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FIXED POINT THEOREM ON SOME ORDERED METRIC SPACES AND ITS APPLICATION

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Abstract. In this paper, we will prove a fixed point theorem for self-mappings on a generalized quasi-ordered metric space which is a generalization of the concept of a generalized metric space with a partial order and we investigate a genralized quasi-ordered metric space related with fuzzy normed spaces. Further, we prove the stability of some functional equations in fuzzy normed spaces as an application of our fixed point theorem.

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1. Introduction

The theory of fuzzy spaces has much progressed as the theory of randomness has developed. Some mathematicians have defined fuzzy norms on a vector space from various points of view([2], [9], [15], [22], [29]). Later, Cheng and Mordeson [5] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [16] and investigated some properties of fuzzy normed spaces [3]. We use the definition of fuzzy normed spaces given in [2], [21], [22].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space.*

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Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X, N) and one denotes it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* in (X, N) if for any $\epsilon > 0$, there is an $m \in N$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$. It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

Definition 1.2. Let X be a non-empty set. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies the following conditions: for any $x, y, z \in X$,
(D1) $d(x, y) = 0$ if and only if $x = y$,
(D2) $d(x, y) = d(y, x)$, and
(D3) $d(x, y) \leq d(x, z) + d(z, y)$.
In case, (X, d) is called *a generalized metric space*.

A sequence $\{x_n\}$ in a generalized metric space (X, d) is called *Cauchy* in (X, d) if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and a generalized metric space (X, d) is called *complete* if every Cauchy sequence in (X, d) is convergent.

We recall the fixed point theorem from [17].

Theorem 1.3. [17] *Suppose that (X, d) is a generalized complete metric space and a function $T : X \longrightarrow X$ is a contraction, that is, there exists a constant L with $0 < L < 1$ such that, whenever $d(x, y) < \infty$,*

$$d(Tx, Ty) \leq Ld(x, y).$$

Let $x_0 \in X$ and consider a sequence $\{T^n x_0\}$ of successive approximations with the initial element x_0. Then the following alternative holds:
either
(i) for all $n \geq 0$, one has $d(T^n x_0, T^{n+1} x_0) = \infty$
or
(ii) the sequence $\{T^n x_0\}$ is convergent to a fixed point of T in (X, d).

Nieto and Rodríguez-López [24] proved a fixed point theorem in a partially ordered set as follows.

Theorem 1.4. [24] *Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a continuous and non-decreasing mapping such that there exists a constant $L \in (0, 1)$ with*

$$d(Tx, Ty) \leq Ld(x, y) \tag{1}$$

for all $x, y \in X$ with $x \geq y$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Moreover, in [13], the following fixed point theorem for a partially ordered generalized complete metric space was proved.

Theorem 1.5. [13] *Let $(X, \leq)$ be a partially ordered set. Suppose that (X, d) is a generalized complete metric space and a function $T : X \longrightarrow X$ is a continuous and non-decreasing mapping such that there exists a constant $L \in (0, 1)$ such that*

$$d(Tx, Ty) \leq Ld(x, y).$$

for all $x, y \in X$ with $x \geq y$. If there exists x_0 in X with $x_0 \leq Tx_0$, then the following alternative holds:
either

(i) for all $n \geq 0$, one has $d(T^n x_0, T^{n+1} x_0) = \infty$

or

(ii) the sequence $\{T^n x_0\}$ is convergent to a fixed point of T in (X, d).

In this paper, we will prove a fixed point theorem for self-mappings on a generalized quasi-ordered metric space $(X, d, \leq_X)$ which is a generalization of the concept of a generalized metric space with a partial order and we investigate a genralized quasi-ordered metric space related with fuzzy normed spaces. Further, we prove the stability of some functional equations in fuzzy normed spaces as an application of our fixed point theorem.

2. Quasi-order and Fixed point theorem

We start with the definition of a quasi-order. A relation $\leq_X$ on a set X is called *a quasi-order on* X if $\leq_X$ satisfies reflexive and transitive. Let $\leq_X$ be a quasi-order on X. Then x and y are called *comparable*, denoted by $x \sim_X y$ or simply $x \sim y$, if $x \leq_X y$ or $y \leq_X x$.

A triple $(X, d, \leq_X)$ is called *a generalized quasi-ordered metric space* if (X, d) is a generalized metric space and $\leq_X$ is a quasi-order on X. A sequence $\{x_n\}$ in a generalized quasi-ordered metric space $(X, d, \leq_X)$ is called *Cauchy* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and a generalized quasi-ordered metric space $(X, d, \leq_X)$ is called *d-complete* if every non-decreasing Cauchy sequence in $(X, d, \leq_X)$ is convergent.

Theorem 2.1. *Let $(X, d, \leq_X)$ be a d-complete space such that*

if $\{x_n\}$ is a non-decreasing sequence in $(X, \leq_X)$ and $x_n \to x$ in (X, d),

then $x_n \leq_X x$ for all $n \in \mathbb{N}$.

Let $T : X \longrightarrow X$ be a non-decreasing mapping such that there exists an $L \in (0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y). \tag{2}$$

for all $x, y \in X$ with $x \sim y$. If there exists an x_0 in X with $x_0 \leq_X Tx_0$, then the following alternative holds:
either

(i) for all $n \geq 0$, one has $d(T^n x_0, T^{n+1}x_0) = \infty$
or
(ii) the sequence $\{T^n x_0\}$ is convergent to a fixed point of T in (X, d). Further, if $d(x_0, Tx_0) < \infty$, then

$$d(x, x_0) \leq \frac{L}{1-L} d(x_0, Tx_0). \tag{3}$$

for all $x \in X$.

Proof. Suppose that there exists an $l \in \mathbb{N}$ such that $d(T^l x_0, T^{l+1}x_0) < \infty$. Since T is non-decreasing and $x_0 \leq_X Tx_0$, $T^{n-1}x_0 \leq_X T^n x_0$ for all $n \in \mathbb{N}$. By (2), we obtain

$$d(T^n x_0, T^{n+1}x_0) \leq L^{n-l} d(T^l x_0, T^{l+1}x_0) < \infty$$

for all $n \in \mathbb{N}$ with $n \geq l$. Hence for $m > n \geq l$, we have

$$\begin{aligned} &d(T^m x_0, T^n x_0) \\ &\leq \sum_{i=n}^{m-1} d(T^i x_0, T^{i+1}x_0) \leq \frac{L^{n-l}(1-L^{m-n})}{1-L} d(T^l x_0, T^{l+1}x_0) \end{aligned} \tag{4}$$

and so the sequence $\{T^n x_0\}$ is a non-decreasing Cauchy sequence in $(X, d, \leq_X)$. Since $(X, d, \leq_X)$ is d-complete, there exists an $y \in X$ such that $T^n x_0 \to y$ in $(X, d, \leq_X)$.

Now, we claim that y is the fixed point of T. Let $\epsilon > 0$ be given. Since $\{T^n x_0\}$ is a non-decreasing sequence in $(X, \leq_X)$ and $T^n x_0 \to y$ in (X, d), $T^n x_0 \leq_X y$ for all $n \in \mathbb{N}$. Since $T^n x_0 \to y$ in $(X, d, \leq_X)$, there exists a $k \in \mathbb{N}$ such that $k > l$ and

$$k \leq n \;\Rightarrow\; d(T^n x_0, y) < \frac{\epsilon}{2}. \tag{5}$$

Since T is a non-decreasing mapping, $T^{n+1}x_0 \leq_X Ty$ for all $n \in \mathbb{N}$ and so by (2) and(5), we have

$$d(Ty, y) \leq d(Ty, T^{k+1}x_0) + d(T^{k+1}x_0, y) \leq Ld(y, T^k x_0) + d(T^{k+1}x_0, y) < \epsilon$$

Thus $Ty = y$. Moreover, if $d(x_0, Tx_0) < \infty$, then, by (4), we have (3). □

For a fuzzy normed space (K, N), define a relation $\leq_K$ on K by

$$x \leq_K y \text{ if } N(x, t) \geq N(y, t), \forall t > 0.$$

We can show the following theorem:

Theorem 2.2. *Let (K, N) be a fuzzy normed space. Then we have the following properties :*
(1) *$\leq_K$ is a quasi-order on (K, N) and*
(2) *if $\{x_n\}$ is a non-decreasing sequence in $(K, \leq_K)$ and $x_n \to x$ in (K, N), then $x_n \leq_K x$ for all $n \in \mathbb{N}$.*

Proof. The proof of (1) is trivial.

(2) Suppose that $\{x_n\}$ is a non-decreasing sequence in $(K, \leq_K)$ and $x_n \to x$ in (K, N).

case 1 : $x = 0$. Let $m \in \mathbb{N}$. Since $\{x_n\}$ is a non-decreasing sequence in $(K, \leq_K)$, $N(x_m, t) \geq N(x_{m+p}, t)$ for all non-negative integer p and all $t > 0$ and since $x_n \to 0$ in (K, N),

$$N(x_m, t) \geq \lim_{p\to\infty} N(x_{m+p}, t) = \lim_{p\to\infty} N(x_{m+p} - 0, t) = 1$$

for all $t > 0$. Hence $N(x_m, t) = 1$ for all $t > 0$ and so $x_m = 0$. Thus $x_m \leq_K x$ for all $m \in \mathbb{N}$.

case 2 : $x \neq 0$. Suppose that there exists an $l \in \mathbb{N}$ such that $x_l \not\leq_K x$. Then

$$N(x_l, t_0) < N(x, t_0) \tag{6}$$

for some $t_0 > 0$. Let $\epsilon = N(x, t_0) - N(x_l, t_0)$. Then $\epsilon > 0$ and since $x_n \to x$ in (K, N), there exists a $k \in \mathbb{N}$ such that $l < k$ and $N(x_n - x, t) \geq 1 - \epsilon$ for all $n \geq k$ and all $t > 0$. Let $n \in \mathbb{N}$ with $n \geq k$ and $s \in \mathbb{R}$ with $0 < s < t_0$. Since $1 - N(x, t_0) \geq 0$,

$$\begin{aligned} N(x_n, t_0) &\geq \min\{N(x_n - x, t_0 - s), N(x, s)\} \\ &\geq \min\{1 - \epsilon, N(x, s)\} \\ &= \min\{1 - N(x, t_0) + N(x_l, t_0), N(x, s)\} \\ &\geq \min\{N(x_l, t_0), N(x, s)\}. \end{aligned} \tag{7}$$

Letting $s \to t_0$ in (7), by (N6) and (6), we have

$$N(x_n, t_0) = \min\{1 - N(x, t_0) + N(x_l, t_0), N(x, t_0)\} = N(x_l, t_0), \tag{8}$$

because $x_l \leq_K x_n$. Suppose that $1 - N(x, t_0) + N(x_l, t_0) \leq N(x, t_0)$. Then by (8), we have

$$N(x_n, t_0) = 1 - N(x, t_0) + N(x_l, t_0) = N(x_l, t_0) \tag{9}$$

for all $n \in \mathbb{N}$ with $n \geq k$ and

$$N(x, t_0) = 1. \tag{10}$$

Take any real number r with $r > t_0$. By (10), we have

$$\begin{aligned} N(x_n, r) &\geq \min\{N(x_n - x, r - t_0), N(x, t_0)\} \\ &= N(x_n - x, r - t_0) \end{aligned} \tag{11}$$

for all $n \in \mathbb{N}$ with $n \geq k$. Since $N - \lim_{n\to\infty} x_n = x$, by (11), we have

$$\lim_{n\to\infty} N(x_n, r) = 1. \tag{12}$$

Let $\gamma > 0$ be given. Since $\{x_n\}$ is non-decreasing, $x_l \leq_K x_k$ and so $N(x_k, t_0) \leq N(x_l, t_0) < 1$. By (N2), $x_k \neq 0$ and by (N6), $N(x_k, \cdot)$ is continuous on $\mathbb{R}$, there is a $\delta > 0$ such that for any real number r with $t_0 < r < t_0 + \delta$,

$$N(x_k, r) < N(x_k, t_0) + \gamma.$$

Let $m \in \mathbb{N}$ with $k \leq m$. Since $\{x_n\}$ is non-decreasing in $(K, \leq_K)$, $x_k \leq_K x_m$ and so $N(x_m, r) \leq N(x_k, r)$. Hence we have

$$N(x_m, r) < N(x_k, t_0) + \gamma.$$

By (12), we get

$$1 \leq N(x_k, t_0) + \gamma$$

and thus $N(x_k, t_0) = 1$. Hence by (9), $N(x_l, t_0) = 1$ which is a contradiction to $N(x_l, t_0) < N(x, t_0)$. Hence $1 - N(x, t_0) + N(x_l, t_0) > N(x, t_0)$. Then by (8), we have

$$N(x, t_0) = N(x_l, t_0)$$

which is a contradiction and thus one has result. □

It is well known that for any normed space $(X, ||\cdot||)$ with $|X| \geq 2$, mappings $N_X, N'_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+||x||}, & \text{if } t > 0 \end{cases}$$

and

$$N'_X(x,t) = \begin{cases} 0, & \text{if } t < \|x\| \\ 1, & \text{if } t \geq \|x\| \end{cases}$$

are fuzzy norms on X. The quasi-order $\leq_X$ on $(X, N_X)((X, N'_X)$, resp.), is not a partially order.

A fuzzy normed space (X, N) is called *d-complete* if every non-decreasing Cauchy sequence in $(X, N, \leq_X)$ is convergent in (X, N), where $\leq_X$ is the quasi-order on X, defined by $x \leq_X y$ if $N(x, t) \geq N(y, t)$ for all $t > 0$.

In the following, assume that X is a linear space, (Y, N) is d-complete, and (Z, N') is a fuzzy normed space.

Let $S = \{g \mid g : X \longrightarrow Y\}$ and define a relation $\leq_s$ on S by

$$g \leq_s h \text{ if } g(x) \leq_Y h(x), \ \forall x \in X.$$

Then clearly, $\leq_s$ is a quasi-order on S. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a mapping and

$$\begin{aligned} &\Phi(x,y,t) \\ &= \min[\{N'(\phi(a_i x, b_i y), p_i t) | 1 \leq i \leq l\} \cup \{N'(\phi(c_i y, d_i x), q_i t) | 1 \leq i \leq m\} \\ &\cup \{N'(\phi(e_i x, k_i x), s_i t) | 1 \leq i \leq n\}] \end{aligned}$$

for some rational numbers $a_i, b_i, c_i, d_i, e_i, k_i$, positive real numbers p_i, q_i, s_i, and natural numbers l, m, n. Define a mapping $d : S^2 \longrightarrow [0, \infty]$ by

$$d(g,h) = \inf\{c \in \mathbb{R}^+ \mid N(f(x) - g(x), ct) \geq \Phi(x, 0, t), \ \forall x \in X, \ \forall t > 0\}.$$

Then (S, d) is a generalized metric space ([18]) and using Theorem 2.2, we have the following theorem.

Theorem 2.3. *$(S, d, \leq_s)$ is a d-complete space such that if $\{g_n\}$ is a non-decreasing sequence in $(S, d, \leq_s)$ and $g_n \to g$ in $(S, d, \leq_s)$, then $g_n \leq_s g$ for all $n \in \mathbb{N}$.*

Proof. Let $\{h_n\}$ be a non-decreasing Cauchy sequence in $(S, d, \leq_s)$. Then for any $x \in X$, $\{h_n(x)\}$ is a non-decreasing Cauchy sequence in Y and since Y is d-complete, there exists a mapping $h : X \longrightarrow Y$ such that $N - \lim_{n\to\infty} h_n(x) = h(x)$. By Theorem 2.2, $h_n(x) \leq_Y h(x)$ for all $n \in \mathbb{N}$ and $x \in X$. Hence $h_n \leq_s h$ for all $n \in \mathbb{N}$.

Let ϵ be a real number with $0 < \epsilon < 1$. Then there exists a $k \in \mathbb{N}$ such that for $n > m \geq k$,

$$d(h_n, h_m) \leq \frac{\epsilon}{4}. \tag{13}$$

Let $x \in X$. For $n > m \geq k$, $h_m \leq_s h_n$ and by (13), we have

$$N\Big(h_n(x) - h_m(x), \frac{\epsilon}{4}t\Big) \geq \Phi(x, 0, t)$$

and

$$\begin{aligned} N\Big(h_k(x) - h(x), \frac{\epsilon}{2}t\Big) &\geq \min\Big\{N\Big(h_k(x) - h_n(x), \frac{\epsilon}{4}t\Big), N\Big(h_n(x) - h(x), \frac{\epsilon}{4}t\Big)\Big\} \\ &\geq \min\Big\{\Phi(x, 0, t), N\Big(h_n(x) - h(x), \frac{\epsilon}{4}t\Big)\Big\} \end{aligned}$$

for all $n \geq k$ and all $t > 0$. Since $N - \lim_{n\to\infty} h_n(x) = h(x)$ in Y,

$$N\Big(h_k(x) - h(x), \frac{\epsilon}{2}t\Big) \geq \Phi(x, 0, t).$$

Hence $d(h_k, h) \leq \frac{\epsilon}{2}$ and so, by (13), we have $d(h_n, h) < \epsilon$ for all $n \geq k$. Thus $h_n \to h$ in $(S, d, \leq_s)$ and so $(S, d, \leq_s)$ is a d-complete space.

Suppose that $\{g_n\}$ is a non-decreasing sequence in $(S, d, \leq_s)$ and $g_n \to g$ in $(S, d, \leq_s)$. By the definition of $\leq_s$ and Theorem 2.2, $g_n \leq_s g$ for all $n \in \mathbb{N}$. ☐

3. Applications

In this section, we will prove the fuzzy stability as an application of our fixed point theorem. We start with the following theorem:

Theorem 3.1. *Let a, k be natural numbers and L, M positive real numbers with $L < 1$. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$N'(\phi(ax, ay), t) \geq N'\Big(\phi(x, y), \frac{1}{a^k L}t\Big) \tag{14}$$

for all $x \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping such that

$$N(a^k f(x) - f(ax), Mt) \geq \Phi(x, 0, t), \;\; N(f(x), t) \geq N\Big(\frac{1}{a^k} f(ax), t\Big) \tag{15}$$

for all $x \in X$ and all $t > 0$. Then there exists an unique mapping $F : X \longrightarrow Y$ such that $N - \lim_{n\to\infty} \frac{1}{a^{kn}} f(a^n) = F(x)$ and

$$N\Big(\frac{1}{a^{kn}} f(a^n x), t\Big) \geq N(F(x), t), \quad F(ax) = a^k F(x)$$
$$N\Big(F(x) - f(x), \frac{ML}{a^k(1-L)}\Big) \geq \Phi(x, 0, t) \tag{16}$$

for all $x \in X$, all $t > 0$, and all $n \in \mathbb{N} \cup \{0\}$.

Proof. Define a mapping $d : S \longrightarrow S$ by

$$d(g,h) = \inf\{c \in \mathbb{R}^+ \mid N(g(x) - h(x), ct) \geq \Phi(x, 0, t), \ \forall x \in X, \ \forall t > 0\}.$$

By Theorem 2.3, $(S, d, \leq_s)$ is a d-complete metric space such that if $\{g_n\}$ is a non-decreasing sequence in $(S, d, \leq_s)$ and that if $g_n \to g$ in $(S, d, \leq_s)$, then $g_n \leq_s g$ for all $n \in \mathbb{N}$.

Define a mapping $T : S \longrightarrow S$ by $Tf(x) = \frac{1}{a^k} f(ax)$. Then T is a non-decreasing mapping. Suppose that $f, g \in S$ with $f \sim g$. For any $c \in \mathbb{R}^+$ with $N(f(x) - g(x), ct) \geq \Phi(x, 0, t)$ for all $x \in X$ and all $t > 0$, by (14), we have

$$N(Tf(x) - Tg(x), Lct) \geq \Phi(ax, 0, La^k t) \geq \Phi(x, 0, t).$$

Hence $d(Tf, Tg) \leq Ld(f, g)$ and by Theorem 2.1, there exists an unique mapping $F : X \longrightarrow Y$ with (16). ☐

In 1940, Ulam proposed the following stability problem ([28]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings, and by Rassias [27] for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias' theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians([6], [7], [8], [23]).

Recently, the stability problems in the fuzzy normed space has been studied ([14], [19], [21], [22]). In 2008, for the first time, Mirmostafaee and Moslehian [19], [21] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of the stability for the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

Rassias [26], Park and Jung [25] investigated the following cubic functional equations

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y) \tag{17}$$

and

$$f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x), \tag{18}$$

and proved the generalized Hyers-Ulam stability for it, respectively. It is easy to see that the function $f(x) = ax^3$ is a solution of the functional equation (17) and (18), which explains why they are called *a cubic functional equation*. Mirmostafaee and Moslehian [20] proved the stability of a cubic functional equation in a fuzzy normed space.

Cádariu and Radu [4] applied the fixed point method to investigate the Jensen functional equation and presented a short and simple proof (different from the direct method initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation.

Define a k-mapping $f : X \longrightarrow Y$ as follows: if $k = 1$, then f is an additive mapping, if $k = 2$, then f is a quadratic mapping, and if $k = 3$, then f is a cubic mapping, $\cdots$, and define a k-functional operator D_k on S as follows: if $D_k h(x, y) = 0$ for all $x, y \in X$, then h is a k-mapping.

By Theorem 3.1, we have the following corollary:

Corollary 3.2. *Let D_k be a k-functional operator on S. Let $\phi : X^2 \longrightarrow [0, \infty)$ with (14). Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(0) = 0$, and*

$$N(f(x), t) \geq N\Big(\frac{1}{a^k} f(ax), t\Big)$$

for all $x \in X$ and all $t > 0$, and

$$N(D_k f(x, y), t) \geq N'(\phi(x, y), t) \tag{19}$$

for all $x, y \in X$ and all $t > 0$. Further, suppose that (19) implies that

$$N(a^k f(x) - f(ax), Mt) \geq \Phi(x, 0, t)$$

for all $x \in X$, all $t > 0$, and some positive real number M. Then there exists an unique k-mapping $F : X \longrightarrow Y$ such that $N - \lim_{n\to\infty} \frac{1}{a^{kn}} f(a^n) = F(x)$ and

$$\begin{aligned} &N\Big(\frac{1}{a^{kn}} f(a^n x), t\Big) \geq N'(F(x), t), \quad f \leq_s F, \\ &N\Big(F(x) - f(x), \frac{ML}{a^k(1-L)} t\Big) \geq \Phi(x, 0, t) \end{aligned} \tag{20}$$

for all $x \in X$, all $t > 0$, and all $n \in \mathbb{N} \cup \{0\}$.

Proof. By Theorem 3.1, there is an unique mapping $F \in S$ with (20). By (14) and (19), we have

$$N\Big(\frac{1}{a^{kn}}D_k f(a^n x, a^n y), t\Big) \geq N'\Big(\frac{1}{a^{kn}}\phi(a^n x, a^n y), t\Big) \geq N'\Big(\phi(x,y), \frac{t}{L^n}\Big)$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, $D_k F(x,y) = 0$ and so F is a k-mapping. □

Now, we will prove the generalized Hyers-Ulam stability of the following cubic functional equation using Corollary 3.2.

$$f(3x+y) + f(3x-y) = f(x+2y) + 2f(x-y) + 2f(3x) - 3f(x) - 6f(y) \quad (21)$$

in fuzzy normed spaces as an application of our fixed point theorem. We can easily shown that the following :

Theorem 3.3. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (21) if and only if f is cubic.*

For any mapping $f : X \longrightarrow Y$, let

$$D_3 f(x,y) = f(3x+y)+f(3x-y)-f(x+2y)-2f(x-y)-2f(3x)+3f(x)+6f(y).$$

By Corollary 3.2, we have the following example:

Example 3.4. Let X be a linear space. Let $\phi : X^2 \longrightarrow Z$ be a function and L a real number such that $0 < L < 1$ and

$$N'(\phi(2x, 2y), t) \geq N'(2^3 L\phi(x,y), t) \quad (22)$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(D_3 f(x,y), t) \geq N'(\phi(x,y), t) \quad (23)$$

for all $x, y \in X$ and all $t > 0$. Further, suppose that

$$N(8f(x), t) \geq N(f(2x), t)$$

for all $x \in X$ and all $t > 0$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$N\Big(f(x) - C(x), \ \frac{1}{48(1-L)}t\Big) \geq \Phi(x, 0, t) \quad (24)$$

for all $x \in X$ and all $t > 0$, where

$$\Phi(x,y,t) = \min\Big\{N'\Big(\phi(y,-x), \frac{t}{15}\Big), N'\Big(\phi(x,x), \frac{t}{15}\Big), N'\Big(\phi(x,-x), \frac{t}{15}\Big), N'\Big(\phi(y,x), \frac{t}{15}\Big), N'\Big(\phi(y,2x), \frac{t}{15}\Big)\Big\}.$$

Proof. By (23), we have

$$\begin{aligned} &N(6f(2x) - 48f(x), t) \\ &= N(D_3f(0,-x) + 2D_3f(x,x) - 3Df(x,-x) - 8D_3f(0,x) - D_3f(0,2x), t) \\ &\geq \min\Big\{N\Big(D_3f(0,-x), \frac{t}{15}\Big), N\Big(2D_3f(x,x), \frac{2t}{15}\Big), N\Big(3D_3f(x,-x), \frac{3t}{15}\Big), \\ &N\Big(8D_3f(0,x), \frac{8t}{15}\Big), N\Big(D_3f(0,2x), \frac{t}{15}\Big)\Big\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence, by (N3), we have

$$N\Big(2^3 f(x) - f(2x), \ \frac{t}{6}\Big) \geq \Phi(x, 0, t)$$

for all $x \in X$ and all $t > 0$. By Corollary 3.2 and Theorem 3.3, there exists an unique cubic mapping C in S with (24). □

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