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GENERATING OPERATORS OF I-TRANSFORM OF THE MELLIN CONVOLUTION TYPE†

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Abstract. In this paper, the I-transform of the Mellin convolution type is presented. Based on the Mellin transform theory, a general integral transform of the Mellin convolution type is introduced. The generating operators for I-transform together with the corresponding operational relations are also presented.

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1. Introduction and Preliminaries

From functional point of view integral transform is a useful technique. We not only deal with mapping properties of the integral transforms but our aim is to show how they can be applied to different problems of mathematics, physical sciences like the solutions of ordinary and partial differential equations [1]. While considering certain problems of mathematical physics all integral transforms are not arbitrary linear operators. For all integral transforms both their inverse operators and the generating operators are known. The classical one-dimensional integral transforms are of the form [2]

$$[Kf](x) = \zeta(x) = \int_{-\infty}^{\infty} k(x,t)f(t)dt,$$

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where $k(x,t)$ is some given function, $f(t)$ is original in certain space of functions and $\zeta(x)$ is image of the function $f(t)$. One of the most important integral transforms which will be the base of our investigations is the Mellin transform [3]

$$f^*(s) = M\{f(t) : s\} = \int_0^\infty f(t)t^{s-1}dt,$$

and its inverse is given by the formula

$$f(x) = M^{-1}\{f(s) : x\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{-s}ds, \qquad \gamma = Re(s).$$

A mathematical theory of transformations of this type can be developed by using the properties of Banach spaces. The initial and boundary value problems in mathematics and mathematical physics can be effectively solved by the use of integral transforms [1]. Various authors have established integrals involving special functions, for instance Srivastava, Kilicman and khan [4, 5, 6], Zemanian and many others [7, 8, 9, 10, 11]. In view of the above mentioned work, this paper gives generating function of I-transform involving Saxena's I-function as a kernel.
The I-function has been defined through a contour integral of the Mellin-Barnes type by Saxena in [12] as

$$\begin{aligned} I(x) &= I^{m,n}_{p_i,q_i:r}[x] \\ &= I^{m,n}_{p_i,q_i:r}\left[x \left| \begin{matrix} (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right.\right] \\ &= \frac{1}{2\pi i}\int_L \Phi(s)x^s ds, \end{aligned} \tag{1}$$

where

$$\Phi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(1-a_{ji}-\alpha_{ji}s)\right\}} \tag{2}$$

$p_i(i=1,2,...,r),\ q_i(i=1,2,...,r)\ m,n$ are integers satisfying $0 \le n \le p_i,\ 0 \le m \le q_i\ (i=1,2,...,r)$; r is finite $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are complex numbers.

defn 1.1. *The I-transform of a function is defined in* [8] *as follows*

$$F(x) = (If)(x)$$

$$= \quad I^{p_i,r}_{\widehat{m_i},\widehat{a_i},\widehat{\alpha_i}} f(x) \tag{3}$$

$$= \quad \frac{1}{2\pi i}\int\limits_{\sigma}\left(\sum_{i=1}^{r} X^{p_i}_{\widehat{m_i},\widehat{a_i},\widehat{\alpha_i}}(s)\right)^{-1} f^*(s)x^{-s}ds,\ x>0 \tag{4}$$

where from [13]

$$X^{p_i}_{\widehat{m_i},\widehat{a_i},\widehat{\alpha_i}}(s)=\prod_{j=1}^{p_i}\Gamma^{m_{ij}}(b_{ij}+\alpha_{ij}s),\ m_{ij}\in\mathbb{Z},\ p_i\in\mathbb{N}.$$

$b_{ij}=\frac{1}{2}-(a_{ij}-\frac{1}{2})sgn(\alpha_{ij}), a_{ij}\in\mathbb{C},\alpha_{ij}\in\mathbb{R},\ \ \widehat{m_i}=(m_{i1},m_{i2},...,m_{ip_i}),$
$\widehat{a_i}=(a_{i1},a_{i2},...,a_{ip_i}),\ \widehat{\alpha_i}=(\alpha_{i1},\alpha_{i2},...,\alpha_{ip_i}),\ \alpha_{ij}+1>(2Re(a_{ij}-1))sgn(\alpha_{ij}),\ j=\widehat{1,p_i}, i=\widehat{1,r}.$
$f^*(s)$ *is the Mellin transform* [14] *of function* $f(x),\ \sigma=\left\{s\in\mathbb{C}, Re(s)=\frac{1}{2}\right\}.$

The parameters $\widehat{a_i},\widehat{\alpha_i}$ are chosen so that $\sum\limits_{i=1}^{r} X^{p_i}_{\widehat{m_i},\widehat{a_i},\widehat{\alpha_i}}(s)\neq 0$ on the contour σ.
A special case of I-transform is H-transform [13, 15] that is $I^{p,1}_{\widehat{m},\widehat{a},\widehat{\alpha}}=H^{p}_{\widehat{m},\widehat{a},\widehat{\alpha}}$
Also, a new version of I-transform is defined by Pankaj and Saxena in [16], but we consider the earlier one.

From [17], we deduce the following definition.

defn 1.2. *Let* $c,\gamma\in\mathbb{R}$ *and* $2sgn(c)+sgn(\gamma)\geq 0$. *Denote by* $M^{-1}_{c,\gamma}(L)$ *the space of functions given in the form*

$$f(x)=\frac{1}{2\pi i}\int\limits_{\sigma} f^*(s)x^{-s}ds$$

where $f^*(s)|s|^{\gamma}e^{\pi c|J(s)|}\in L(\sigma)$ *and in this case above integral converges if* $c\geq 0,\gamma\in\mathbb{R}$ *or* $c=0,\ \gamma\geq 0$.

The space $M^{-1}_{c,\gamma}(L)$ is a Banach Space with the norm

$$||f||_{M^{-1}_{c,\gamma}(L)}=\frac{1}{2\pi}\int\limits_{\sigma} e^{\pi c|J(s)|}\left|s^{\gamma}f^*(s)ds\right|.$$

Convolution: Consider the integral transform $I: U(X)\to V(Y)$ where $U(X)$ is a linear space, $V(Y)$ is an algebraic space. The convolution of two functions f and g denoted by the symbol $f*g$, is an operator such that $I(f*g)(y)=(If)(y)\cdot(Ig)(y),\ y\in Y$ holds (see [8]).
The convolution of functions f,g for the Fourier transform defined by Churchill

[18] is as follows,

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt,$$

and the Mellin and Laplace transform has been investigated in [18] as

$$(f*g)(x) = \int_{0}^{\infty} f\left(\frac{x}{t}\right) g(t)\frac{dt}{t},$$

$$(f*g)(x) = \int_{0}^{x} f(x-t)g(t)dt.$$

A generalized convolution of functions f and g with some weight function γ is denoted by $f*g$ such that $I(f*^{\gamma} g)(x) = \gamma(x)(I_1 f)(x)(I_2 g)(x)$ under three under three operators I, I_1, I_2.

Generating operators in the space $_{\lambda}M^{-1}_{c,\gamma}$: We consider the special space of functions, as considered by B.L.J. Braaksma and A. Schuitman and suitable for studying generating operators of the I-transforms. The term generating operators of the integral transform has been introduced by M.V. Fedoryuk [19].

defn 1.3. *Let $\lambda \geq 0$ and $c \geq 0$, $\gamma \in R$ be such that $2sgn(c) + sgn(\gamma) \geq 0$. $_{\lambda}M^{-1}_{c,\gamma}$ denotes the space of functions $f(x)$, $x > 0$ representable in the form*

$$f(x) = \frac{1}{2\pi i} \int_{\sigma\left(\frac{1}{2}\right)} f^*(s)x^{-s}ds, \quad x > 0 \tag{5}$$

$$f^*(s) = \left|\frac{1}{2} + iJ(s)\right|^{-\gamma} e^{-\pi c|J(s)|}F(s),$$

where $\sigma(t) = \{\tau \in C,\ Re(\tau) = t\}$ and if $\left|Re(s) - \frac{1}{2}\right| \leq \lambda$, then $\int\limits_{\sigma(Re(s))} |F(s)ds| \leq C$, C is an absolute constant, $F(s) \to 0$ if $J(s) \to \infty$ uniformly with respect to $Re(s)$ and $f^*(s)$ is an analytic function in this strip.

It follows from the family of spaces $M^{-1}_{c,\gamma}$ that $M^{-1}_{c_1,\gamma_1}(L) \subset M^{-1}_{c,\gamma}(L)$ holds if and only if $2sgn(c_1 - c) + sgn(\gamma_1 - \gamma) \geq 0$.
where c_1, γ_1 are characteristic numbers.
The set $_{\lambda}M^{-1}_{c,\gamma}$ of spaces is a partially ordered one $_{\lambda_1}M^{-1}_{c_1,\gamma_1} \subseteq\ _{\lambda}M^{-1}_{c,\gamma}$ under the conditions

$$\lambda_1 \geq \lambda,\ 2sgn(c_1 - c) + sgn(\gamma_1 - \gamma) \geq 0. \tag{6}$$

Also, the space ${}_{\lambda}M^{-1}_{c,\gamma}$ is a subspace of the space $M^{-1}_{c,\gamma}(L)$.

2. Main results:

thm 2.1. *If $f(x) \in {}_{\lambda}M^{-1}_{c,\gamma}$, for any $\rho \in R$, such that $|\rho| \leq \lambda$, we have*

$$x^{\rho} f(x) \in {}_{\lambda_1}M^{-1}_{c,\gamma}$$

where $\lambda_1 = min\{\lambda - \rho, \lambda + \rho\}$ and

$$x^{\rho} f(x) = \frac{1}{2\pi i} \int_{\sigma\left(\frac{1}{2}\right)} f^*(s+\rho) x^{-s} ds, \quad x > 0. \tag{7}$$

Proof. Let us consider the integral representation (5) for the function $f(x)$. we have,

$$|f^*(s)| = \left|\frac{1}{2} + iJ(s)\right|^{-\gamma} e^{-\pi c|J(s)|} |F(s)| \leq C_1 |F(s)|, \tag{8}$$

where C_1 does not depend on $Re(s)$ in the strip $\left|Re(s) - \frac{1}{2}\right| \leq \lambda$ by virtue of relation $2sgn(c) + sgn(\gamma) \geq 0$. Consequently, $f^*(s) \to 0$ if $J(s) \to \infty$ uniformly with respect to $Re(s)$ in this strip.
Let us consider the following integral

$$\frac{1}{2\pi i} \int_{L_N} f^*(s) x^{-s} ds, \quad L_N = L_{N_1} \cup L_{N_2} \cup L_{N_3} \cup L_{N_4} \tag{9}$$

where

$$\left\{ \begin{array}{r} L_{N_1} = \{s \in \mathbb{C} : Re(s) = \frac{1}{2}, |J(s)| \leq N\}, \\ L_{N_2} = \{s \in \mathbb{C} : \frac{1}{2} \leq Re(s) \leq \rho + \frac{1}{2}, J(s) = N\}, \\ L_{N_3} = \{s \in \mathbb{C} : Re(s) = \frac{1}{2} + \rho, |J(s)| \leq N\}, \\ L_{N_4} = \{s \in \mathbb{C} : \frac{1}{2} \leq Re(s) \leq \rho + \frac{1}{2}, J(s) = -N\}, \end{array} \right\}.$$

The function $f(s)$ is analytic in the domain which is bounded by the contour L_N, thus we can apply Cauchy theorem to obtain

$$\int_{L_N} f(s) x^{-s} ds = 0, \quad N \in R_+. \tag{10}$$

we have also

$$\int_{L_N} f(s) x^{-s} ds$$

$$= \int_{L_{N_1}} f(s)x^{-s}ds + \int_{L_{N_2}} f(s)x^{-s}ds + \int_{L_{N_3}} f(s)x^{-s}ds + \int_{L_{N_4}} f(s)x^{-s}ds$$
$$= I_{N_1}(x) + I_{N_2}(x) + I_{N_3}(x) + I_{N_4}(x) \tag{11}$$

Let us consider $\lim\limits_{N\to\infty} \int_{L_N} f(s)x^{-s}ds$. From relation (10) we have

$$\int_{L_N} f(s)x^{-s}ds = 0 \tag{12}$$

and from relation (11)

$$\lim_{N\to\infty} \int_{L_N} f(s)x^{-s}ds = \lim_{N\to\infty} (I_{N_1}(x) + I_{N_2}(x) + I_{N_3}(x) + I_{N_4}(x)). \tag{13}$$

Using relation (8), Definition 1.3, and the mean value theorem we obtain

$$\lim_{N\to\infty} I_{N_2}(x) = \lim_{N\to\infty} I_{N_4}(x) = 0. \tag{14}$$

Finally, using relations (12)-(14) and changing variables, we have

$$f(x) = \frac{1}{2\pi i}\int_{\sigma(\frac{1}{2})} f^*(s)x^{-s}ds = \frac{1}{2\pi i}\int_{\sigma(\frac{1}{2}+\rho)} f^*(s)x^{-s}ds \tag{15}$$
$$= x^{-\rho}\frac{1}{2\pi i}\int_{\sigma(\frac{1}{2})} f^*(s+\rho)x^{-s}ds. \tag{16}$$

We obtain relation (7) and the fact that $x^\rho f(x) \in {}_{\lambda_1}M^{-1}_{c,\gamma}$, where $\lambda_1 = min\{\lambda-\rho, \lambda+\rho\}$ follows easily from it and Definition 1.3.

□

thm 2.2. *If $f(x) \in {}_{\lambda}M^{-1}_{c,\gamma}$. Then the I-transform (4) of the function $f(x)$ is a function from ${}_{\lambda}M^{-1}_{c_1,\gamma_1}$, where*

$$c_1 = c + k,$$
$$\gamma_1 = min\left\{\gamma+\mu-\lambda\left|\sum_{j=1}^{n} a_j - \sum_{j=1}^{m} b_j\right|, \gamma+\mu-\lambda\left|\sum_{j=1}^{p_i} a_{ji} - \sum_{j=1}^{q_i} b_{ji}\right|\right\},$$
$$i = 1,2,...,r, \quad k = \frac{1}{2}\left(\sum_{j=1}^{m} b_j + \sum_{j=1}^{n} a_j - \sum_{j=n+1}^{p_i} a_{ji} - \sum_{j=m+1}^{q_i} b_{ji}\right) \tag{17}$$

and

$$\mu = Re\left(\sum_{j=1}^{p_i}\alpha_{ji} - \sum_{j=1}^{q_i}\beta_{ji}\right) + \frac{1}{2}\left(\sum_{j=1}^{p_i} a_{ji} - \sum_{j=1}^{q_i} b_{ji}\right) - \frac{p_i - q_i}{2} \tag{18}$$

under the conditions

$$2sgn(c_1) + sgn(\gamma_1) \geq 0,\ \lambda < min\{\frac{1}{2} - \rho_l,\ \rho_h - \frac{1}{2}\}, \tag{19}$$

where

$$\rho_l = \max_{1\leq j\leq q_i}\left\{-\frac{\beta_{ji}}{b_{ji}},\right\},\quad \rho_h = \min_{1\leq j\leq p_i}\left\{\frac{1-\alpha_{ji}}{a_{ji}}\right\}. \tag{20}$$

Proof. In accordance with the Definition 1.3, we need to consider the function $\Phi(s)f^*(s)$, where $\Phi(s)$ is determined by relation (2) and $f^*(s)$ by relation (5). Using the properties of Euler's $\Gamma-$function we obtain that the function $\Phi(s)$ is analytic in the strip $\rho_l < Re(s) < \rho_h$ and consequently, in the strip $\left|Re(s) - \frac{1}{2}\right| \leq \lambda$ and we have the following inequality in this strip

$$\begin{aligned}|\Phi(s)| &\leq C_2\left|\frac{1}{2}+iJ(s)\right|^{-\mu-\left(Re(s)-\frac{1}{2}\right)\left(\sum\limits_{j=1}^{n}a_j-\sum\limits_{j=1}^{m}b_j\right)} e^{-\pi k|J(s)|}\\ &\leq C_3\left|\frac{1}{2}+iJ(s)\right|^{-\mu+\lambda\left|\sum\limits_{j=1}^{n}a_j-\sum\limits_{j=1}^{m}b_j\right|} e^{-\pi k|J(s)|}\end{aligned}$$

where C_3 is an absolute constant. Thus we obtain that $\Phi(s)f^*(s)$ is an analytic function in the strip $\left|Re(s) - \frac{1}{2}\right| \leq \lambda$ and in the same strip

$$\begin{aligned}&|f^*(s)\Phi(s)||\frac{1}{2}+iJ(s)|^{\gamma_1}e^{\pi c_1|J(s)|}\\ &\leq C_3|F(s)||\frac{1}{2}+iJ(s)|^{-\gamma} * e^{-\pi c}|J(s)|\left|\frac{1}{2}+iJ(s)\right|^{-\mu+\lambda|\sum\limits_{j=1}^{n}a_j-\sum\limits_{j=1}^{m}b_j|} e^{-\pi k|J(s)|}\\ &\times\left|\frac{1}{2}+iJ(s)\right|^{\gamma_1} e^{-\pi c_1|J(s)|} = C_3|F(s)|\end{aligned}$$

where $\int\limits_{\sigma(Re(s))} |F(s)ds| \leq C$, C is an absolute constant, $F(s) \to 0$ if $J(s) \to \infty$ uniformly with respect to $Re(s)$. □

rem 2.1. *If the conditions of above theorem are fulfilled for the function* $f(x) \in {}_\lambda M_{c,\gamma}^{-1}$ *and the I-transform , then in accordance with theorem 1 for any* $\rho \in \mathbb{R}$ *such that* $|\rho| \leq \lambda$, *we can represent this I-transform in the form*

$$(If)(x) = x^{-\rho}\frac{1}{2\pi i}\int\limits_{\sigma\left(\frac{1}{2}\right)} \Phi(s+\rho)f^*(s+\rho)x^{-s}ds \tag{21}$$

and $x^{-\rho}(If)(x) \in {}_{\lambda_1}M^{-1}_{c_1,\gamma_1}$, *where* $\lambda_1 = min\{\lambda-\rho, \lambda+\rho\}$, $c_1 = c+k$,
$\gamma_1 = min\left\{\gamma+\mu-\lambda_1\left|\sum_{j=1}^{n} a_j - \sum_{j=1}^{m} b_j\right|, \gamma+\mu-\lambda_1\left|\sum_{j=1}^{p_i} a_{ji} - \sum_{j=1}^{q_i} b_{ji}\right|\right\}$,
$i = 1, 2, ..., r$.

The generating operator of I-transform is introduced as

defn 2.3. *Let* $f(x) \in {}_{\lambda}M^{-1}_{c,\gamma}$. *The operator of the form*

$$(L_\eta f)(x) = \frac{1}{2\pi i}\int_{\sigma\left(\frac{1}{2}\right)} \frac{\Phi(s+\eta)}{\Phi(s)} f^*(s+\eta)x^{-s}ds \tag{22}$$

where $|\eta| \leq \lambda, \rho_l - \frac{1}{2} < \eta < \rho_h - \frac{1}{2}, \rho'_l < \frac{1}{2} < \rho'_h$, $\Phi(s)$ *is determined by the relation (2),* ρ_l, ρ_h *by the relation (20),*

$$\rho'_l = \max_{n+1\leq i\leq p_i}\left\{-\frac{\alpha_{ji}}{a_{ji}}\right\}, \quad \rho'_h = \min_{m+1\leq i\leq q_i}\left\{\frac{1-\beta_{ji}}{b_{ji}}\right\} \tag{23}$$

is called the generating operator of I-transform with the kernel $\Phi(s)$.

thm 2.4. *Let* $f(x) \in {}_{\lambda}M^{-1}_{c,\gamma}$ *and let the generating operator* $(L_\eta f)(x)$ *be determined by relation* (21). *Then* $(L_\eta f)(x) \in {}_{\lambda_2}M^{-1}_{c,\gamma_2}$, $\lambda_2 = min\{\lambda-\eta, \lambda+\eta\}$,
$\gamma_2 = min\{\gamma+\eta\left(\sum_{j=1}^{n} a_j - \sum_{j=1}^{m} b_j\right), \gamma+\eta\left(\sum_{j=1}^{p_i} a_{ji} - \sum_{j=1}^{q_i} b_{ji}\right)\}$ *under conditions*

$$2sgn(c)+sgn(\gamma_2) \geq 0,\ \lambda_2 < min\{\frac{1}{2}-\rho'_l,\ \rho'_h-\frac{1}{2},\ \frac{1}{2}-\rho_l+\eta,\ \rho_h-\eta-\frac{1}{2}\}. \tag{24}$$

If moreover, the condition $2|\eta| \leq \lambda$ *holds, then the generating operator (22) is represented in the following form*

$$(L_\eta f)(x) = x^\eta \frac{1}{2\pi i}\int_{\sigma\left(\frac{1}{2}\right)} \frac{\Phi(s)}{\Phi(s-\eta)} f^*(s)x^{-s}ds. \tag{25}$$

Proof. If $f(x) \in {}_{\lambda}M^{-1}_{c,\gamma}$ and $|\eta| \leq \lambda$, then using Theorem 2.1 we have that $x^\eta f(x) \in {}_{\lambda_2}M^{-1}_{c,\gamma}$, where $\lambda_2 = min\{\lambda-\eta, \lambda+\eta\}$ and

$$x^\eta f(x) = \frac{1}{2\pi i}\int_{\sigma\left(\frac{1}{2}\right)} f^*(s+\eta)x^{-s}ds, x > 0. \tag{26}$$

we see that the generating operator $(L_\eta f)(x)$ for the function $f(x)$ is a I-transform with the kernel $\frac{\Phi(s+\eta)}{\Phi(s)}$ for the function $x^\eta f(x)$. This I-transform has the index $\left(0, \eta\left(\sum\limits_{j=1}^{n} a_j - \sum\limits_{j=1}^{m} b_j\right)\right)$, it exists for the space ${}_{\lambda_2}M_{c,\gamma}^{-1}$ and maps it into ${}_{\lambda_2}M_{c,\gamma_2}^{-1}$ in accordance with Theorem 2.2. Again, using Theorem 2.1 for the function $(L_\eta f)(x) \in {}_{\lambda_2}M_{c,\gamma_2}^{-1}$ and $\rho = -\eta$, we obtain representation (25) under the condition $|\eta| \leq \lambda_2$, which is equivalent to the condition $2|\eta| \leq \lambda$ of the theorem. □

rem 2.2. *The generating operator in the form (25) is a particular case of the I-transform with the power weight*

$$(L_\eta f)(x) = x^\eta I_{q_i+p_i,q_i+p_i}^{p_i-n+m,q_i-m+n}\left(\begin{array}{c} (\alpha,a)_{1,n},(\beta-\eta b,b)_{m+1,q_i},(\alpha,a)_{n+1,p_i},(\beta-\eta b,b)_{1,m} \\ (\beta,b)_{1,m},(\alpha-\eta a,a)_{n+1,p_i},(\beta,b)_{m+1,q_i},(\alpha-\eta a,a)_{1,n} \end{array}\right) o[f(u)](x).$$

We can represent the kernel of this transform in the following forms for $\eta > 0$:

$$\begin{aligned} &\frac{\Phi(s)}{\Phi(s-\eta)} \\ &= \prod_{j=1}^{m} \frac{\Gamma(b_j - \beta_j s)}{\Gamma(b_j + \beta_j \eta) - \beta_j s)} \sum_{i=1}^{r} \prod_{j=m+1}^{q_i} \frac{\Gamma(1 - b_{ji} - \beta_{ji}\eta + \beta_{ji}s)}{\Gamma(1 - b_{ji} + \beta_{ji}s)} \\ &\times \prod_{j=1}^{n} \frac{\Gamma(1 - a_j + \alpha_j s)}{\Gamma(1 - a_j + \alpha_j s - \alpha_j \eta)} \sum_{i=1}^{r} \prod_{j=m+1}^{p_i} \frac{\Gamma(1 - a_{ji} - \alpha_{ji}s + \alpha_{ji}\eta)}{\Gamma(1 - a_{ji} - \alpha_{ji}s)}. \quad (27) \end{aligned}$$

Now we will formulate and prove the main result, which is the base of solution of differential, integral and integro-differential equations equations involving the generating operators.

thm 2.5. *Let $f(x) \in {}_{\lambda}M_{c,\gamma}^{-1}$ the I-transform with the kernel $\Phi(s)$ and the generating operator $(L_\eta f)(x)$ of this transform be such that the following conditions hold*

$2sgn(c) + sgn(\gamma_2) \geq 0,\ 2sgn(c+k) + sgn(\mu + \gamma_3) \geq 0,\ 2|\eta| \leq \lambda$

$\lambda_2 < min\left\{\frac{1}{2} - \rho_l',\ \rho_h' - \frac{1}{2},\ \frac{1}{2} - \rho_l + \eta,\ \rho_h - \eta - \frac{1}{2}, \frac{1}{2} - \rho_l, \rho_h - \frac{1}{2}\right\}$

where

$\gamma_2 = min\left\{\gamma + \eta\left(\sum\limits_{j=1}^{n} a_j - \sum\limits_{j=1}^{m} b_j\right), \gamma + \eta\left(\sum\limits_{j=1}^{p_i} a_{ji} - \sum\limits_{j=1}^{q_i} b_{ji}\right)\right\}$,

$\gamma_3 = \gamma_2 + \mu - \lambda_2\left|\sum\limits_{j=1}^{n} a_j - \sum\limits_{j=1}^{m} b_j\right|$, $\lambda_2 = min\{\lambda - \eta, \lambda + \eta\}$, *$k, \mu$ are determined by relations (17) and (18), ρ_l, ρ_h by relation (20) and ρ_l', ρ_h' by relation (23) Then the following relation is valid*

$$I(L_\eta f)(x) = x^\eta (If)(x), \quad (28)$$

that is, the I-transform is the similarity which translates the generating operator $(L_\eta f)(x)$ into the operation of multiplication by the power function x^η.

Proof. The generating operators $(L_\eta f)(x)$ and $I(L_\eta f)(x)$ exists under the conditions in the statement of the Theorem and with the help of Theorem 2.2 and Theorem 2.4. Then using the Remark 2.2, we have

$$\begin{aligned} I(L_\eta f)(x) &= \frac{1}{2\pi i}\int_{\sigma\left(\frac{1}{2}\right)} \Phi(s)\frac{\Phi(s+\eta)}{\Phi(s)} f^*(s+\eta)x^{-s}ds \\ &= \frac{1}{2\pi i}\int_{\sigma\left(\frac{1}{2}\right)} \Phi(s+\eta) f^*(s+\eta)x^{-s}ds \\ &= x^\eta \frac{1}{2\pi i}\int_{\sigma\left(\frac{1}{2}\right)} \Phi(s) f^*(s)x^{-s}ds \\ &= x^\eta (If)(x). \end{aligned}$$

Hence completes the proof. □

3. Conclusion

In this article, we obtained an important technique of Integral transform to obtain generating relation for Saxena's I-function. With the aid of generating functions many problems related to dual space, orthogonality, combinatorial, recurrence equations and problems in physics can be solved and an investigation of useful properties of the sequences, which they generate. The results proved in this paper appear to be new and are likely to have useful applications to a wide range of problems of mathematics and physical sciences.

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