathbf{h} \in \mathbb{H}).$$

Put $\Theta_1 = \Gamma_1 - \mathbf{p} \otimes \mathbf{p}$, $\Theta_2 = \Gamma_2 - \mathbf{q} \otimes \mathbf{q}$, $\Omega_{\mathbb{K}} : \mathbb{K}\widehat{\otimes}\mathbb{K} \to \mathbb{K}$ and $\Omega_{\mathbb{H}} : \mathbb{H}\widehat{\otimes}\mathbb{H} \to \mathbb{H}$, thus we obtain

$$\mathbf{k}.\Theta_1 = \chi(\mathbf{k})\Theta_1, \qquad \langle \Omega^{**}_{\mathbb{K}}(\Theta_1), \chi \rangle = 1 \tag{2.8}$$

and

$$\mathbf{h}.\Theta_2 = \eta(\mathbf{h})\Theta_2, \qquad \langle \Omega_{\mathbb{H}}^{**}(\Theta_2), \eta \rangle = 1 \tag{2.9}$$

for $\mathbf{k} \in \mathbb{K}$ and $\mathbf{h} \in \mathbb{H}$. Now, we put $\Theta = \Theta_1 \otimes \Theta_2 := \Gamma_1 \otimes \Gamma_2 - (\mathbf{p} \otimes \mathbf{q}) \otimes (\mathbf{p} \otimes \mathbf{q}) \in \mathcal{L}^2_{w^*}(\mathbb{K}, \mathbb{C})^*$. Then we have

$$\chi \otimes \eta(\mathbf{p} \otimes \mathbf{q}) = \chi(\mathbf{p})\eta(\mathbf{q}) = 1, \qquad \mathbf{p} \otimes \mathbf{q} \in \mathbb{K}\widehat{\otimes}\mathbb{H}.$$

Set $\mathcal{D} := \mathcal{D}_1 \otimes \mathcal{D}_2$. Define $\mathcal{D} : \mathbb{K}\widehat{\otimes}\mathbb{H} \to \big((\mathbb{K}\widehat{\otimes}\mathbb{H})\widehat{\otimes}(\mathbb{K}\widehat{\otimes}\mathbb{H})\big)^*)^*$ by

$$\mathbf{k} \otimes \mathbf{h} \mapsto \big((\mathbf{p} \otimes \mathbf{q}) \otimes (\mathbf{p} \otimes \mathbf{q})\big).\mathbf{k} \otimes \mathbf{h} - \chi \otimes \eta(\mathbf{k} \otimes \mathbf{h})\big((\mathbf{p} \otimes \mathbf{q}) \otimes (\mathbf{p} \otimes \mathbf{q})\big).$$

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are inner w^*-continuous derivations that their ranges, consists of w^*-elements, it is easy to see that $\mathcal{D}$ has such properties. On the other hand, by relations (2.8) and (2.9), we obtain

$$\begin{aligned}
\chi \otimes \eta(\mathbf{k} \otimes \mathbf{h})\Theta &= \chi \otimes \eta(\mathbf{k} \otimes \mathbf{h})\big(\Gamma_1 \otimes \Gamma_2 - (\mathbf{p} \otimes \mathbf{q}) \otimes (\mathbf{p} \otimes \mathbf{q})\big) \\
&= \chi(\mathbf{k})\eta(\mathbf{h})(\Gamma_1 - \mathbf{p} \otimes \mathbf{p}) \otimes (\Gamma_2 - \mathbf{q} \otimes \mathbf{q}) \\
&= \chi(\mathbf{k})(\Gamma_1 - \mathbf{p} \otimes \mathbf{q}) \otimes \psi(\mathbf{h})(\Gamma_2 - \mathbf{p} \otimes \mathbf{q}) \\
&= \chi(\mathbf{k})\Theta_1 \otimes \eta(\mathbf{h})\Theta_2 \\
&= \mathbf{k}.\Theta_1 \otimes \mathbf{h}.\Theta_2 \\
&= (\mathbf{k} \otimes \mathbf{h}).(\Theta_1 \otimes \Theta_2) \\
&= (\mathbf{k} \otimes \mathbf{h}).\Theta.
\end{aligned}$$

Then

$$(\mathbf{k} \otimes \mathbf{h}).\Theta = \chi \otimes \eta(\mathbf{k} \otimes \mathbf{h})\Theta \tag{2.10}$$

Since $\Omega_{\mathbb{K}\widehat{\otimes}\mathbb{H}}$ is a homomorphism onto $(\mathbb{K}\widehat{\otimes}\mathbb{H})\widehat{\otimes}(\mathbb{K}\widehat{\otimes}\mathbb{H})$, then

$$\langle \Omega^{**}_{\mathbb{K}\widehat{\otimes}\mathbb{H}}(\Theta), \chi \otimes \eta \rangle = \langle \Omega_{\mathbb{K}}^{**}(\Theta_1), \chi \rangle \langle \Omega_{\mathbb{H}}^{**}(\Theta_2), \eta \rangle = 1. \tag{2.11}$$

Using (2.10) and (2.11), we conclude that Θ is a $\chi \otimes \eta$-normal virtual diagonal for $\mathbb{K}\widehat{\otimes}\mathbb{H}$. Therefore, $\mathbb{K}\widehat{\otimes}\mathbb{H}$ has desirable properties.

The converse obtain by Corollary 2.16 and [4, Theorem 2.7]. □

The next result shows that for a Banach algebra $\mathbb{K}$ with special conditions, there exists a close relationship between the χ-normal virtual diagonal and the χ-σwc virtual diagonal, where $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$.

Corollary 2.21. *Let $\mathbb{K}$ be a unital dual Banach algebra, and let $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$. If $\mathbb{K}$ has a χ-normal virtual diagonal, then $\mathbb{K}$ has a χ-σwc virtual diagonal and vice versa.*

Proof. Let $\mathbb{K}$ has a χ-σwc virtual diagonal. Thus by combining Lemma 2.11 and [4, Theorem 2.7], we can find a χ-normal virtual diagonal for $\mathbb{K}$. The converse is obvious. $\square$

Corollary 2.22. *Let $\mathbb{K}$ and $\mathbb{H}$ be unital dual Banach algebras, and let $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\eta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. Then $\mathbb{K}\widehat{\otimes}\mathbb{H}$ has a $\chi \otimes \eta$-normal virtual diagonal if $\mathbb{K}$ has a χ-normal virtual diagonal and $\mathbb{H}$ has an η-normal virtual diagonal.*

Proof. Let $\mathbb{K}$ has a χ-normal virtual diagonal and $\mathbb{H}$ has an η-normal virtual diagonal, respectively. By Corollary 2.21, $\mathbb{K}$ has a χ-σwc virtual diagonal and $\mathbb{H}$ has an η-σwc virtual diagonal. Then by Theorem 2.14, $\mathbb{K}\widehat{\otimes}\mathbb{H}$ is $\chi \otimes \eta$-strong Connes amenable. Now from [4, Theorem 2.7], there exists a $\chi \otimes \eta$-normal virtual diagonal for $\mathbb{K}\widehat{\otimes}\mathbb{H}$. The converse is obvious. $\square$

2.2. (χ, θ)-strong Connes amenability of θ-Lau product of dual Banach algebras

In this subsection, let $\mathbb{K}$ and $\mathbb{H}$ be two dual Banach algebras. Using the notion of χ-invariant mean we study (χ, θ)-strong Connes amenability of Banach algebra $\mathbb{K} \times_\theta \mathbb{H}$, where $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\theta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. For this purpose, first we present some preliminary notations.

For Banach algebras $\mathbb{K}$ and $\mathbb{H}$ the θ-Lau product $\mathbb{K} \times_\theta \mathbb{H}$ is defined with

$$(\mathbf{k}, \mathbf{h}) \times_\theta (\mathbf{k}', \mathbf{h}') = (\mathbf{k}.\mathbf{k}' + \mathbf{k}.\theta(\mathbf{h}') + \theta(\mathbf{h}).\mathbf{k}', \mathbf{h}\mathbf{h}') \tag{2.12}$$

and the norm

$$\|(\mathbf{k}, \mathbf{h})\|_{\mathbb{K}\times_\theta\mathbb{H}} = \|\mathbf{k}\|_{\mathbb{K}} + \|\mathbf{h}\|_{\mathbb{H}}, \qquad (\mathbf{k}, \mathbf{k}' \in \mathbb{K}, \mathbf{h}, \mathbf{h}' \in \mathbb{H}).$$

This definition is a certain case of the product that is presented in [10, 13]. Many basic properties of $\mathbb{K} \times_\theta \mathbb{H}$ are investigated in [13]. Since $\theta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$ so, $\mathbb{K} \times_\theta \mathbb{H} = (\mathbb{K}_* \times \mathbb{H}_*)^*$ is a dual Banach algebra. In a natural way the dual space $(\mathbb{K} \times_\theta \mathbb{H})^*$ can be identified with $\mathbb{K}^* \times \mathbb{H}^*$ via

$$(f, g)((\mathbf{k}, \mathbf{h})) = f(\mathbf{k}) + g(\mathbf{h}) \tag{2.13}$$

for every $\mathbf{k} \in \mathbb{K}$, $\mathbf{h} \in \mathbb{H}$, $f \in \mathbb{K}^*$ and $g \in \mathbb{H}^*$. The second dual of $(\mathbb{K} \times_\theta \mathbb{H})^{**}$ is identified with $\mathbb{K}^{**} \times_{\theta^{**}} \mathbb{H}^{**}$. If we consider $\mathbb{K}$ with $\mathbb{K} \times \{0\}$, then $\mathbb{K}$ is a closed ideal in $\mathbb{K} \times_\theta \mathbb{H}$ and also, $(\mathbb{K} \times_\theta \mathbb{H})/\mathbb{K}$ is isometric isomorphism with $\mathbb{H}$, i.e $\frac{\mathbb{K}\times_\theta \mathbb{H}}{\mathbb{K}} \simeq \mathbb{H}$, see [13].

Example 2.23. The unitization $\mathbb{K}^\sharp = \mathbb{K} \times_i \mathbb{C}$ of a dual Banach algebra $\mathbb{K}$ can be regarded as the i-Lau product of $\mathbb{K}$ and $\mathbb{C}$ where $i \in \Delta(\mathbb{C})$ is the identity character.

Definition 2.24. Suppose that $\mathbb{K} = (\mathbb{K}_*)^*$ is a dual Banach algebra, and $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$. A linear functional μ on $\mathbb{K}^*$ is called a *mean* if $\mu(\chi) = 1$. Also, μ is called *χ-invariant mean* if $\mu(f.\mathbf{k}) = \chi(\mathbf{k})\mu(f)$ for all $\mathbf{k} \in \mathbb{K}$ and $f \in \mathbb{K}_*$, see [4].

In the next lemma we characterize the θ-Lau product of dual Banach algebras through the existence of (χ, θ)-σwc virtual diagonals.

Lemma 2.25. *Let $\mathbb{K}$ and $\mathbb{H}$ be dual Banach algebras with preduals $\mathbb{K}_*$ and $\mathbb{H}_*$, respectively. Suppose that $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$, and $\theta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. If $\mathbb{K}$ has a χ-σwc virtual diagonal, then $\mathbb{K} \times_\theta \mathbb{H}$ has a (χ, θ)-σwc virtual diagonal.*

Proof. Let $\mathbb{K}$ has a χ-σwc virtual diagonal. By Lemma 2.11, $\mathbb{K}$ is χ-strong Connes amenable. Thus, $\mathbb{K}$ is χ-Connes amenable. We equip $\mathbb{K}^{**}$ by the first Arens product. From [4, Theorem 2.3], it follows that $\mathbb{K}^{**}$ has an χ-invariant mean on predual of $\mathbb{K}$, say μ, and define $\nu \in \mathbb{K}^{**} \times_{\theta^{**}} \mathbb{H}^{**}$ by $\nu := (\mu, 0)$. Then $\langle \mu, \chi\rangle = \langle (\mu, 0), (\chi, \theta)\rangle = 1$. On the other hand, choose a net $\{\mu_\alpha\}$ in $\mathbb{K}$ such that $\mu_\alpha \stackrel{w^*}{\to} \mu$. Suppose that $\mathbb{K}$ is a normal χ-bimodule, whose underling space is itself, and on $\mathbb{K}$ acts by $\mathbf{x}.\mathbf{k} = \chi(\mathbf{k})\mathbf{x}$ for every $\mathbf{x}, \mathbf{k} \in \mathbb{K}$. Now, by (2.12) and (2.13), we have

$$\begin{aligned}
\langle (\mu, 0) \times_\theta (\mathbf{k}, \mathbf{h}), (f, g)\rangle &= \lim_\alpha \langle (\mu_\alpha, 0) \times_\theta (\mathbf{k}, \mathbf{h}), (f, g)\rangle \\
&= \lim_\alpha \langle (\mu_\alpha .\ \mathbf{k} + \mu_\alpha .\ \theta(\mathbf{h}) + \theta(0) .\ \mathbf{k}, 0 .\ \mathbf{h}), (f, g)\rangle \\
&= \lim_\alpha \langle (\chi(\mathbf{k})\ \mu_\alpha + \theta(\mathbf{h})\ \mu_\alpha, 0), (f, g)\rangle \\
&= \lim_\alpha (\chi(\mathbf{k}) + \theta(\mathbf{h}))\langle (\mu_\alpha, 0), (f, g)\rangle \\
&= (\chi, \theta)(\mathbf{k}, \mathbf{h})\langle (\mu, 0), (f, g)\rangle,
\end{aligned}$$

for every $(\mathbf{k}, \mathbf{h}) \in \mathbb{K} \times_\theta \mathbb{H}$ and $(f, g) \in \mathbb{K}_* \times \mathbb{H}_*$ in w^*-topology. So, $(\mu, 0)$ is an (χ, θ)-invariant mean for $\mathbb{K}^{**} \times_{\theta^{**}} \mathbb{H}^{**}$. Therefore, $\mathbb{K} \times_\theta \mathbb{H}$ has a (χ, θ)-σwc virtual diagonal. □

Example 2.26. Suppose that $\mathbb{K}$, $\mathbb{H}$, χ and $\theta = \eta$ are such as in Example 2.15. We saw that $\mathbb{K}$ has a χ-σwc virtual diagonal, say $\Gamma_\mathbb{K}$. Therefore $\mathbb{K}^{**} = \mathbb{K}$ has a χ-invariant mean, say μ. So, $\langle \mu, \chi\rangle = 1$ and $\mu(f.\mathbf{k}) = \chi(\mathbf{k})\mu(f)$ for every $\mathbf{k} \in \mathbb{K}$ and

$f \in \mathbb{K}_* = \mathbb{K}$. Now we show that $(\mu, 0)$ is an (χ, θ)-invariant mean for $\mathbb{K}^{**} \times_{\theta^{**}} \mathbb{H}^{**}$. It is clear that $\langle(\mu, 0), (\chi, \theta)\rangle = 1$. Choose $\mathbf{k} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{K}$, $\mathbf{f} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{K}_*$, $\mathbf{h} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in \mathbb{H}$ and $\mathbf{g} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in \mathbb{H}_*$. One can see that $\mathbb{K}$ becomes a normal χ-bimodule, whose underling space is itself, and on $\mathbb{K}$ acts by $\mathbf{x}.\mathbf{k} = \chi(\mathbf{k})\mathbf{x}$ for every $\mathbf{x}, \mathbf{k} \in \mathbb{K}$. By (2.12) and (2.13), we have

$$\begin{aligned}
\Big\langle (\mu, 0) \times_\theta (\mathbf{k}, \mathbf{h}), (\mathbf{f}, \mathbf{g}) \Big\rangle &= \Big\langle \Big(\mu. \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \mu.\theta \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + 0, 0\Big), (\mathbf{f}, \mathbf{g}) \Big\rangle \\
&= \Big\langle \Big(\chi \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mu + \theta \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mu, 0\Big), (\mathbf{f}, \mathbf{g}) \Big\rangle \\
&= \Big(\chi \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \theta \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\Big) \langle(\mu, 0), (\mathbf{f}, \mathbf{g})\rangle \\
&= (\chi, \theta)\Big(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\Big) \langle(\mu, 0), (\mathbf{f}, \mathbf{g})\rangle \\
&= (\chi, \theta)(\mathbf{k}, \mathbf{h}) \langle(\mu, 0), (\mathbf{f}, \mathbf{g})\rangle
\end{aligned}$$

So, $(\mu, 0)$ is an (χ, θ)-invariant mean for $\mathbb{K}^{**} \times_{\theta^{**}} \mathbb{H}^{**}$. Indeed, $\mathbb{K} \times_\theta \mathbb{H}$ has a (χ, θ)-σwc virtual diagonal.

In the following we investigate the $(0, \theta)$-strong Connes amenability of $\mathbb{K} \times_\theta \mathbb{H}$ through the θ-strong Connes amenability of Banach algebra $\mathbb{H}$.

Theorem 2.27. *Let $\mathbb{K} = (\mathbb{K}_*)^*$, and $\mathbb{H} = (\mathbb{H}_*)^*$ be dual Banach algebras. Suppose that $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\theta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$, then:*

(i) *If $\mathbb{K} \times_\theta \mathbb{H}$ is $(0, \theta)$-strong Connes amenable, then $\mathbb{H}$ is θ-strong Connes amenable;*

(ii) *If $\mathbb{K}$ is unital and $\mathbb{H}$ is θ-strong Connes amenable, then $\mathbb{K} \times_\theta \mathbb{H}$ is $(0, \theta)$-strong Connes amenable.*

Proof.

(i) The process of the proof is analogous to the argument of Lemma 2.25. Suppose that (μ_1, μ_2) is an $(0, \theta)$-invariant mean on $\mathbb{K}_* \times \mathbb{H}_*$. Hence,

$$\langle(\mu_1, \mu_2) \times_\theta (\mathbf{k}, \mathbf{h}), (f, g)\rangle = \theta(\mathbf{h})\langle(\mu_1, \mu_2), (f, g)\rangle, \qquad \langle(\mu_1, \mu_2), (0, \theta)\rangle = 1$$

for every $f \in \mathbb{K}_*$, $g \in \mathbb{H}_*$, $\mathbf{k} \in \mathbb{K}$ and $\mathbf{h} \in \mathbb{H}$. It is easy to see that

$$\langle(\mu_1.\mathbf{k} + \mu_1.\theta(\mathbf{h}) + \theta(\mu_2).\mathbf{k}, \mu_2.\mathbf{h}), (f, g)\rangle = \langle(\theta(\mathbf{h})\mu_1, \theta(\mathbf{h})\mu_2), (f, g)\rangle.$$

So, $\langle \mu_2.\mathbf{h}, g\rangle = \theta(\mathbf{h})\langle \mu_2, g\rangle$ and $\langle \mu_2, \theta\rangle = 1$. It follows that, μ_2 is an θ-invariant mean on $\mathbb{H}_*$. By Lemma 2.25 the proof is complete.

(ii) let $\mathbb{H}$ be θ-strong Connes amenable, and let $e_{\mathbb{K}}$ be the unit of $\mathbb{K}$. Let μ be an θ-invariant mean on $\mathbb{H}_*$. We claim that $(-e_{\mathbb{K}}, \mu)$ is an $(0,\theta)$-invariant mean on $\mathbb{K}_* \times \mathbb{H}_*$. Consider the net $\{\mu_\alpha\} \subseteq \mathbb{K}$ such that $\mu_\alpha \xrightarrow{w^*} \mu$. So

$$\begin{aligned}\langle(-e_{\mathbb{K}}, \mu) \times_\theta (\mathbf{k}, \mathbf{h}), (f, g)\rangle &= w^* - \lim_\alpha \langle(-e_{\mathbb{K}}, \mu_\alpha) \times_\theta (\mathbf{k}, \mathbf{h}), (f, g)\rangle \\ &= \langle(-e_{\mathbb{K}}.\mathbf{k} - e_{\mathbb{K}}.\theta(\mathbf{h}) + \theta(\mu).e_{\mathbb{K}}.\mathbf{k}, \mu.\mathbf{h}), (f, g)\rangle \\ &= \langle(-e_{\mathbb{K}}\theta(\mathbf{h}), \theta(\mathbf{h})\mu), (f, g)\rangle \\ &= (0,\theta)(\mathbf{k}, \mathbf{h})\langle(-e_{\mathbb{K}}, \mu), (f, g)\rangle\end{aligned}$$

for every $f \in \mathbb{K}_*, g \in \mathbb{H}_*$, $\mathbf{k} \in \mathbb{K}$ and $\mathbf{h} \in \mathbb{H}$. Thus $\mathbb{K} \times_\theta \mathbb{H}$ is $(0,\theta)$-strong Connes amenable. □

In the following example, in the general case, we show that if the Banach algebra $\mathbb{K}$ in Theorem 2.27, is not unital then the clause *(ii)* is not holds.

Example 2.28. Let $\mathbb{K} = (\mathbb{K}_*)^*$ and $\mathbb{H} = (\mathbb{H}_*)^*$ be dual Banach algebras. Suppose that $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$ and $\theta \in \Delta_{w^*}(\mathbb{H}) \cap \mathbb{H}_*$. Let $\mathbb{K}$ be non-unital, and let $\mathbb{H}$ be θ-strong Connes amenable. Then $\mathbb{K} \times_\theta \mathbb{H}$ can not be $(0,\theta)$-strong Connes amenable. Assume towards a contradiction that (μ_1, μ_2) is an $(0,\theta)$-invariant mean on $\mathbb{K}_* \times \mathbb{H}_*$, in which μ_2 is an θ-invariant mean on $\mathbb{H}_*$. So, $\mu_2(\theta) = 1$ and $\langle(\mu_1, \mu_2), (0, \theta)\rangle = 1$. Also,

$$\begin{aligned}\langle(\mu_1, \mu_2) \times_\theta (\mathbf{k}, \mathbf{h}), (f, g)\rangle &= (0,\theta)(\mathbf{k}, \mathbf{h})\langle(\mu_1, \mu_2), (f, g)\rangle \\ &= \theta(\mathbf{h})\langle(\mu_1, \mu_2), (f, g)\rangle\end{aligned}$$

for every $f \in \mathbb{K}_*$, $g \in \mathbb{H}_*$, $\mathbf{k} \in \mathbb{K}$ and $\mathbf{h} \in \mathbb{H}$. Then

$$\langle(\mu_1.\mathbf{k} + \mu_1.\theta(\mathbf{h}) + \theta(\mu_2).\mathbf{k}, \mu_2.\mathbf{h}), (f, g)\rangle = \langle(\theta(\mathbf{h})\mu_1, \theta(\mathbf{h})\mu_2), (f, g)\rangle.$$

It follows that $\mu_1.\mathbf{k} + \overbrace{\theta(\mu_2)}^{1}.\mathbf{k} = 0$. Hence, $\mathbb{K}$ must be unital, and this is contadiction with assumption.

3. Characterization of Module Extension of Banach Algebras

In this section, we investigate χ-strong Connes amenability of module extension of dual Banach algebras. Let $\mathbb{K} = (\mathbb{K}_*)^*$ be a dual Banach algebra, and let $\mathbb{X} = (\mathbb{X}_*)^*$ be a normal Banach $\mathbb{K}$-bimodule. We denote the l^1-direct sum of dual Banach algebra $\mathbb{K}$ with a nonzero Banach $\mathbb{K}$-bimodule $\mathbb{X}$ by $\mathbb{K} \oplus \mathbb{X}$. In the following we define the algebraic product and norm for $\mathbb{K} \oplus \mathbb{X}$:

$$(\mathbf{k}, \mathbf{x}) \cdot (\mathbf{k}', \mathbf{x}') = (\mathbf{kk}', \mathbf{kx}' + \mathbf{xk}'), \qquad (\mathbf{k}, \mathbf{k}' \in \mathbb{K}, \mathbf{x}, \mathbf{x}' \in \mathbb{X})$$

and

$$\|(\mathbf{k}, \mathbf{x})\|_{\mathbb{K}\oplus\mathbb{X}} = \|\mathbf{k}\|_{\mathbb{K}} + \|\mathbf{x}\|_{\mathbb{X}}.$$

With above structure, $\mathbb{K} \oplus \mathbb{X}$ is called module extension of dual Banach algebra $\mathbb{K}$. Some algebras of this form have been discussed in [17]. It is known that $\mathbb{K} \oplus \mathbb{X} = (\mathbb{K}_* \oplus_\infty \mathbb{X}_*)^*$ is a dual Banach algebra, where $\oplus_\infty$ denotes l_∞-direct sum of Banach $\mathbb{K}$-bimodules. The first and the second dual of $\mathbb{K} \oplus \mathbb{X}$ are identified with $\mathbb{K}^* \oplus_\infty \mathbb{X}^*$ and $\mathbb{K}^{**} \oplus \mathbb{X}^{**}$, respectively. If we equip $\mathbb{K}^{**}$ with the first Arens product, then it follows that $\mathbb{X}^{**}$ is Banach $\mathbb{K}^{**}$-bimodule (for more details see [17]).

Example 3.1. Let G be a locally compact group, and let $L^1(G)$ and $M(G)$ be its group algebra and measure algebra, respectively. Then $L^1(G) \oplus M(G)$, $M(G) \oplus L^1(G)$, $M(G) \oplus M(G)$, $L^1(G) \oplus L^1(G)$ and $L^1(G) \oplus L^\infty(G)$ are module extension of Banach algebras.

Example 3.2. $l^1 \oplus l_0^p$, for $1 \leq p \leq \infty$ and $c_0 \oplus c_0$ with pointwise multiplication are module extension of Banach algebras.

Example 3.3. Let $\mathbb{K}$ be a commutative Banach algebra, and let $\mathbb{X}$ be a non-zero symmetric $\mathbb{K}$-bimodule, then $\mathbb{K} \oplus \mathbb{X}$ is a module extension of Banach algebra.

In [1], Vishki and Khoddami showed that $\Delta(\mathbb{K}\oplus\mathbb{X}) = \Delta(\mathbb{K}) \times \{0\}$ and they proved that if $\mathbb{K} \oplus \mathbb{X}$ is $(\chi, 0)$-amenable then $\mathbb{K}$ is χ-amenable, where $\chi \in \Delta_{w^*}(\mathbb{K}) \cap \mathbb{K}_*$. The converse also holds if $\mathbb{XK} = 0$. One might ask whether the result extends to $(\chi, 0)$-strong Connes amenability. We give an affirmative answer to this question.

The purpose of the next theorem is to study the relationship between the $(\chi, 0)$-σwc virtual diagonal of $\mathbb{K} \oplus \mathbb{X}$ and the χ-σwc virtual diagonal of $\mathbb{K}$.

Theorem 3.4. *Let $\mathbb{K} = (\mathbb{K}_*)^*$ be a dual Banach algebra, and let $\chi \in \Delta_{\omega^*}(\mathbb{K}) \cap \mathbb{K}_*$. Suppose that $\mathbb{X}$ is a normal Banach $\mathbb{K}$-bimodule with predual $\mathbb{X}_*$. If $\mathbb{K} \oplus \mathbb{X}$ is $(\chi, 0)$-strong Connes amenable, then $\mathbb{K}$ is χ-strong Connes amenable.*

Proof. Let $\mathbb{K} \oplus \mathbb{X}$ be $(\chi, 0)$-strong Connes amenable. By Lemma 2.11, $\mathbb{K} \oplus \mathbb{X}$ has a $(\chi, 0)$-σwc virtual diagonal. Let $\gamma : \mathbb{K} \oplus \mathbb{X} \to \mathbb{K}$ be a projection map with adjoint γ^*, and let $\mathbb{K}$ be the normal χ-bimodule. The idea of the proof is similar to that of the proofs of Lemma 2.25 and Theorem 2.27. Suppose that $(S, T) \in \sigma wc((\mathbb{K} \oplus \mathbb{X})^*)^*$ is an $(\chi, 0)$-invariant mean on $\mathbb{K}_* \oplus \mathbb{X}_*$. It is clear that $\gamma^*(\mathbf{k}.f) = \mathbf{k}.\gamma^*(f)$ for every $\mathbf{k} \in \mathbb{K}$ and $f \in \mathbb{K}_*$. Then

$$\begin{aligned} \langle \gamma^{**}(S,T), \mathbf{k}.f \rangle &= \langle (S,T), \gamma^*(\mathbf{k}.f) \rangle \\ &= \langle (S,T), (\mathbf{k},0)\gamma^*(f) \rangle \\ &= (\chi,0)(\mathbf{k},0)\langle (S,T), \gamma^*(f) \rangle \\ &= (\chi,0)(\mathbf{k},0)\langle \gamma^{**}(S,T), f \rangle \end{aligned} \tag{3.1}$$

for every $\mathbf{k} \in \mathbb{K}$ and $f \in \mathbb{K}_*$. Also,

$$\langle \gamma^{**}(S,T), \chi \rangle = \langle (S,T), \gamma^*(\chi) \rangle = \langle (S,T), (\chi,0) \rangle = 1. \tag{3.2}$$

By (3.1) and (3.2), $\gamma^{**}(S,T)$ is an χ-invariant mean on $\mathbb{K}_*$. Hence, there exists a χ-σwc virtual diagonal for $\mathbb{K}$. Now, using Lemma 2.11, $\mathbb{K}$ is χ-strong Connes amenable. □

Now, we show that by applying an important condition, the converse of above theorm is hold.

Corollary 3.5. *Let $\mathbb{K} = (\mathbb{K}_*)^*$ be a unital dual Banach algebra, and $\chi \in \Delta_{\omega^*}(\mathbb{K}) \cap \mathbb{K}_*$. Suppose that $\mathbb{X} = (\mathbb{X}_*)^*$ is a normal Banach $\mathbb{K}$-bimodule. If $\mathbb{K}$ is χ-strong Connes amenable and $\mathbb{X}\mathbb{K} = 0$. Then $\mathbb{K} \oplus \mathbb{X}$ is $(\chi, 0)$-strong Connes amenable.*

Proof. Let $\mathbb{K}$ be χ-strong Connes amenable. By [4, Theorem 2.7], we obtain a χ-normal virtual diagonal for $\mathbb{K}$. Using Corollary 2.21, $\mathbb{K}$ has a χ-σwc virtual diagonal. By using Remark 2.10, it is clear that $\mathbb{K}$ is χ-Connes amenable. Now, from [4, Theorem 2.3], it follows that $\mathbb{K}^{**}$ has an χ-invariant mean, say $\mu \in \mathbb{K}^{**}$ such that

$$\mu(\chi) = 1, \qquad \mu(f.\mathbf{k}) = \chi(\mathbf{k})\mu(f)$$

for every $\mathbf{k} \in \mathbb{K}$ and $f \in \mathbb{K}_*$. Let $\mathbb{E}_*$ be a $\mathbb{K} \oplus \mathbb{X}$-bimodule with module action

$$\mathbf{e}.(\mathbf{k}, \mathbf{x}) = (\chi, 0)(\mathbf{k}, \mathbf{x})\mathbf{e}$$

for every $\mathbf{k} \in \mathbb{K}$, $\mathbf{x} \in \mathbb{X}$, and $\mathbf{e} \in \mathbb{E}_*$. Let $\mathcal{D} : \mathbb{K} \oplus \mathbb{X} \to (\mathbb{E}_*)^*$ be a bounded w^*-continuous derivation such that $\mathcal{D}(\mathbb{K} \oplus \mathbb{X})$ consists of w^*-elements. Suppose that $\mathcal{D}^* : \mathbb{E}^* \to (\mathbb{K} \oplus \mathbb{X})^*$ denotes the adjoint of $\mathcal{D}$. Then $\mathcal{D}^*$ maps $\mathbb{E}_* \subseteq \mathbb{E}^*$ into $\mathbb{K}_* \oplus \mathbb{X}_*$. Let $\pi : (\mathbb{K} \oplus \mathbb{X})^{**} \to \mathbb{K} \oplus \mathbb{X}$ be a Dixmier projection. Suppose that

$(\mathbf{k}, \mathbf{x}), (\mathbf{k}', \mathbf{x}') \in \mathbb{K} \oplus \mathbb{X}$ and $\mathbf{e} \in \mathbb{E}_*$. We have

$$\begin{aligned}\langle \mathcal{D}^*(\mathbf{e}.(\mathbf{k}, \mathbf{x}), (\mathbf{k}', \mathbf{x}')) \rangle &= \langle (\mathbf{e}.(\mathbf{k}, \mathbf{x}), \mathcal{D}(\mathbf{k}', \mathbf{x}')) \rangle \\ &= (\chi, 0)(\mathbf{k}, \mathbf{x}) \langle \mathbf{e}, \mathcal{D}(\mathbf{k}', \mathbf{x}')) \rangle \\ &= (\chi, 0)(\mathbf{k}, \mathbf{x}) \langle \mathcal{D}^*(\mathbf{e}), (\mathbf{k}', \mathbf{x}') \rangle.\end{aligned}$$

Therefore, $\mathcal{D}^*(\mathbf{e}.(\mathbf{k}, \mathbf{x})) = (\chi, 0)(\mathbf{k}, \mathbf{x})\mathcal{D}^*(\mathbf{e})$. A similar argument shows that

$$\mathcal{D}^*((\mathbf{k}, \mathbf{x}).\mathbf{e}) = (\mathbf{k}, \mathbf{x})\mathcal{D}^*(\mathbf{e}) - (\chi, 0)\langle \mathbf{e}, \mathcal{D}(\mathbf{k}, \mathbf{x})) \rangle.$$

Now by hypothesis $\mathbb{X}\mathbb{K} = 0$ and put $q := \mathcal{D} \circ \pi(\mu, 0)$ we have,

$$\begin{aligned}\mathcal{D}(\mathbf{k}, \mathbf{x}) &= (\chi, 0)(\mathbf{k}, \mathbf{x}).q - q.(\mathbf{k}, \mathbf{x}) \\ &= (\mathbf{k}, \mathbf{x}).q - q.(\mathbf{k}, \mathbf{x}).\end{aligned}$$

Thus $\mathcal{D}$ is inner. □

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