J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
https://doi.org/10.7468/jksmeb.2024.31.1.103
Volume 31, Number 1 (February 2024), Pages 103–117

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

F-CONTRACTION IN PARTIALLY ORDERED b-METRIC LIKE SPACES

Om Prakash Chauhan [a,*], Vishal Joshi [b] and Saurabh Singh [c]

Abstract. In this article, we utilize the concepts of hybrid rational Geraghty type generalized F-contraction and to prove some fixed point results for such mappings are in the perspective of partially ordered b-metric like space. Some innovative examples are also presented which substantiate the validity of obtained results. The example is also authenticated with the help of graphical representations.

1. Introduction

Numerous extensions and variations of the metric space concept have been explored. Bakhtin [8] introduced the notion of a b-metric space, and subsequently, Czerwik ([11], [12]) extensively utilized the concept of b-metric spaces.The concept of partial metric spaces was introduced by Matthews [16], who replaced the conventional metric with a partial metric as part of the investigation into denotational semantics for data flow networks. Notably, this approach exhibits the intriguing property that the self-distance of any point within the space need not be zero. Matthews [16] also obtained the partial metric version of Banach contraction theorem. Subsequently, many authors studied partial metric spaces and their topological properties and obtained a number of fixed point theorems [3, 4, 6, 7, 13, 18, 19]. In 2012, Amini-Harandi [5] introduced a distinct generalization of partial metric spaces known as metric-like spaces. Building upon this, Alghamdi et al. [2] introduced b-metric-like spaces, which are purported to extend partial metric spaces, b-metric spaces, and metric-like spaces. Concurrently, Wardowski [21] introduced the novel concept of an F-contraction, which constitutes a broader framework than the classical Banach contraction principle, leading to new fixed point results. Expanding

Received by the editors September 11, 2023. Accepted October 6, 2023.
2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. fixed point, partially ordered b-metric-like spaces, F-contraction.
*Corresponding author.

on this, Abbas et al. [1] further generalized the notion of F-contraction and established a collection of fixed point and common fixed point theorems. More recently, Secelean et al. [20] outlined a comprehensive set of functions by incorporating condition $(F2')$ in lieu of the conventional condition $(F2)$ within the definition of the F-contraction, as introduced by Wardowski [21]. In a recent advancement, Piri et al. [17] enhanced the findings of Secelean et al. [20] by replacing condition $(F3')$ with the original condition $(F3)$.
The forthcoming sections will make use of elementary definitions and essential outcomes, which are expounded upon in this section.

Definition 1.1 ([11])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a *b-metric* if for all $x, y, z \in X$ the following conditions are satisfied:
(b_1) $d(x, y) = 0$ iff $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a *b-metric space.* The number $s \geq 1$ is called the *coefficient* of (X, d).

Definition 1.2 ([16])**.** A partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,
(p1) $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Definition 1.3 ([2])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : X \times X \to [0, \infty)$ is called a *b-metric-like* if for all $x, y, z \in X$ the following conditions are satisfied:
$(\sigma_b 1)$ $\sigma_b(x, y) = 0$ implies $x = y$;
$(\sigma_b 2)$ $\sigma_b(x, y) = \sigma_b(y, x)$;
$(\sigma_b 3)$ $\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.
The pair (X, σ_b) is called a *b-metric-like space.* The number $s \geq 1$ is called the *coefficient* of (X, σ_b).

Example 1.4 ([2]). Let $X = \mathbb{R}^+$ and the mapping $\sigma_b : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma_b(x, y) = [\max\{x, y\}]^2,$$

for all $x, y \in X$. Then (X, σ_b) is a b-metric-like space with the coefficient $s = 2 > 1$, but it is neither a b-metric nor a metric-like space.

Remark 1.5. The class of b-metric-like spaces (X, σ_b) encompasses a broader scope than that of metric-like spaces, as a metric-like space becomes a distinct instance of a b-metric-like space (X, σ_b) when $s = 1$. Similarly, the category of b-metric-like spaces (X, σ_b) extends beyond that of b-metric spaces, given that a b-metric space emerges as a specific manifestation of a b-metric-like space (X, σ_b) in cases where the self-distance $\sigma_b(x, x) = 0$.

Each b-metric-like σ_b on X generalizes a topology τ_{σ_b} on X whose base is the family of open σ_b-balls $B - \sigma_b(x, \epsilon) = \{y \in X : |\sigma_b(x, y) - \sigma_b(x, x)| < \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 1.6 ([2]). A b-metric-like space (X, σ_b) is said to be σ_b-*complete* if every σ_b-Cauchy sequence $\{x_n\}$ in X, σ_b-converges to a point $x \in X$, such that $\sigma_b(x, x) = \lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x, x_n)$.

Lemma 1.7 ([2]). *Let $\{y_n\}$ be a sequence in a b-metric like space (X, σ_b) such that*

$$\sigma_b(y_n, y_{n+1}) \leq \lambda \sigma_b(y_{n-1}, y_n)$$

for some λ, $0 < \lambda < \frac{1}{s}$ and each $n \in N$. Then $\{y_n\}$ is a Cauchy sequence in X and $\lim_{n,m\to\infty} \sigma_b(y_n, y_m) = 0$.

Remark 1.8 ([2]). Let (X, σ_b) be a b-metric-like space with constant $s \geq 1$. Then it is clear that

$$\sigma_b^s(x, y) = |2\sigma_b(x, y) - \sigma_b(x, x) - \sigma_b(y, y)|$$

satisfies $\sigma_b^s(x, x) = 0$, for all $x \in X$. Then $\sigma_b^s(x, y)$ is considered to be a b-metric space.

Remark 1.9 ([10]). Let (X, σ_b) be a b-metric-like space and let $T : X \to X$ be a continuous mapping. Then $\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) \Rightarrow \lim_{n\to\infty} \sigma_b(Tx, Tx_n) = \sigma_b(x, x)$.

Definition 1.10 ([15]). If a nonempty set X is equipped with a partial order $\preceq$ such that (X, σ) is a metric-like space, then the $(X, \sigma, \preceq)$ is called a *partially ordered metric-like space.*

The proof of following lemma is similar as for the metric case.

Lemma 1.11 ([15])**.** *Let (X, σ) be a metric-like space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ converges to some $x \in X$ with $\sigma(x, x) = 0$ then $\lim_{n\to\infty}(x_n, y) = \sigma(x, y)$ for all $x \in X$.*

Definition 1.12 ([21])**.** Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:
(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Denote the set of all functions satisfying (F1)-(F3) by $\Im$. In [20], Secelean et al. changed the condition $(F2)$ by an equivalent but a more simple condition $(F2^{'})$.
$(F2^{'})$ $\inf F = -\infty$,
or, also by
$(F2^{''})$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.
Recently, Piri et al. [17] used the following condition $(F3^{'})$ instead of $(F3)$.
$(F3^{'})$ F is continuous on $(0, \infty)$.
In our subsequent discussion, condition $(F2^{'})$ is dropped out. Thus we utilize the functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy $(F1)$ and $(F3^{'})$. Class of all such functions satisfying $(F1)$, and $(F3^{'})$ is denoted by Δ_F.
Wardowski in [21] introduced the F-contraction as follows:

Definition 1.13. Let (X, ρ) be a metric space. A mapping $T : X \to X$; is said to be an *F-contraction* if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$, $\rho(Tx, Ty) > 0$ we have

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)).$$

Definition 1.14 ([14])**.** Let Θ denotes the class of the functions $\theta : [0, +\infty) \to [0, 1)$ which satisfy the condition $\theta(t_n) \to 1 \Rightarrow t_n \to 0$.

Definition 1.15 ([9])**.** Let X be a non-empty set,$T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. We say that T is an (α, β)-*admissible* if $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$, for all $x, y \in X$.

Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that

(1) ϕ is non-decreasing;

(2) ϕ is continuous;

(3) $\phi(t) = 0 \iff t = 0$.

Let Ψ denote the set of all decreasing functions $\psi : (0, \infty) \to (0, \infty)$.

2. Main Results

In this section, we proved some results on partially ordered b metric like spaces via Geraghty type generalized F-contraction and their consequences.

Definition 2.1. Let $(X, \sigma_b, \precsim)$ be a partially ordered b-metric like space. Let $T : X \to X$ be a self-mapping. if there exist $F \in \Delta_F$, $\theta \in \Theta$, $\phi \in \Phi$ and $\psi \in \Psi$ such that for all $x, y \in X$ and $s > 1$ with $\sigma_b(Tx, Ty) > 0$,

(2.1)

$$F\Big(s\Big(\frac{1 + s\sigma_b(x, y)}{1 + \frac{1}{2}\sigma_b(x, Tx)}\Big)\sigma_b(Tx, Ty)\Big) \leq \theta\Big(\phi\Big(M_s(x, y)\Big)\Big)F(M_s(x, y)) - \psi\Big(\sigma_b(x, y)\Big),$$

$$\textit{where} \quad M_s(x, y) = \max\Big\{\sigma_b(x, y), \sigma_b(x, Tx), \frac{\sigma_b(x, Tx)\sigma_b(y, Ty)}{1 + \sigma_b(Tx, Ty)}\Big\}.$$

Theorem 2.2. *Let $(X, \sigma_b, \precsim)$ be a complete partially ordered b-metric like space. Let T be a self mapping on X satisfying the Geraghty type generalized F-contraction on $(X, \sigma_b, \precsim)$. Then T has a unique fixed point $u \in X$, moreover $\sigma_b(u, u) = 0$.*

Proof. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n+1} = x_n$ for any $n \in \mathbb{N}$, then x_n is a fixed point of T. Consequently, assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Since $\frac{1+s\sigma_b(x_{n-1},x_n)}{1+\frac{1}{2}\sigma_b(x_{n-1},Tx_{n-1})} = \frac{1+s\sigma_b(x_{n-1},x_n)}{1+\frac{1}{2}\sigma_b(x_{n-1},x_n)} \geq \frac{1+\sigma_b(x_{n-1},x_n)}{1+\frac{1}{2}\sigma_b(x_{n-1},Tx_{n-1})} \geq 1$.

By the Definition 2.1 with $x = x_{n-1}$ and $y = x_n$ in (2.1) and due to $\Delta_{(F1)}$, property of θ and ϕ, we arrive at

$$\begin{aligned} F\big(\sigma_b(x_n, x_{n+1})\big) \leq& F\big(s\sigma_b(Tx_{n-1}, Tx_n)\big) \\ \leq& \theta\Big(\phi\big(M_s(x_{n-1}, x_n)\big)\Big)F\big(M_s(x_{n-1}, x_n)\big) - \psi\big(\sigma_b(x_{n-1}, x_n)\big) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M_s(x_{n-1}, x_n) =& \max\Big\{\sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, Tx_{n-1}), \frac{\sigma_b(x_{n-1}, Tx_{n-1})\sigma_b(x_n, Tx_n)}{1 + \sigma_b(Tx_{n-1}, Tx_n)}\Big\} \\ =& \max\Big\{\sigma_b(x_{n-1}, x_n), \sigma_b(x_{n-1}, x_n), \frac{\sigma_b(x_{n-1}, x_n)\sigma_b(x_n, x_{n+1})}{1 + \sigma_b(x_n, x_{n+1})}\Big\} \end{aligned}$$

$$\begin{aligned}&= \max\left\{\sigma_b(x_{n-1},x_n), \frac{\sigma_b(x_{n-1},x_n)\sigma_b(x_n,x_{n+1})}{1+\sigma_b(x_n,x_{n+1})}\right\}\\ &= \sigma_b(x_{n-1},x_n).\end{aligned}$$

From (2.2) and by the definition of ϕ, ψ and θ, we have

$$\begin{aligned}F(\sigma_b(x_n,x_{n+1})) \leq &\theta\Big(\phi\big(\sigma_b(x_{n-1},x_n)\big)\Big)F\big(\sigma_b(x_{n-1},x_n)\big) - \psi\big(\sigma_b(x_{n-1},x_n)\big)\\ &\leq F\big(\sigma_b(x_{n-1},x_n)\big) - \psi\big(\sigma_b(x_{n-1},x_n)\big).\end{aligned} \tag{2.3}$$

Which gives

$$F\big(\sigma_b(x_n,x_{n+1})\big) < F\big(\sigma_b(x_{n-1},x_n)\big)$$

$$\implies \sigma_b(x_n,x_{n+1}) < \sigma_b(x_{n-1},x_n). \tag{2.4}$$

Hence, $\{\sigma_b(x_n,x_{n+1})\}$ is a decreasing sequence of positive real numbers. Repeated use of (2.3) gives

$$\begin{aligned}F\big(\sigma_b(x_n,x_{n+1})\big) \leq &F\big(\sigma_b(x_{n-1},x_n)\big) - \psi\big(\sigma_b(x_{n-1},x_n)\big)\\ \leq &F\big(\sigma_b(x_{n-2},x_{n-1})\big) - \psi\big(\sigma_b(x_{n-2},x_{n-1})\big) - \psi\big(\sigma_b(x_{n-1},x_n)\big).\end{aligned}$$

As ψ is a decreasing function, we get

$$F\big(\sigma_b(x_n,x_{n+1})\big) \leq F\big(\sigma_b(x_{n-2},x_{n-1})\big) - 2\psi\big(\sigma_b(x_{n-2},x_{n-1})\big),$$

it follows by successive application that

$$F\big(\sigma_b(x_n,x_{n+1})\big) \leq F\big(\sigma_b(x_0,x_1)\big) - n\psi\big(\sigma_b(x_0,x_1)\big). \tag{2.5}$$

Since, $F \in \Delta_F$, letting the limit as $n \to \infty$ in inequality (2.5) we must have

$$\lim_{n\to\infty} F\big(\sigma_b(x_n,x_{n+1})\big) = -\infty \iff \lim_{n\to\infty} \sigma_b(x_n,x_{n+1}) = 0. \tag{2.6}$$

Next, we show that $\{x_n\}$ is a σ_b-Cauchy sequence in X. If is not, then there exists $\epsilon > 0$ for which we can find sub-sequences $x_{m(k)}$ and $x_{n(k)}$ of $\{x_n\}$ where $x_{n(k)}$ is the smallest index for which $n(k) > m(k) > k$, with

$$\sigma_b(x_{m(k)},x_{n(k)}) \geq \epsilon, \tag{2.7}$$

then

$$\sigma_b(x_{m(k)},x_{n(k)-1}) < \epsilon. \tag{2.8}$$

Using (2.7) and (2.8), we obtain

$$\begin{aligned}\epsilon \leq \sigma_b(x_{m(k)},x_{n(k)}) &\leq s\sigma_b(x_{m(k)},x_{n(k)-1}) + s\sigma_b(x_{n(k)-1},x_{n(k)})\\ &< s\epsilon + s\sigma_b(x_{n(k)-1},x_{n(k)}).\end{aligned} \tag{2.9}$$

Taking the upper and lower limits as $k \to \infty$, we conclude

$$\epsilon \leq \lim_{k\to\infty} \inf \sigma_b(x_{m(k)}, x_{n(k)}) \leq \lim_{k\to\infty} \sup \sigma_b(x_{m(k)}, x_{n(k)}) \leq s\epsilon \tag{2.10}$$

and

$$\begin{aligned}\sigma_b(x_{m(k)+1}, x_{n(k)}) &\leq s\sigma_b(x_{m(k)+1}, x_{m(k)}) + s\sigma_b(x_{m(k)}, x_{n(k)}) \\ &\leq s\sigma_b(x_{m(k)+1}, x_{m(k)}) + s^2\sigma_b(x_{m(k)}, x_{n(k)-1}) + s^2\sigma_b(x_{n(k)-1}, x_{n(k)}) \\ &\leq s\sigma_b(x_{m(k)+1}, x_{m(k)}) + s^2\epsilon + s^2\sigma_b(x_{n(k)-1}, x_{n(k)}).\end{aligned} \tag{2.11}$$

With taking the upper limit as $k \to \infty$ in (2.11), we obtain

$$\lim_{k\to\infty} \sup \sigma_b(x_{m(k)+1}, x_{n(k)}) \leq s^2\epsilon. \tag{2.12}$$

Furthermore,

$$\sigma_b(x_{m(k)+1}, x_{n(k)-1}) \leq s\sigma_b(x_{m(k)+1}, x_{m(k)}) + s\sigma_b(x_{m(k)}, x_{n(k)-1}). \tag{2.13}$$

By taking the upper limit as $k \to \infty$ in (2.13), we get

$$\lim_{k\to\infty} \sup \sigma_b(x_{m(k)+1}, x_{n(k)-1}) \leq s\epsilon. \tag{2.14}$$

On the other hand

$$\begin{aligned}\sigma_b(x_{m(k)}, x_{n(k)}) &\leq s\sigma_b(x_{m(k)}, x_{m(k)+1}) + s\sigma_b(x_{m(k)+1}, x_{n(k)}) \\ &\leq s\sigma_b(x_{m(k)}, x_{m(k)+1}) + s^2\sigma_b(x_{m(k)+1}, x_{n(k)-1}) \\ &\quad + s^2\sigma_b(x_{n(k)-1}, x_{n(k)}).\end{aligned} \tag{2.15}$$

Using (2.6) and (2.10), we acquire

$$\frac{\epsilon}{s^2} \leq \lim_{k\to\infty} \inf \sigma_b(x_{m(k)+1}, x_{n(k)-1}). \tag{2.16}$$

Moreover

$$\epsilon \leq \sigma_b(x_{m(k)}, x_{n(k)}) \leq s\sigma_b(x_{m(k)}, x_{m(k)+1}) + s\sigma_b(x_{m(k)+1}, x_{n(k)}). \tag{2.17}$$

With taking the upper limit as $k \to \infty$ in (2.17), we have

$$\frac{\epsilon}{s} \leq \lim_{k\to\infty} \sup \sigma_b(x_{m(k)+1}, x_{n(k)}). \tag{2.18}$$

By using (2.2), we have

$$\begin{aligned}&F\big(s\sigma_b(x_{m(k)+1}, x_{n(k)})\big) \\ &= F\big(s\sigma_b(Tx_{m(k)}, Tx_{n(k)-1})\big) \\ &\leq \theta\Big(\phi\big(M_s(x_{m(k)}, x_{n(k)-1})\big)\Big)F\big(M_s(x_{m(k)}, x_{n(k)-1})\big) - \psi\big(\sigma_b(x_{m(k)}, x_{n(k)-1})\big)\end{aligned} \tag{2.19}$$

where

(2.20)

$$M_s(x_{m(k)}, x_{n(k)-1})$$
$$= \max\Big\{\sigma_b(x_{m(k)}, x_{n(k)-1}), \sigma_b(x_{m(k)}, x_{m(k)+1}), \frac{\sigma_b(x_{m(k)}, x_{m(k)+1})\sigma_b(x_{n(k)-1}, x_{n(k)})}{1+\sigma_b(x_{m(k)+1}, x_{n(k)+1})}\Big\}$$

on taking the upper limit as $k \to \infty$, from (2.6),(2.8),(2.10) and (2.14), we obtain

$$\lim_{k\to\infty} \sup M_s(x_{m(k)}, x_{n(k)-1}) = \max\{\epsilon, 0, 0\} = \epsilon. \tag{2.21}$$

Undoubtedly, $\frac{1+s\sigma_b(x_{m(k)}, x_{n(k)-1})}{1+\frac{1}{2}\sigma_b(x_{m(k)}, x_{m(k)+1})} \geq 1$ as $n \to \infty$. Thus

$$\lim_{k\to\infty} F\big(s \sup \sigma_b(x_{m(k)+1}, x_{n(k)})\big)$$
$$\leq \lim_{k\to\infty} F\Big(s\frac{1+s\sigma_b(x_{m(k)}, x_{n(k)-1})}{1+\frac{1}{2}\sigma_b(x_{m(k)}, x_{m(k)+1})}\Big)\sigma_b(x_{m(k)+1}, x_{n(k)})$$

therefore, we have from (2.1)

$$F\Big(s\frac{\epsilon}{s}\Big) \leq \lim_{k\to\infty} F\big(s \sup \sigma_b(Tx_{m(k)}, Tx_{n(k)-1})\big)$$
$$\leq \lim_{k\to\infty} \theta\Big(\phi\big(M_s(x_{m(k)}, x_{n(k)-1})\big)\Big)F\big(M_s(x_{m(k)}, x_{n(k)-1})\big)$$
$$- \psi\big(\sigma_b(x_{m(k)}, x_{n(k)-1})\big),$$
$$F(\epsilon) \leq \lim_{k\to\infty} \theta\Big(\phi\big(M_s(x_{m(k)}, x_{n(k)-1})\big)\Big)F(\epsilon) - \psi(\epsilon)$$

which is a contradiction in view of the definition of θ and ψ as $\theta \in [0,1)$ and $\psi \in (0,\infty)$. Thus $\{x_n\}$ is a σ_b Cauchy sequence. As $(X, \sigma_b, \precsim)$ is complete therefore, the sequence $\{x_n\}$ converges to some point $u \in X$.

Since T is continuous, u is a fixed point of T,

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tu.$$

This amount to say that u is a fixed point of T.

Uniqueness: To prove the uniqueness of the fixed point u, let v be the another fixed point of T i.e. $Tv = v$ such that $p_b(u,v) > 0$. From (2.1), we obtain that

$$F\big(\sigma_b(u,v)\big) = F\big(\sigma_b(Tu,Tv)\big) \leq F\big(s\sigma_b(Tu,Tv)\big) \leq F\Big(s\frac{1+s\sigma_b(u,v)}{1+\frac{1}{2}\sigma_b(u,Tu)}\Big)\sigma_b(Tu,Tv)$$
$$\leq \theta\Big(\phi\big(M_s(u,v)\big)\Big)F\big(M_s(u,v)\big) - \psi\big(\sigma_b(u,v)\big)$$

where

$$M_s(u,v) = \max\Big\{\sigma_b(u,v), \sigma_b(u,Tu), \frac{\sigma_b(u,Tu)\sigma_b(v,Tv)}{1+\sigma_b(Tu,Tv)}\Big\} = \sigma_b(u,v).$$

Therefore, using the definition of θ and ψ along with the value of $M_s(u,v)$ the above inequality turns into the following

$$\begin{aligned} F\big(\sigma_b(u,v)\big) &\leq \theta\Big(\phi\big(\sigma_b(u,v)\big)\Big)F\big(\sigma_b(u,v)\big) - \psi\big((\sigma_b(u,v)\big) \\ F\big(\sigma_b(u,v)\big) &\leq F\big(\sigma_b(u,v)\big) - \psi\big((\sigma_b(u,v)\big) \end{aligned}$$

is a contradiction. Thus, $\sigma_b(Tu,Tv) = \sigma_b(u,v) = 0$, i.e., $u = v$, this shows that the fixed point is unique. This complete the proof of theorem. □

Theorem 2.3. *Under the same hypothesis of Theorem 2.2 and without assuming the continuity of T, assume that whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x \in X$ implies $x_n \precsim x$ for all $n \in N$, then T has a fixed point $u \in X$.*

Proof. Following similar arguments to those given in Theorem 2.2, we construct a nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u$ for some $u \in X$. Using the assumption of X, we have $x_n \preceq u$ for every $n \in N$. Now, we show that $Tu = u$.

Suppose $F(Tu,u) = \lim_{n\to\infty} F(Tu, x_{n+1}) = \lim_{n\to\infty} F(Tu, Tx_n) > 0$.
Suppose that there exist $n_0 \in N_1$, such that

$$\frac{1}{2}\sigma_b(x_{n_0}, Tx_{n_0}) > s\sigma_b(x_{n_0}, u)$$

and

$$\frac{1}{2}\sigma_b(x_{n_0+1}, Tx_{n_0+1}) > s\sigma_b(x_{n_0+1}, u).$$

Then from (2.4), it follows that

$$\begin{aligned} \sigma_b(x_{n_0+1}, x_{n_0}) &\leq s\sigma_b(x_{n_0}, u) \\ &< \frac{1}{2}\sigma_b(x_{n_0}, Tx_{n_0}) + \frac{1}{2}\sigma_b(x_{n_0+1}, Tx_{n_0+1}) \\ &= \frac{1}{2}\sigma_b(x_{n_0}, x_{n_0+1}) + \frac{1}{2}\sigma_b(x_{n_0+1}, x_{n_0+2}) \\ &\leq \frac{1}{2}\sigma_b(x_{n_0}, x_{n_0+1}) + \frac{1}{2}\sigma_b(x_{n_0}, x_{n_0+1}) \\ &= \sigma_b(x_{n_0+1}, x_{n_0}) \end{aligned}$$

which is a contradiction. Hence either

$$\frac{1}{2}\sigma_b(x_n, Tx_n) \leq s\sigma_b(x_n, u)$$

and

$$\frac{1}{2}\sigma_b(x_{n+1}, Tx_{n+1}) \leq s\sigma_b(x_{n+1}, u).$$

for all $n \in N_1$. It is not restrictive to assume that one of these inequalities holds for all $n \in N_1$, for example

$$\frac{1}{2}\sigma_b(x_n, Tx_n) \leq s\sigma_b(x_n, u). \tag{2.22}$$

By (2.1) and (2.22), we have

$$\begin{aligned} F\big(s\sigma_b(Tu, Tx_{n-1})\big) &\leq F\Big(s\frac{1+\sigma_b(u,x_n)}{1+\frac{1}{2}\sigma_b(x_n,Tx_n)}\Big)\sigma_b(Tu,x_n) \\ &\leq \theta\Big(\phi\big(M_s(u,x_n)\big)\Big)F\big(M_s(u,x_n)\big) - \psi\big(\sigma_b(u,x_n)\big), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \sigma_b(Tu,u) &\leq M_s(u,x_n) \\ &= \max\Big\{\sigma_b(u,x_n), \sigma_b(Tu,u), \frac{\sigma_b(u,Tu)\sigma_b(x_n,Tx_n)}{1+\sigma_b(Tu,Tx_n)}\Big\} \\ &= \max\Big\{\sigma(u,x_n), \sigma(Tu,u), \frac{\sigma_b(u,Tu)\sigma_b(x_n,x_{n+1})}{1+\sigma_b(Tu,x_{n+1})}\Big\} \end{aligned}$$

taking limit as $n \to \infty$, by Definition 1.6, we obtain

$$\sigma_b(Tu,u) \leq \lim_{n\to\infty} M_s(u,x_n) \leq \sigma_b(u,Tu)$$

$$\Rightarrow \quad M_s(u,x_n) = \sigma_b(u,Tu)$$

therefore, letting $n \to \infty$ in (2.23), we get

$$F\big(\sigma_b(Tu,u)\big) \leq \theta\Big(\phi\big(\sigma_b(Tu,u)\big)\Big)F\big(\sigma_b(Tu,u)\big) - \psi\big(\sigma_b(Tu,u)\big),$$

which is a contradiction in view of $F1, \theta, \phi$ and ψ. Then $\sigma(Tu,u) = 0$. Thus $Tu = u$. □

Some consequences:

By choosing $\psi(\sigma_b(x,y)) = \tau \geq 0$ in Theorem 2.2, Berinde-Wardowiski type result is obtained in the setting of partially ordered b-metric like spaces as follows.

Corollary 2.4. *Theorem 2.2 remains true, if we replace the assumption* (2.1) *by the following (besides retaining the rest of the hypotheses):*

$$F\Big(s\Big(\frac{1+s\sigma_b(x,y)}{1+\frac{1}{2}\sigma_b(x,Tx)}\Big)\sigma_b(Tx,Ty)\Big) \leq \theta\Big(\phi\big(M_s(x,y)\big)\Big)F\big(M_s(x,y)\big) - \tau.$$

Further by taking $\phi(t) = t$ in Corollary 2.4, we have the following Corollary as a consequence of Theorem 2.2.

Corollary 2.5. *Let (X, σ_b) be a complete partially ordered b-metric like space with $s > 1$. Let T be a continuous self mapping on X. If there exist, $\theta \in \Theta, F \in \Delta_F, \tau > 0$, such that for all $x, y \in X$ with $\sigma_b(Tx, Ty) \geq 0$,*

$$F\Big(s\Big(\frac{1 + s\sigma_b(x, y)}{1 + \frac{1}{2}\sigma_b(x, Tx)}\Big)\sigma_b(Tx, Ty)\Big) \leq \theta\big(M_s(x, y)\big)F\big(M_s(x, y)\big) - \tau,$$

where

$$M_s(x, y) = \max\Big\{\sigma_b(x, y), \sigma_b(x, Tx), \frac{\sigma_b(x, Tx)\sigma_b(y, Ty)}{1 + \sigma_b(Tx, Ty)}\Big\}.$$

Then T has a unique fixed point.

Example 2.6. Let $X = [0, 10]$ be equipped with the partial order relation $\preceq$ defined by $x \preceq y \iff x > y$ and the function $\sigma_b : X \times X \to [0, \infty)$ is defined by

$$\sigma_b(x, y) = [\max\{x, y\}]^2; \quad \text{for all } \ x, y \in X, \ \text{ where } \ s = 2.$$

It is obvious that, $(X, \sigma_b, \precsim)$ is a complete partially ordered b-metric like space. Let the mapping $T : X \to X$ is defined by $Tx = \frac{1}{8}x^4 e^{\frac{-3x}{2}}$. Define $\theta : [0, \infty) \to [0, 1)$ by $\theta(t) = \frac{1}{t+2}$, and let $\phi : [0, \infty) \to [0, \infty)$ be given by $\phi(t) = \frac{t}{10}$, also let $\psi : (0, \infty) \to (0, \infty)$ by $\psi(t) = \frac{1}{t^2+1}$. Let $F(t) = \log t$ for all $t \in \mathbb{R}^+$.
We are now verifying the contractive condition (2.1) of Theorem 2.2, which requires assessing the following cases:

Case I. If $x, y \in [0, 1]$, then

$$\begin{aligned} L.H.S. &= F\Big(s\Big(\frac{1 + s\sigma_b(x, y)}{1 + \frac{1}{2}\sigma_b(x, Tx)}\Big)\sigma_b(Tx, Ty)\Big) \\ &= F\left(2\Big(\frac{1 + 2x^2}{1 + \frac{1}{2}x^2}\Big) \max\Big\{\frac{1}{8}x^4 e^{\frac{-3c}{2}}, \frac{1}{8}y^4 e^{\frac{-3x}{2}}\Big\}^2\right) \\ &= F\left(2\Big(\frac{1 + 2x^2}{1 + \frac{1}{2}x^2}\Big)\Big(\frac{1}{64}x^8 e^{-3x}\Big)\right) \\ &= log\Big[\Big(\frac{2 + x^2}{1 + \frac{1}{2}x^2}\Big)\Big(\frac{1}{64}x^8 e^{-3x}\Big)\Big]. \end{aligned} \tag{2.24}$$

Clearly when $x, y \in [0, 1]$

$$M_s(x, y) = x^2.$$

Then R.H.S. becomes

$$R.H.S. = \theta\Big(\phi\Big(x^2\Big)\Big)F\Big(x^2\Big) - \psi(x^2)$$

$$= \theta\Big(\frac{x^2}{10}\Big)(\log x^2) - \frac{1}{x^4+1} \tag{2.25}$$

$$= \frac{10}{x^2+20}(\log x^2) - \frac{1}{x^4+1}.$$

From the following figure it is clear that R.H.S. expression (with Violet curve) dominates the L.H.S. expression (with Brown curve). This implies that (2.1) is true for $x, y \in [0, 1]$.

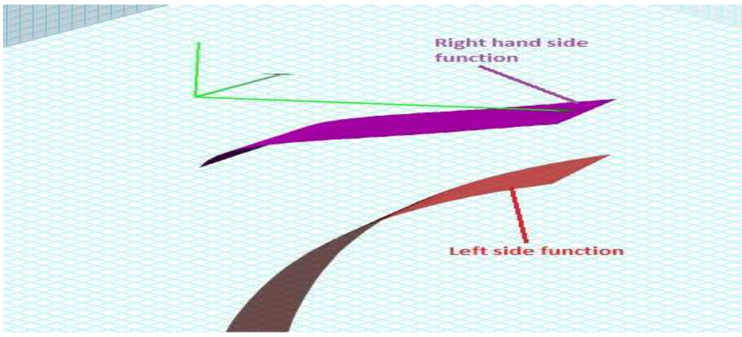

Figure 1. Plot of inequality for Case I

CASE II. If $x, y \in (1, 10]$, then

(2.26)

$$L.H.S. = F\Big(s\Big(\frac{1+s\sigma_b(x,y)}{1+\frac{1}{2}\sigma_b(x,Tx)}\Big)\sigma_b(Tx,Ty)\Big) = log\Big[\Big(\frac{2+x^2}{1+\frac{1}{2}x^2}\Big)\Big(\frac{1}{64}x^8e^{-3x}\Big)\Big].$$

$$Now, \quad M_s(x,y) = \max\Big\{\sigma_b(x,y), \sigma_b(x,Tx), \frac{\sigma_b(x,Tx)\sigma_b(y,Ty)}{1+\sigma_b(Tx,Ty)}\Big\}$$

$$= \max\Big\{x^2, x^2, \frac{x^2y^2}{1+\frac{1}{64}x^2e^{-3x}}\Big\}$$

$$= \frac{x^2y^2}{1+\frac{1}{64}x^2e^{-3x}}, \quad \text{when} \;\; x, y \in (1, 10].$$

$$Thus, \quad R.H.S. = \frac{10}{\frac{x^2y^2}{1+\frac{1}{8}x^4e^{-3x}}+20} log\Big[\frac{x^2y^2}{1+\frac{1}{8}x^4e^{-3x}}\Big] - \frac{1}{x^4+1}.$$

Again, following figure shows that R.H.S. expression (with Violet curve) is superimposing the L.H.S. expression (with Brown curve), which authenticates our inequality. On the basis of above two cases one can easily verify that in all other possibilities

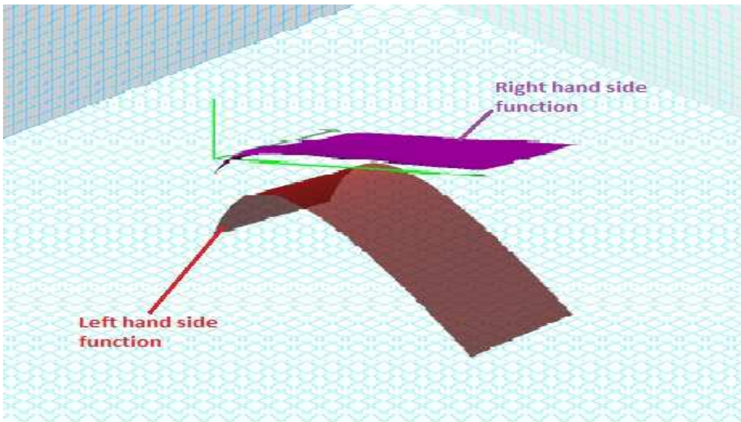

Figure 2. Plot of inequality for Case II.

and cases inequality (2.1) is true. Thus, all the conditions of Theorem 2.2 are fulfilled. Then Tx has a unique fixed point as $x = 0$, which is demonstrated by the following figure.

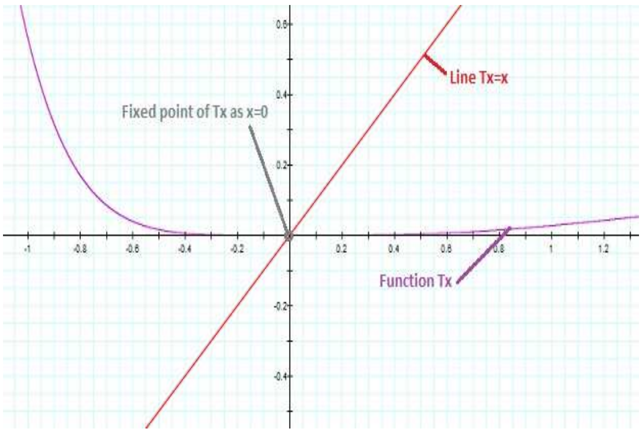

Figure 3. Fixed point of the mapping T

3. Conclusion

In this study, recognizing the concept of F-contraction, some fixed point theorems for Geraghty type generalized F- contraction in partially ordered b-metric-like spaces are established. The illustrative example show the high degree of reliability to other authors to generalize and improve these results for future research.

References

1. M. Abbas, T. Nazir & S. Romaguera: Fixed point results for generalized cyclic contraction mappings in partial metric spaces. RACSAM **106** (2012), 287-297. https://doi.org/10.1007/s13398-011-0051-5
2. M.A. Alghamdi, N. Hussain & P. Salimi: Fixed point and coupled fixed point theorems on b-metric-like spaces. J. Inequal. Appl. 2013, **402** (2013). https://doi.org/10.1186/1029-242X-2013-402
3. I. Altun, F. Sola & H. Simsek: Generalized contractions on partial metric spaces. Topology and Its Applications **157** (2010), no. 18, 2778-2785. https://doi.org/10.1016/j.topol.2010.08.017
4. I. Altun & S. Romaguera: Characterizations of partial metric completeness in terms of weakly contractive mappings having fixed point. Applicable Analysis and Discrete Mathematics **6** (2012), no. 2, 247-256. doi:10.2298/AADM120322009A
5. A. Amini-Harandi: Metric-like spaces, partial metric spaces and fixed points. Fixed Point Theory and Applications **2012**, 2012:204. https://doi.org/10.1186/1687-1812-2012-204
6. H. Aydi, S.H. Amor & E. Karapnar: Berinde-type generalized contractions on partial metric spaces. Abstract and Applied Analysis **2013**, Article ID 312479. https://doi.org/10.1155/2013/312479
7. H. Aydi, M. Jellali & E. Karapnar: Common fixed points for generalized α-implicit contractions in partial metric spaces, consequences and application. RACSAM **109** (2015), 367-384. https://doi.org/10.1007/s13398-014-0187-1
8. I.A. Bakhtin: The contraction mapping principle in quasi metric spaces. Funct. Anal. Unianowsk Gos. Ped. Inst. **30** (1989), 26-37.
9. S. Chandok: Some fixed point theorems for (α, β)-admissible geraghty type contractive mappings and related results. Math Sci. **9** (2015), 127-135. https://doi.org/10.1007/s40096-015-0159-4
10. C.F. Chen, J. Dong & C.X. Zhu: Some fixed point theorems in b-metric-like spaces. Fixed Point Theory Appl. **2015** (2015). DOI 10.1186/s13663-015-0369-3
11. S. Czerwik: Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostrav. **1** (1993), 5-11. https://eudml.org/doc/23748
12. S. Czerwik: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46** (1998), 263-276.
13. R.C. Dimri & G. Prasad: Coincidence theorems for comparable generalized nonlinear contractions in ordered partial metric spaces. Commun. Korean Math. Soc. **32** (2017), no. 2, 375-387. https://doi.org/10.4134/CKMS.c160127
14. M. Geraghty: On contractive mappings. Proc. Amer. Math. Soc. **40** (1973), 604-608.

15. S.K. Malhotra, S. Radenovic & S. Shukla: Some fixed point results without monotone property in partially ordered metric-like spaces. Journal of the Egyptian Mathematical Society **22** (2014), 83-89. `https://doi.org/10.1016/j.joems.2013.06.010`
16. S.G. Matthews: Partial metric topology, in Proceeding of the 8th summer conference on General Topology and Application. Ann. New York Acad. Sci. **728** (1994), 183-197.
17. H. Piri & P. Kumam: Some fixed point theorems concerning F-contraction in complete metric spaces. Fixed Point Theory and Appl. **2014** (2014), 210. `https://doi.org/10.1186/1687-1812-2014-210`
18. G. Prasad: Fixed points of Kannan contractive mappings in relational metric spaces. J. Anal **29** (2021), 669-684. `https://doi.org/10.1007/s41478-020-00273-7`
19. G. Prasad & H. Isik: On Solution of Boundary Value Problems via Weak Contractions. Journal of Function Spaces **2022** (2022), Article ID 6799205. `https://doi.org/10.1155/2022/6799205`
20. N.A. Secelean: Iterated function system consisting of F-contractions. Fixed Point Theory and Appl. **2013** (2013), 277. `doi:10.1166/1687-1812-2013-277`
21. D. Wardowski: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory and Appl. **2012** (2012), 94. `https://doi.org/10.1186/1687-1812-2012-94`

[a]Professor: Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur, India
Email address: `chauhaan.op@gmail.com, opchauhan@jecjabalpur.ac.in`

[b]Professor: Department of Applied Mathematics, Jabalpur Engineering College, Jabalpur, India
Email address: `joshnvishal76@gmail.com`

[c]Professor: Department of Computer Science and Engineering, Jabalpur Engineering College, Jabalpur, India
Email address: `ssingh@jecjabalpur.ac.in`