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ON THE BOUNDS FOR WAVE STABILITY OF STRATIFIED SHEAR FLOWS

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Abstract. We consider incompressible, inviscid, stratified shear flows in β plane. First, we obtained an unbounded instability region intersect with semi-ellipse region. Second, we obtained a bounded instability regions depending on Coriolis, stratification parameters and basic velocity profile. Third, we obtained a criterion for wave stability. This has been illustrated with standard examples. Also, we obtained upper bound for growth rate.

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1. Introduction

The study of stability analysis of stratified shear flows under normal mode approach has been studied extensively (see [23], [2] & [17]). Parallel shear flow problem is a standard classical problem of hydrodynamic stability. [8] studied the problem with Coriolis parameter. For this problem, [8] derived inflexion point theorem. [14] proved that the complex phase velocity lies inside upper half of semicircle which is an extension of [6]'s work. [5], [10] obtained upper bounds for the growth rate. [13], [22], [9], [10] derived parabolic instability regions. [3] obtained neutral modes for the case of Bickley jet. [10] also obtained upper bound for the amplification factor.

For incompressible, inviscid, stratified shear flows, [12] derived a sufficient condition for stability. [6] derived semicircle theorem. [7], [20] proved a semiellipse theorems. [4] derived unbounded instability region for Taylor- Goldstein problem. [21], [19] obtained unbounded parabolic instability regions for extended Taylor Goldstein problem. [15], [16] obtained criterion for long wave stability.

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The following results are known for Taylor Goldstein Problem in β plane and Taylor Goldstein problem of hydro dynamic stability.

(1) The instability region is a semi-circle whose diameter given by range of basic velocity profile (cf. [6]).

(2) The instability region is a semi-ellipse whose major axis depends on basic velocity profile and minor axis depends on Richardson number (cf. [7]).

(3) The instability region is a semi-ellipse whose major axis depends on basic velocity profile and minor axis depends on Richardson number, curvature. (cf. [20]).

(4) If $c = c_r + ic_i$ with $c_i > 0$, then $c_i^2 \leq \lambda\left[c_r - (3U_{\min} - 2U_{\max})\right]$, where

$$\lambda = \left(\frac{2\rho\left(\frac{\left(U'\right)^2}{4} - g\beta\right)}{\phi\left(y\right)}\right)_{\max},$$

$$\phi\left(y\right) = \left(\rho U'\right)' + \left(\frac{2\rho_{\min}\left(b-a\right)\pi^2}{\left(y_2-y_1\right)^2}\right) > 0, \forall y \in [y_1, y_2]$$

(cf. [4]).

(5) If $c = c_r + ic_i$, with $c_i > 0$ then $c_i^2 < \lambda^*\left[3U_{\max} - 2U_{\min} - c_r\right]$, where

$$\lambda^* = \left(\frac{2\rho\left(\frac{\left(U'\right)^2}{4} - g\beta\right)}{\left|\psi\left(y\right)\right|}\right)_{\max},$$

$$\psi\left(y\right) = \left(\rho U'\right)' - \left(\frac{2\rho_{\max}\left(b-a\right)\pi^2}{\left(y_2-y_1\right)^2}\right) < 0, \forall y \in [y_1, y_2]$$

(cf. [4]).

(6) The phase velocity of non singular neutral mode is given by

$$U_{\min} + \frac{\left[\left(D^2\left(U\right) - \beta\right)\right]_{\min} - \sqrt{\left[\left(D^2\left(U\right) - \beta\right)\right]_{\max}^2 + 4\alpha^2 N_{\max}^2}}{2\left(\alpha^2 + k^2\right)} \leq c_r \leq$$

$$U_{\max} + \frac{\left[\left(D^2\left(U\right) - \beta\right)\right]_{\max} + \sqrt{\left[\left(D^2\left(U\right) - \beta\right)\right]_{\max}^2 + 4\alpha^2 N_{\max}^2}}{2\left(\alpha^2 + k^2\right)}$$

(cf. [18]).

(7) A necessary condition for instability is that $\left(D^2(U)-\beta\right)(U-c_r) \le N^2$ (cf. [18]).

In this paper, we study Taylor-Goldstein problem in β plane. For this problem, we obtained first an unbounded instability region which intersect with Kochar & Jain semi ellipse under certain condition. Second, we obtained a bounded instability regions depending on Coriolis, stratification parameters and basic velocity profile. Third, we obtained a criterion for wave stability. The result has been illustrated with three standard examples. Also, we obtained an upper bound for the growth rate. In the absence of coriolis force, the results will reduce to new results for Taylor - Goldstein problem of hydrodynamic stability. The methodology we adopted are standard method of complex eigen function and integral inequality method (refer [23], [2], [17]).

2. Taylor-Goldstein Problem in β Plane

The Taylor-Goldstein problem in β plane is given by

$$D^2(\phi) + \left[\frac{N^2}{(U-c)^2} - \frac{D^2(U)-\beta}{U-c} - k^2\right]\phi = 0, \tag{1}$$

with boundary conditions

$$\phi(z_1) = 0 = \phi(z_2). \tag{2}$$

Where U is the basic velocity profile, ϕ is the eigen function, $c = c_r + ic_i$ phase velocity, $k > 0$ is the wave number, $\beta = \frac{2\Omega}{a}\cos\theta$ is Coriolis parameter (a is the radius of earth, Ω is the earth's rotation rate, θ is the latitude (cf. [14], [11]). Using $\phi = (U-c)^{\frac{1}{2}}\psi$ in (1), (2), we get

$$\begin{array}{l} D\left[(U-c)\,D(\psi)\right] - \frac{1}{4}\frac{[D(U)]^2}{(U-c)}\psi + \frac{N^2}{(U-c)}\psi - k^2(U-c)\,\psi \\ -\left(\frac{D^2(U)}{2} - \beta\right)\psi = 0 \end{array}, \tag{3}$$

with boundary conditions

$$\psi(z_1) = 0 = \psi(z_2). \tag{4}$$

3. Unbounded Region:

Theorem 3.1. *If* (c,ψ) *is a solution of* (3),(4) *and* $U_m = \frac{U_{\min}+U_{\max}}{2}$, *then* $c_i^2 \le \lambda\left[c_r + \frac{3U_{\max}}{2} - \frac{U_{\min}}{2}\right]$, *where*

$$\lambda = \frac{[D(U)]^2_{\max}\left[\frac{1}{4} - J_m\right]}{\left(\frac{3U_{\min}+U_{\max}}{2}\right)\left[\frac{\pi^2}{(z_2-z_1)^2} + k^2\right]}, J_m = \left[\frac{N^2}{\left[D(U)^2\right]}\right]_{min}.$$

Proof. Multiplying (3) by ψ^*, integrating, applying (4) and comparing real and imaginary parts, we get

$$\int_{z_1}^{z_2} (U-c_r)\left[|D(\psi)|^2+k^2|\psi|^2\right]dz+\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2\,dz \\ +\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(U-c_r)|\psi|^2\,dz=0, \tag{5}$$

and

$$-c_i\int_{z_1}^{z_2}\left[|D(\psi)|^2+k^2|\psi|^2\right]dz+c_i\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}|\psi|^2\,dz=0. \tag{6}$$

Multiplying (6) by $\left(\frac{c_r+U_m}{c_i}\right)$ and subtracting from (5), we get

$$\int_{z_1}^{z_2} (U+U_m)\left[|D(\psi)|^2+k^2|\psi|^2\right]dz+\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2\,dz \\ +\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(U-2c_r-U_m)|\psi|^2\,dz=0. \tag{7}$$

Multiplying (6) by $\left(\frac{U_{\max}-U_{\min}}{c_i}\right)$ and adding with (5), we get

$$\int_{z_1}^{z_2}(U-c_r-U_{\max}+U_{\min})\left[|D(\psi)|^2+k^2|\psi|^2\right]dz \\ +\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2\,dz \\ +\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(U-c_r+U_{\max}-U_{\min})|\psi|^2\,dz=0.$$

Since $(U-c_r-U_{\max}+U_{\min})<0$ dropping the term, we get

$$\int_{z_1}^{z_2}\left(\frac{D^2(U)}{2}-\beta\right)|\psi|^2\,dz \\ \geq\int_{z_1}^{z_2}\frac{\left(\frac{(D(U))^2}{4}-N^2\right)}{|U-c|^2}(c_r-U+U_{\min}-U_{\max})|\psi|^2\,dz. \tag{8}$$

Substituting (8) in (7), applying Rayleigh- Ritz inequality, and $\frac{1}{|U-c|^2} \leq \frac{1}{c_i^2}$, we get

$$(U_{\min} + U_m)\left[\frac{\pi^2}{(z_2 - z_1)^2} + k^2\right]\int_{z_1}^{z_2} |\psi|^2\, dz$$
$$\leq \frac{\left[\frac{(D(U))^2}{4} - N^2\right]_{\max}}{c_i^2}\left[c_r + U_m + U_{\max} - U_{\min}\right]\int_{z_1}^{z_2} |\psi|^2\, dz;$$

Let $U_m = \frac{U_{\min}+U_{\max}}{2}$, we have

$$c_i^2 \leq \lambda\left[c_r + \frac{3U_{\max}}{2} - \frac{U_{\min}}{2}\right], \tag{9}$$

where

$$\lambda = \frac{[D\,(U)]^2_{\max}\left[\frac{1}{4} - J_m\right]}{\left(\frac{3U_{\min}+U_{\max}}{2}\right)\left[\frac{\pi^2}{(z_2-z_1)^2} + k^2\right]},\, J_m = \left[\frac{N^2}{\left[D\,(U)^2\right]}\right]_{min}.$$

When $\beta = 0$, then above instability region reduces to instability region far standard Taylor- Goldstein problem. Unlike [[4]] result, the new result does not depend on any conditions. □

Theorem 3.2. *If $\lambda < \lambda_c$, then the parabola given by* (9) *intersect with Kochar & Jain semi ellipse region*

$$\left[c_r - \frac{U_{\min} + U_{\max}}{2}\right]^2 + \frac{2c_i^2}{1 + \sqrt{1 - 4J_m}} \leq \left[\frac{U_{\max} - U_{\min}}{2}\right]^2.$$

Proof. Kochar & Jain semi ellipse region [cf. [7]] is given by

$$\left[c_r - \frac{U_{\min} + U_{\max}}{2}\right]^2 + \frac{2c_i^2}{1 + \sqrt{1 - 4J_m}} \leq \left[\frac{U_{\max} - U_{\min}}{2}\right]^2. \tag{10}$$

Substituting (9) in (10), we get

$$c_r^2 + \left[\frac{2\lambda}{1 + \sqrt{1 - 4J_m}} - U_{\min} - U_{\max}\right] c_r$$
$$+ \left[U_{\min}U_{\max} + \frac{2\lambda}{1 + \sqrt{1 - 4J_m}}\left[\frac{3U_{\max}}{2} - \frac{U_{\min}}{2}\right]\right] \leq 0.$$

For real roots, we have

$$\left(\frac{2\lambda}{1 + \sqrt{1 - 4J_m}}\right)^2 - \frac{16}{1 + \sqrt{1 - 4J_m}} U_{\max}\lambda + (U_{\max} - U_{\min})^2 \geq 0.$$

solving, we get

$$\lambda = \frac{\left(1 + \sqrt{1 - 4J_m}\right)}{2}\left[4U_{\max} \pm \sqrt{15U_{\max}^2 - U_{\min}^2 + 2U_{\max}U_{\min}}\right].$$

λ value with positive sign leads to $c_r < U_{\min}$ and hence

$$\lambda_c = \frac{\left(1+\sqrt{1-4J_m}\right)}{2}\left[4U_{\max} - \sqrt{15U_{\max}^2 - U_{\min}^2 + 2U_{\max}U_{\min}}\right]$$

then the parabola (9) intersects semi-ellipse (10). □

Example: 1 Let us consider the example $U = \left(z - \frac{1}{2}\right)^2$,$z \in [0, 1]$.
In this case

$$U_{\min} = 0, U_{\max} = 0.25, \left(D\left(U\right)\right)_{\max} = 1$$

$$\lambda = \frac{0.16}{[\pi^2 + k^2]}, \lambda_c = 0.5936987,$$

the parabola intersects the semi ellipse for all values of $k \geq 1$.
Region of J$_m$ for intersection :
$0.15988 \leq J_m \leq 0.25$, for k=1
$0.14181 \leq J_m \leq 0.25$, for k=2

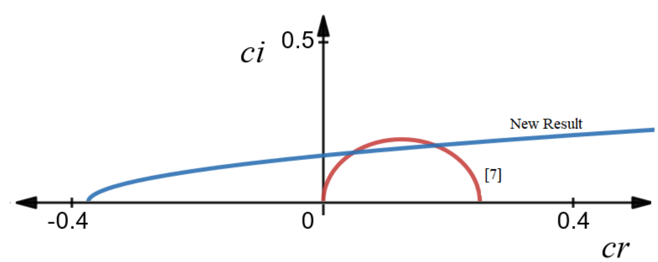

FIGURE 1. c_r vs c_i Intersection of parabola with semi-ellipse.

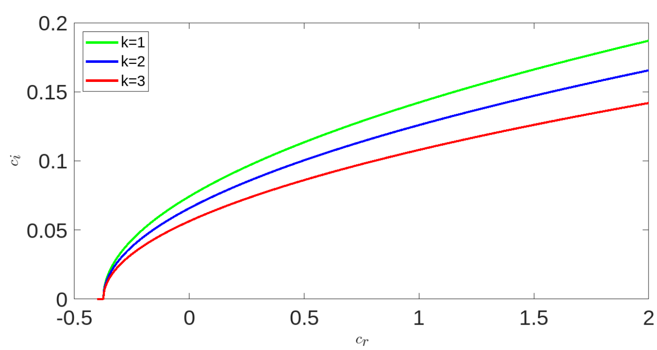

FIGURE 2. c_r vs c_i parabola for distinct values of k.

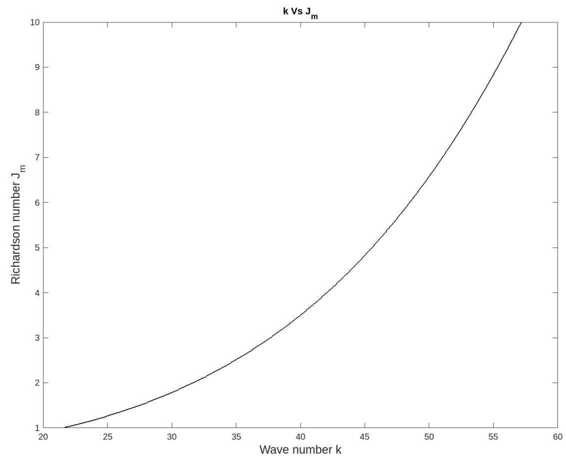

FIGURE 3. k vs J_m wave number versus Richardson number.

Example: 2 Let us consider the example $U = \left(z - \frac{1}{2}\right), z \in [0,1]$.
In this case

$$U_{\min} = -0.5, U_{\max} = 0.5, \left(D\left(U\right)\right)_{\max} = 1$$
$$\lambda = \frac{0.2}{[\pi^2 + k^2]}, \lambda_c = 1.6324555,$$

the parabola intersects the semi ellipse for all values of $k \geq 1$. Region of J_m for intersection :
$0 \leq J_m \leq 0.25$, for k=1
$0 \leq J_m \leq 0.25$, for k=2

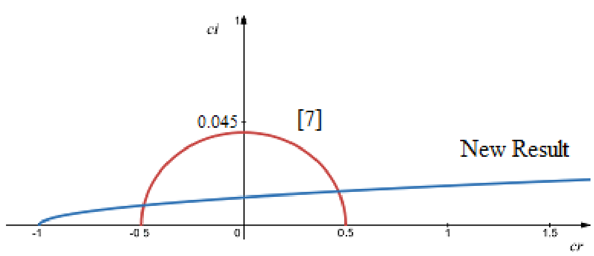

FIGURE 4. c_r vs c_i Intersection of parabola with semi-ellipse.

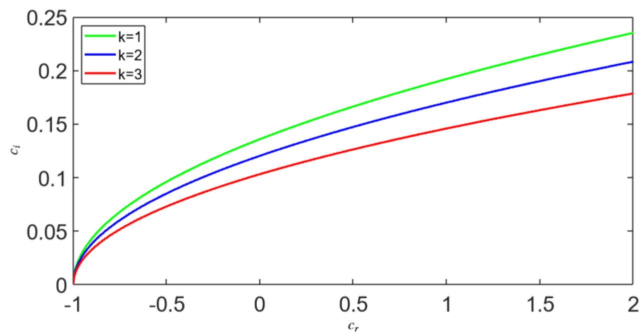

FIGURE 5. c_r vs c_i parabola for distinct values of k.

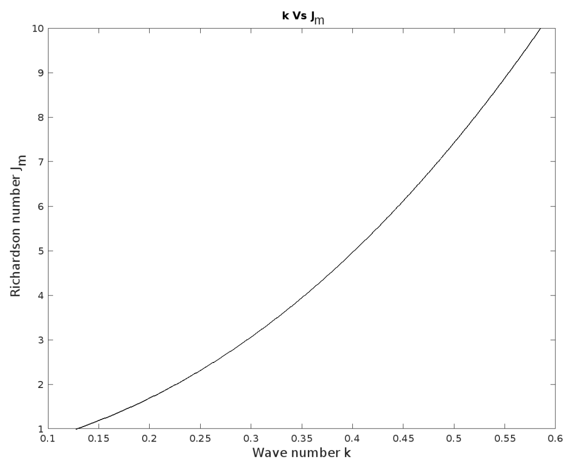

FIGURE 6. k vs J_m wave number versus Richardson number.

Example: 3
Let us consider the example $U = \sin z$, $z \in [0, 1]$.
In this case

$$U_{\min} = 0, U_{\max} = 0.84147, (D(U))_{\max} = 1$$

$$\lambda = \frac{0.1188}{[\pi^2 + k^2]}, \lambda_c = 0.07733,$$

the parabola intersects the semi ellipse for all values of $k \geq 1$.
Region of J_m for intersection :
$0 \leq J_m \leq 0.25$, for k=1
$0 \leq J_m \leq 0.25$, for $k = 2$

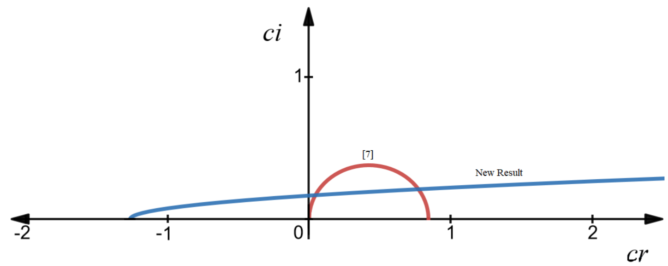

FIGURE 7. c_r vs c_i Intersection of parabola with semi-ellipse.

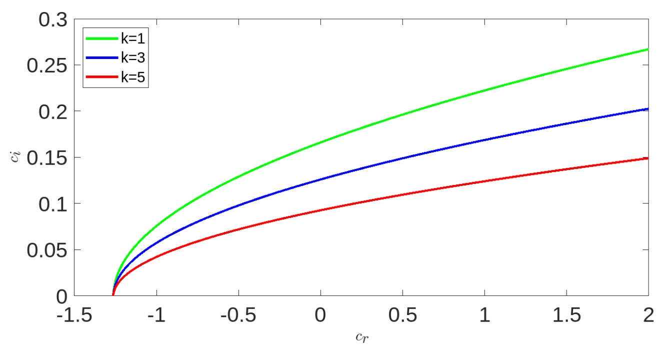

FIGURE 8. c_r vs c_i parabola for distinct values of k.

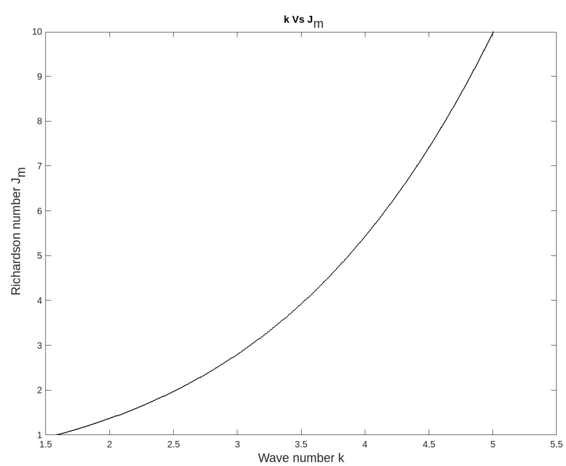

FIGURE 9. k vs J$_m$ wave number versus Richardson number.

4. Bounded Instability Region

Theorem 4.1. *If $|D(U)|^2_{\min} > 0$ then the range of (c_r, c_i) is given by*

$$\left[c_r - \left(\frac{U_{\min} + U_{\max}}{2}\right)\right]^2 + c_i^2 + \frac{J_m}{\left(1 + \frac{A_1}{c_i}\right)^2} c_i^2 \leq \left[\frac{U_{\max} - U_{\min}}{2}\right]^2,$$

where

$$A_1^2 = \frac{[U_{\max} - U_{\min}]^2 \left[\left|D^2(U) - \beta\right|_{\max} \left|\frac{U_{\max} - U_{\min}}{4}\right| + \left|N^2\right|_{\max} - \frac{|N^2|_{\min}}{4}\right]}{|D(U)|^2_{\min}}.$$

Proof. Multiplying (1) by, ϕ^*integrating, applying (2) and equating real parts, we get

$$\int_{z_1}^{z_2} \left[|D(\phi)|^2 + k^2 |\phi|^2\right] dz + \int_{z_1}^{z_2} \frac{\left(D^2(U) - \beta\right)(U - c_r)}{|U - c|^2} |\phi|^2 dz - \int_{z_1}^{z_2} \frac{N^2 \left[(U - c_r)^2 - c_i^2\right]}{|U - c|^4} |\phi|^2 dz = 0.$$

Using triangular inequality, we get

$$\int_{z_1}^{z_2} \left[|D(\phi)|^2 + k^2 |\phi|^2\right] dz \leq \int_{z_1}^{z_2} \frac{\left(D^2(U) - \beta\right)(U - c_r)}{|U - c|^2} |\phi|^2 dz + \int_{z_1}^{z_2} \frac{N^2}{|U - c|^4} \left[(U - c_r)^2 - c_i^2\right] |\phi|^2 dz. \tag{11}$$

Using the inequalities $\frac{c_i^2}{|U-c|^2} \leq 1$, $|U - c_r| \leq |U_{\max} - U_{\min}|$, $c_i \leq \frac{U_{\max} - U_{\min}}{2}$, we have

$$\int_{z_1}^{z_2} \left[|D(\phi)|^2 + k^2 |\phi|^2\right] dz \leq \frac{\left[\left|D^2(U) - \beta\right|_{\max} \frac{(U_{\max} - U_{\min})^3}{4} + \left|N^2\right|_{\max} (U_{\max} - U_{\min})^2 - \left|N^2\right|_{\min} \frac{(U_{\max} - U_{\min})^2}{4}\right]}{c_i^2} \int_{z_1}^{z_2} \frac{|\phi|^2}{|U-c|^2} dz. \tag{12}$$

Using

$$\phi = (U - c)\varphi, \tag{13}$$

we get

$$|D(\phi)|^2 \geq |U - c|^2 |D(\varphi)|^2 - 2|U - c| |D(U)| |\varphi| |D(\varphi)| + |D(U)|^2 |\varphi|^2. \tag{14}$$

Using Cauchy-Schwartz inequality, we get

$$\int_{z_1}^{z_2} |U - c| |D(U)| |\varphi| |D(\varphi)| dz \leq BC, \tag{15}$$

where

$$B^2 = \int_{z_1}^{z_2} |D(U)|^2 |\varphi|^2 \, dz; \tag{16}$$

$$C^2 = \int_{z_1}^{z_2} |U - c|^2 \left[|D(\varphi)|^2 + k^2 |\varphi|^2 \right] dz. \tag{17}$$

Substituting (13), (14), (15), (16) and (17) in (12), we get

$$[C - B]^2 \leq (U_{\max} - U_{\min})^2$$
$$\frac{\left[\left| D^2(U) - \beta \right|_{\max} \left| \frac{U_{\max} - U_{\min}}{4} \right| + \left| N^2 \right|_{\max} - \frac{\left| N^2 \right|_{\min}}{4} \right]}{c_i^2} \int_{z_1}^{z_2} |\varphi|^2 \, dz. \tag{18}$$

From (16), we have

$$\frac{B^2}{|D(U)|^2_{\min}} \geq \int_{z_1}^{z_2} |\varphi|^2 \, dz. \tag{19}$$

Substituting (19) in (18), we get

$$[C - B]^2 \leq A_1^2 B^2,$$

where

$$A_1^2 = \frac{[U_{\max} - U_{\min}]^2 \left[\left| D^2(U) - \beta \right|_{\max} \left| \frac{U_{\max} - U_{\min}}{4} \right| + \left| N^2 \right|_{\max} - \frac{\left| N^2 \right|_{\min}}{4} \right]}{|D(U)|^2_{\min}}.$$

From the above equation, we have

$$B^2 \geq \frac{C^2}{\left(1 + \frac{A_1}{c_i} \right)^2}. \tag{20}$$

Now,

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq J_m B^2, \tag{21}$$

where

$$J_m = \frac{N^2}{|D(U)|^2_{\min}}.$$

Substituting (21) in (20), we get

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq \frac{J_m C^2}{\left(1 + \frac{A_1}{c_i} \right)^2}. \tag{22}$$

Using the inequality $|U - c|^2 \geq c_i^2$ in (17), we have

$$C^2 \geq c_i^2 \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2 \right] dz. \tag{23}$$

Substituting (23) in (22), we have

$$\int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \geq \frac{J_m c_i^2}{\left(1+\frac{A_1}{c_i}\right)^2} \int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz; \tag{24}$$

From [6], we have

$$\left[\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 - \left[\frac{U_{\max}-U_{\min}}{2}\right]^2\right]$$
$$\int_{z_1}^{z_2} \left[|D(\varphi)|^2 + k^2 |\varphi|^2\right] dz + \int_{z_1}^{z_2} N^2 |\varphi|^2 \, dz \leq 0. \tag{25}$$

Substituting (24) in (25), we get

$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + \frac{J_m}{\left(1+\frac{A_1}{c_i}\right)^2} c_i^2 \leq \left[\frac{U_{\max}-U_{\min}}{2}\right]^2.$$

When β =0, the result will reduce to sharpen result for standard Taylor-Goldstein problem. The results of Howard semi-circle, [[7]],[[20]] depends on velocity profile, Richardson number and curvature. But the above result depends on velocity profile, Richardson number, vorticity function, curvature and stratification parameter. □

Theorem 4.2. *The range of* (c_r, c_i) *is given by*

$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_m\left(1+\frac{A_2}{c_i^2}\right)^2 c_i^2 \leq \left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

where

$$A_2^2 = \frac{[U_{\max}-U_{\min}]^2 \left[\left|D^2(U)-\beta\right|_{\max} \left|\frac{U_{\max}-U_{\min}}{4}\right| + \left|N^2\right|_{\max} - \frac{|N^2|_{\min}}{4}\right]}{\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]}.$$

Proof. Using Rayleigh Ritz inequality and $|U-c|^2 \geq c_i^2$ in (17), we get

$$\int_{z_1}^{z_2} |\varphi|^2 \, dz \leq \frac{C^2}{c_i^2\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]}. \tag{26}$$

Substituting (26) in (18), we get

$$[C-B]^2 \leq \frac{A_2^2 C^2}{c_i^4},$$

where

$$A_2^2 = \frac{[U_{\max}-U_{\min}]^2 \left[\left|D^2(U)-\beta\right|_{\max} \left|\frac{U_{\max}-U_{\min}}{4}\right| + \left|N^2\right|_{\max} - \frac{|N^2|_{\min}}{4}\right]}{\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]}.$$

From the above equation, we have

$$1-\frac{A_2}{c_i^2}\leq\frac{B}{C};$$

If $C^2\leq\frac{B^2}{\left(1+\frac{A_2}{c_i^2}\right)^2}\leq\frac{B^2}{\left(1-\frac{A_2}{c_i^2}\right)^2}$ then

$$C^2\left(1+\frac{A_2}{c_i^2}\right)^2\leq B^2. \tag{27}$$

Substituting (27) in (22), we get

$$\int_{z_1}^{z_2}N^2\left|\varphi\right|^2dz\geq J_mC^2\left(1+\frac{A_2}{c_i^2}\right)^2. \tag{28}$$

Substituting (28) in (25), we get

$$\left[c_r-\left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2+c_i^2+J_m\left(1+\frac{A_2}{c_i^2}\right)^2c_i^2\leq\left[\frac{U_{\max}-U_{\min}}{2}\right]^2.$$

□

When $\beta=0$, the result will reduce to a sharpen result for standard Taylor-Goldstein problem. [[7]] result depends on condition like minimum curvature should be positive. The new instability region does not depend on any condition. It also depends on basic velocity profile, vorticity function, Coriolis force ,stratification parameter, wave number.

Theorem 4.3. *The range of* $(c_r,\,c_i)$ *is given by*

$$\left[c_r-\left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2+c_i^2+J_m\left(1+\frac{A_3}{c_i^2}\right)^2c_i^2\leq\left[\frac{U_{\max}-U_{\min}}{2}\right]^2,$$

where

$$A_3^2=\frac{\left[U_{\max}-U_{\min}\right]^2\left[\left|D^2(U)-\beta\right|_{\max}\left|\frac{U_{\max}-U_{\min}}{4}\right|+\left|N^2\right|_{\max}-\frac{\left|N^2\right|_{\min}}{4}\right]}{k^2}.$$

Proof. From (17), dropping the first term which is positive from R.H.S., we

$$\int_{z_1}^{z_2}\left|\varphi\right|^2dz\leq\frac{C^2}{c_i^2k^2}. \tag{29}$$

Substituting (29) in (18), we get

$$\left[C-B\right]^2\leq\frac{A_3^2C^2}{c_i^4},$$

where

$$A_3^2=\frac{\left[U_{\max}-U_{\min}\right]^2\left[\left|D^2(U)-\beta\right|_{\max}\left|\frac{U_{\max}-U_{\min}}{4}\right|+\left|N^2\right|_{\max}-\frac{\left|N^2\right|_{\min}}{4}\right]}{k^2}.$$

If
$$C^2 \le \frac{B^2}{\left(1+\frac{A_3}{c_i^2}\right)^2} \le \frac{B^2}{\left(1-\frac{A_3}{c_i^2}\right)^2},$$
then
$$C^2\left(1+\frac{A_3}{c_i^2}\right)^2 \le B^2. \tag{30}$$
Substituting (30) in (22), we get
$$\int_{z_1}^{z_2} N^2 \left|\varphi\right|^2 dz \ge J_m C^2 \left(1+\frac{A_3}{c_i^2}\right)^2. \tag{31}$$
Substituting (31) in (25), we get
$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_m\left(1+\frac{A_3}{c_i^2}\right)^2 c_i^2 \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2. \tag{32}$$
□

5. Criterion for Wave Number:

Theorem 5.1. *If $k \le k_c$, where*
$$k_c^2 = \left[\frac{2J_m\left(U_{\max}-U_{\min}\right)\sqrt{\left|D^2(U)-\beta\right|\left|\frac{U_{\max}-U_{\min}}{4}\right| + \left|N^2\right|_{\max} - \frac{\left|N^2\right|_{\min}}{4}}}{\left(\frac{U_{\max}-U_{\min}}{2}\right)^2}\right]^2$$
implies stability.

Proof. From (32), we have
$$\left[c_r - \left(\frac{U_{\min}+U_{\max}}{2}\right)\right]^2 + c_i^2 + J_m\left(1+\frac{A_3^2}{c_i^2}+2A_3\right) \le \left[\frac{U_{\max}-U_{\min}}{2}\right]^2;$$
For stability,
$$2A_3 J_m \ge \left(\frac{U_{\max}-U_{\min}}{2}\right)^2;$$
i.e;
$$2J_m\left[\frac{\left(U_{\max}-U_{\min}\right)\sqrt{\left|D^2(U)-\beta\right|_{\max}\left|\frac{U_{\max}-U_{\min}}{4}\right| + \left|N^2\right|_{\max} - \frac{\left|N^2\right|_{\min}}{4}}}{k}\right] \le \left(\frac{U_{\max}-U_{\min}}{2}\right)^2$$
From the above equation, we get
$$k \le \frac{\left[2J_m\left(U_{\max}-U_{\min}\right)\sqrt{\left|D^2(U)-\beta\right|_{\max}\left|\frac{U_{\max}-U_{\min}}{4}\right| + \left|N^2\right|_{\max} - \frac{\left|N^2\right|_{\min}}{4}}\right]}{\left(\frac{U_{\max}-U_{\min}}{2}\right)^2}.$$

From above, we have the statement of the theorem.

[1] derived condition for stability for the case of homogeneous shear flows. The new result is the stability criterion for heterogeneous shear flows. When β =0, it will be a new result for standard Taylor- Goldstein problem.

Example 1.
1. $U = 1 - z^2$, $N^2 = \beta = z$, $z \in [0, 1]$
for the above flow $k \leq 2.4495$ implies stability.
2. $U = 1 - z^2$, $N^2 = z$, $\beta = a\, constant$, $z \in [0, 1]$
for the above flow $k \leq 2.6458$ implies stability.
3. $U = 1 - z^2$, $N^2 = \beta = a\,\text{constant}$, $z \in [0, 1]$
for the above flow $k \leq 2.4495$ implies stability.
4. $U = 1 - z^2$, $N^2 = a\, constant$, $\beta = z$, $z \in [0, 1]$
for the above flow $k \leq 2.2361$ implies stability.
5. $U = z$, $N^2 = \beta = a\, constant$, $z \in [0, 1]$
for the above flow $k \leq 8$ implies stability.
6. $U = z$, $N^2 = a\ constant, \beta = z$, $z \in [0, 1]$
for the above flow $k \leq 6.9282$ implies stability. □

6. Growth Rate

Theorem 6.1. *The upper bound for the growth rate of an unstable mode is*

$$k^2 c_i^2 \leq \frac{\left[\left|D^2(U) - \beta\right|_{\max} \left|\frac{U_{\max} - U_{\min}}{4}\right| + \frac{\left|N^2\right|_{\max}}{2} - \left|N^2\right|_{\min}\right]}{\left[\frac{\pi^2}{k^2 (z_2 - z_1)^2} + 1\right]}.$$

Proof. Using $\frac{U - c_r}{|U - c|^2} \leq \frac{1}{2c_i}$ and $(U - c_r)^2 - c_i^2 = 2\,(U - c_r)^2 - |U - c|^2$ and Rayleigh-Ritz inequality in (11), we get

$$\left[\frac{\pi^2}{(z_2 - z_1)^2} + k^2\right] \int_{z_1}^{z_2} |\phi|^2\, dz$$

$$\leq \frac{\left[\frac{\left|D^2(U) - \beta\right|_{max}}{2} c_i + \frac{\left|N^2\right|_{\max}}{2} - \left|N^2\right|_{\min}\right]}{c_i^2} \int_{z_1}^{z_2} |\phi|^2\, dz.$$

Using $c_i \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)$, we have

$$k^2 c_i^2 \leq \frac{\left[\left|D^2(U) - \beta\right|_{\max} \left|\frac{U_{\max} - U_{\min}}{4}\right| + \frac{\left|N^2\right|_{\max}}{2} - \left|N^2\right|_{\min}\right]}{\left[\frac{\pi^2}{k^2 (z_2 - z_1)^2} + 1\right]}.$$

□

7. Concluding Remarks:

In this paper, we consider Taylor-Goldstein problem in β plane. For this problem, we obtained first an unbounded instability region which intersect with Kochar & Jain semi ellipse under certain condition. Unlike [[4]], the instability region does not depend on any conditions. Second, we obtained a bounded instability regions depending on Coriolis, stratification parameters and basic velocity profile. The bounded instability region is a generalized result for heterogeneous shear flows. Third, we obtained a criterion for wave stability. The result has been illustrated with Couette flow and plane Poiseuille basic flow examples. The result can be extended to standard Taylor- Goldstein problem. Also, we obtained an upper bound for growth rate. Finally, we want to add that some of the above results can be extended to incompressible, inviscid, stratified shear flows with variable bottom topography and will be reported later.

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