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SOME REMARKS ON PAIRWISE FUZZY SEMI VOLTERRA SPACES

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ABSTRACT. The purpose of this paper is to introduce the concept of pairwise fuzzy semi door spaces and study its properties and applications. The conditions for a pairwise fuzzy semi door space to become a pairwise fuzzy semi Volterra space and for a pairwise fuzzy semi Volterra space together with a pairwise fuzzy semi door space to become a pairwise fuzzy semi Baire space are established. Also, the inter-relations between pairwise fuzzy semi Volterra spaces and other fuzzy bitopological spaces such as pairwise fuzzy semi Baire space, pairwise fuzzy semi σ-Baire space, pairwise fuzzy semi D-Baire space, pairwise fuzzy semi GID-space, pairwise fuzzy semi door space are also discussed in this paper.

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1. Introduction

In 1965, the notion of fuzzy sets introduced by L.A.Zadeh [9] as a body of concepts and techniques aimed at providing a systematic framework for dealing with the *vagueness* and *imprecision* inherent in human *thought* processes, inspired mathematicians to fuzzify mathematical structures. General topology is one of the important branches of mathematics in which fuzzy set theory has been applied systematically. The theory of general topology is based on the set operations of unions, intersections and complementation. Fuzzy sets were assumed to have a set theoretic behaviour almost identical to that of ordinary sets. It is therefore natural to extend the concept of point set topology to fuzzy sets resulting in a theory of fuzzy topology. Using fuzzy sets introduced by Zadeh, C.L.Chang [2] advanced the concept of fuzzy topological spaces in 1968. In 1989, the concept of fuzzy bitopological spaces was introduced by

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A.Kandil[3]. The notion of pairwise fuzzy semi Volterra spaces was introduced and studied by G.Thangaraj and V.Chandiran [5] in 2020. The purpose of this paper is to introduce the concept of pairwise fuzzy semi door spaces and study its properties and applications. The conditions for a pairwise fuzzy semi door space to become a pairwise fuzzy semi Volterra space and for a pairwise fuzzy semi Volterra space together with a pairwise fuzzy semi door space to become a pairwise fuzzy semi Baire space are established. Also, the inter-relations between pairwise fuzzy semi Volterra spaces and other fuzzy bitopological spaces such as pairwise fuzzy semi Baire space, pairwise fuzzy semi σ-Baire space, pairwise fuzzy semi D-Baire space, pairwise fuzzy semi GID-space, pairwise fuzzy semi door space are also discussed in this paper.

2. Preliminaries

A fuzzy bitopological space (Kandil, 1989) or fbts in short we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are two fuzzy topologies on a non-empty set X. Throughout this paper, the indices i and j take values in $\{1, 2\}$ and $i \neq j$.

Definition 2.1. [4] A fuzzy set ν in a fbts (X, T_1, T_2) is called a pairwise fuzzy semi open set or pfso set in short if $\nu \leq scl_{T_i} sint_{T_j}(\nu)$, $(i \neq j$ and $i, j = 1, 2)$.

Definition 2.2. [4] A fuzzy set ν in a fbts (X, T_1, T_2) is called a pairwise fuzzy semi closed set or pfsc set in short if $sint_{T_i} scl_{T_j}(\nu) \leq \nu$, $(i \neq j$ and $i, j = 1, 2)$.

Definition 2.3. [5] A fuzzy set ν is called a pairwise fuzzy semi G_δ-set or pfsG_δ-set in short in a fbts (X, T_1, T_2) if $\nu = \wedge_{k=1}^{\infty}(\nu_k)$, where (ν_k)'s are pfso sets.

Definition 2.4. [5] A fuzzy set ν is called a pairwise fuzzy semi F_σ-set or pfsF_σ-set in short in a fbts (X, T_1, T_2) if $\nu = \vee_{k=1}^{\infty}(\nu_k)$, where (ν_k)'s are pfsc sets.

Definition 2.5. [5] A fuzzy set ν is called a pairwise fuzzy semi dense set or pfsd set in short in a fbts (X, T_1, T_2) if $scl_{T_i} scl_{T_j}(\nu) = 1$, $(i \neq j$ and $i, j = 1, 2)$.

Definition 2.6. [8] A fuzzy set ν is called a pairwise fuzzy semi nowhere dense set or pfsnd set in short in a fbts (X, T_1, T_2) if $sint_{T_i} scl_{T_j}(\nu) = 0$, $(i \neq j$ and $i, j = 1, 2)$.

Definition 2.7. [8] A fuzzy set ν is called a pairwise fuzzy semi first category set or pfsfc set in short in a fbts (X, T_1, T_2) if $\nu = \vee_{k=1}^{\infty}(\nu_k)$, where (ν_k)'s are pfsnd sets. Any other fuzzy set is said to be a pairwise fuzzy semi second category set or pfssc set in short.

Definition 2.8. [5] If ν is a pfsfc set in a fbts (X, T_1, T_2), then the fuzzy set $1 - \nu$ is called a pairwise fuzzy semi residual set or pfsr set in short.

Definition 2.9. [5] A fuzzy set ν in a fbts (X, T_1, T_2) is called a pairwise fuzzy semi σ-nowhere dense set or pfsσ-nd set in short if ν is a pfsF_σ-set such that $sint_{T_i} sint_{T_j}(\nu) = 0$, $(i \neq j$ and $i, j = 1, 2)$.

Definition 2.10. [7] A fuzzy set ν in a fbts (X, T_1, T_2) is called a pairwise fuzzy semi σ-first category set or pfsσ-fc set in short if $\nu = \vee_{k=1}^{\infty}(\nu_k)$, where (ν_k)'s are pfsσ-nd sets. Any other fuzzy set is said to be a pairwise fuzzy semi σ-second category set or pfsσ-sc set in short.

Definition 2.11. [5] A fbts (X, T_1, T_2) is called a pairwise fuzzy semi Volterra space or pfsVs in short if $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1,\ (i = 1, 2)$ where (ν_k)'s are pfsd and pfsG_δ-sets.

Definition 2.12. [8] Let (X, T_1, T_2) be a fbts. Then (X, T_1, T_2) is called a pairwise fuzzy semi Baire space or pfsBs in short if $sint(\vee_{k=1}^{\infty}(\nu_k)) = 0$, where (ν_k)'s are pfsnd sets.

Definition 2.13. [7] A fbts (X, T_1, T_2) is called a pairwise fuzzy semi σ-Baire space or pfsσ-Bs in short if $sint_{T_i}(\vee_{k=1}^{\infty}(\nu_k)) = 0,\ (i = 1, 2)$ where (ν_k)'s are pfsσ-nd sets in (X, T_1, T_2).

Definition 2.14. [6] A fbts (X, T_1, T_2) is called a pairwise fuzzy semi D-Baire space or pfs$D$-Bs in short if every pfsfc set is a pfsnd set. That is, if ν is a pfsfc set, then $sint_{T_i} scl_{T_j}(\nu) = 0,\ (i, j = 1, 2 \text{ and } i \neq j)$.

Definition 2.15. [6] A fbts (X, T_1, T_2) is called a pairwise fuzzy semi strongly irresolvable space or pfssis in short if $scl_{T_i} sint_{T_j}(\nu) = 1,\ (i \neq j \text{ and } i, j = 1, 2)$ for each pfsd set ν.

Theorem 2.16. [8] *If a fuzzy set ν is a pfsnd set in a fbts (X, T_1, T_2), then the fuzzy set $1 - \nu$ is a pfsd set.*

Theorem 2.17. [5] *A fuzzy set ν is a pfsσ-nd set if and only if $1 - \nu$ is a pfsd and pfsG_δ-set in a fbts (X, T_1, T_2).*

Theorem 2.18. [8] *Let (X, T_1, T_2) be a fbts. Then the following are equivalent:*
(1). *(X, T_1, T_2) is a pfsBs.*
(2). *$sint_{T_i}(\nu) = 0,\quad (i = 1, 2)$, for every pfsfc set ν.*
(3). *$scl_{T_i}(\gamma) = 1,\quad (i = 1, 2)$, for every pfsr set γ.*

Theorem 2.19. [6] *If $scl_{T_i} scl_{T_j}(\nu) = 1,\ (i, j = 1, 2 \text{ and } i \neq j)$ for a fuzzy set ν in a pfssis (X, T_1, T_2), then $scl_{T_i}(\nu) = 1$ in (X, T_1, T_2).*

Theorem 2.20. [5] *If ν is a pfsd and pfsG_δ-set in a pfssis (X, T_1, T_2), then $1 - \nu$ is a pfsfc set.*

Theorem 2.21. [5] *If a pfsG_δ-set ν in a fbts (X, T_1, T_2) such that $scl_{T_i}(\nu) = 1,\ (i = 1, 2)$, then $1 - \nu$ is a pfsfc set.*

3. Pairwise fuzzy semi Volterra spaces and pairwise fuzzy semi GID-spaces

Definition 3.1. A fbts (X, T_1, T_2) is said to a pairwise fuzzy semi GID-space or pfsGID-s in short if for each pfsd and pfsG_δ-set ν, $scl_{T_i} sint_{T_j}(\nu) = 1,\ (i \neq j \text{ and } i, j = 1, 2)$.

Example 3.2. Let $X = \{a, b, c\}$. Consider the fuzzy sets λ_l, $(l = 1$ to $5)$ defined on X as follows:

$\lambda_1 : X \to [0,1]$ is defined as $\lambda_1(a) = 0.6$; $\lambda_1(b) = 0.4$; $\lambda_1(c) = 0.5$,
$\lambda_2 : X \to [0,1]$ is defined as $\lambda_2(a) = 0.4$; $\lambda_2(b) = 0.7$; $\lambda_2(c) = 0.6$,
$\lambda_3 : X \to [0,1]$ is defined as $\lambda_3(a) = 0.5$; $\lambda_3(b) = 0.3$; $\lambda_3(c) = 0.7$,
$\lambda_4 : X \to [0,1]$ is defined as $\lambda_4(a) = 0.5$; $\lambda_4(b) = 0.4$; $\lambda_4(c) = 0.6$,
$\lambda_5 : X \to [0,1]$ is defined as $\lambda_5(a) = 0.5$; $\lambda_5(b) = 0.2$; $\lambda_5(c) = 0.7$.

Then, $T_1 = \{0, \lambda_1, \lambda_2, \lambda_3, \lambda_1 \vee \lambda_2, \lambda_1 \vee \lambda_3, \lambda_2 \vee \lambda_3, \lambda_1 \wedge \lambda_2, \lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_3, \lambda_3 \wedge (\lambda_1 \vee \lambda_2), \lambda_2 \wedge (\lambda_1 \vee \lambda_3), \lambda_1 \wedge (\lambda_2 \vee \lambda_3), \lambda_3 \vee (\lambda_1 \wedge \lambda_2), \lambda_2 \vee (\lambda_1 \wedge \lambda_3), \lambda_1 \vee (\lambda_2 \wedge \lambda_3), \lambda_1 \wedge \lambda_2 \wedge \lambda_3, \lambda_1 \vee \lambda_2 \vee \lambda_3, 1\}$ and $T_2 = \{0, \lambda_1, \lambda_2, \lambda_5, \lambda_1 \vee \lambda_2, \lambda_1 \vee \lambda_5, \lambda_2 \vee \lambda_5, \lambda_1 \wedge \lambda_2, \lambda_1 \wedge \lambda_5, \lambda_2 \wedge \lambda_5, \lambda_5 \wedge (\lambda_1 \vee \lambda_2), \lambda_2 \wedge (\lambda_1 \vee \lambda_5), \lambda_1 \wedge (\lambda_2 \vee \lambda_5), \lambda_5 \vee (\lambda_1 \wedge \lambda_2), \lambda_2 \vee (\lambda_1 \wedge \lambda_5), \lambda_1 \vee (\lambda_2 \wedge \lambda_5), \lambda_1 \wedge \lambda_2 \wedge \lambda_5, \lambda_1 \vee \lambda_2 \vee \lambda_5, 1\}$ are fuzzy topologies on X. Clearly λ_1, λ_2, $\lambda_1 \vee \lambda_2$, $\lambda_1 \vee \lambda_3$, $\lambda_2 \vee \lambda_3$, $\lambda_1 \wedge \lambda_2$, $\lambda_2 \wedge (\lambda_1 \vee \lambda_3)$, $\lambda_1 \wedge (\lambda_2 \vee \lambda_3)$, $\lambda_3 \vee (\lambda_1 \wedge \lambda_2)$, $\lambda_2 \vee (\lambda_1 \wedge \lambda_3)$, $\lambda_1 \vee (\lambda_2 \wedge \lambda_3)$, $\lambda_1 \vee \lambda_2 \vee \lambda_3$, $\lambda_1 \vee \lambda_5$, $\lambda_2 \vee \lambda_5$, $\lambda_2 \wedge (\lambda_1 \vee \lambda_5)$, $\lambda_1 \wedge (\lambda_2 \vee \lambda_5)$, $\lambda_5 \vee (\lambda_1 \wedge \lambda_2)$, $\lambda_2 \vee (\lambda_1 \wedge \lambda_5)$, $\lambda_1 \vee (\lambda_2 \wedge \lambda_5)$, $\lambda_1 \vee \lambda_2 \vee \lambda_5$ are pfso sets. Now $\lambda_4 = [\lambda_1 \vee (\lambda_2 \wedge \lambda_3)] \wedge [\lambda_2 \vee (\lambda_1 \wedge \lambda_3)] \wedge [\lambda_3 \vee (\lambda_1 \wedge \lambda_2)]$. Then λ_4 is a pfsG_δ-set. Also, $scl_{T_1} scl_{T_2}(\lambda_4) = 1$ and $scl_{T_2} scl_{T_1}(\lambda_4) = 1$ and hence λ_4 is a pfsd set. Thus, λ_4 is a pfsd and pfsG_δ-set. Also, $scl_{T_2} sint_{T_1}(\lambda_4) = scl_{T_2}[\lambda_2 \wedge (\lambda_1 \vee \lambda_3)] = 1$ and $scl_{T_1} sint_{T_2}(\lambda_4) = scl_{T_1}[\lambda_2 \wedge (\lambda_1 \vee \lambda_5)] = 1$. Also, $\lambda_1 \wedge \lambda_2 = [\lambda_1 \wedge \lambda_2] \wedge [\lambda_1 \wedge \lambda_3] \wedge [\lambda_2 \wedge \lambda_3] \wedge [\lambda_1 \wedge (\lambda_2 \wedge \lambda_3)] \wedge [\lambda_2 \wedge (\lambda_1 \vee \lambda_5)]$ and hence $\lambda_1 \wedge \lambda_2$ is a pfsG_δ-set. This implies that $scl_{T_2} scl_{T_1}(\lambda_1 \wedge \lambda_2) = scl_{T_2}(1 - \lambda_1) = 1 - \lambda_1 \neq 1$ and $scl_{T_1} scl_{T_2}(\lambda_1 \wedge \lambda_2) = scl_{T_1}(1 - \lambda_1) = 1 - \lambda_1 \neq 1$ and hence $\lambda_1 \wedge \lambda_2$ is not a pfsd set. So, $\lambda_1 \wedge \lambda_2$ is a pfsG_δ-set but not a pfsd set. Therefore, for a pfsd and pfsG_δ-set λ_4, $scl_{T_i} sint_{T_j}(\lambda_4) = 1$, $(i \neq j$ and $i, j = 1, 2)$. So, (X, T_1, T_2) is a pfsGID-s.

The following Proposition shows the inter-relations among the three fbts namely, pfsGID-s, pfsBs and pfsVs.

Proposition 3.3. *If a pfsGID-s (X, T_1, T_2) is a pfsBs, then (X, T_1, T_2) is a pfsVs.*

Proof. Let (ν_k)'s $(k = 1$ to $N)$ be the pfsd and pfsG_δ-sets. Since (X, T_1, T_2) is a pfsGID-s, $scl_{T_i} sint_{T_j}(\nu_k) = 1$, $(i \neq j$ and $i, j = 1, 2)$ for the pfsd and pfsG_δ-sets (ν_k)'s. Now, $sint_{T_i} scl_{T_j}(1 - \nu_k) = 1 - scl_{T_i} sint_{T_j}(\nu_k) = 1 - 1 = 0$. This implies that $(1 - \nu_k)$'s are pfsnd sets. Let (γ_k)'s $(k = 1$ *to* $\infty)$ be pfsnd sets in which the first N pfsnd sets be $(1 - \nu_k)$'s. Since (X, T_1, T_2) is a pfsBs, $sint_{T_i}(\vee_{k=1}^{\infty}(\gamma_k)) = 0$ where (γ_k)'s are pfsnd sets. Now $sint_{T_i}(\vee_{k=1}^{N}(1 - \nu_k)) \leq sint_{T_i}(\vee_{k=1}^{\infty}(\gamma_k))$. Then, $sint_{T_i}(\vee_{k=1}^{N}(1 - \nu_k)) \leq 0$. That is, $sint_{T_i}(\vee_{k=1}^{N}(1 - \nu_k)) = 0$. This implies that $1 - scl_{T_i}(\wedge_{k=1}^{N}(\nu_k)) = 0$ and hence $scl_{T_i}(\wedge_{k=1}^{N}(\nu_k)) = 1$, where (ν_k)'s are pfsd and pfsG_δ-sets. So, (X, T_1, T_2) is a pfsVs. □

Proposition 3.4. *A fuzzy set ν is a pfso set in a fbts (X, T_1, T_2) if and only if $1 - \nu$ is a pfsc set.*

Proof. Let ν be a pfso set. Then, $\nu \leq scl_{T_i} sint_{T_j}(\nu)$, $(i \neq j$ and $i, j = 1, 2)$. Now, $sint_{T_i} scl_{T_j}(1 - \nu) = 1 - scl_{T_i} sint_{T_j}(\nu) \leq 1 - \nu$. So, $1 - \nu$ is a pfsc set.

Conversely, let ν be a pfsc set. Then, $sint_{T_i}scl_{T_j}(\nu) \leq \nu$. Now, $scl_{T_i}sint_{T_j}(1-\nu) = 1 - sint_{T_i}scl_{T_j}(\nu) \geq 1-\nu$. So, $1-\nu$ is a pfso set. □

Proposition 3.5. *A fuzzy set ν is a pfsG_δ-set in a fbts (X, T_1, T_2) if and only if $1-\nu$ is a pfsF_σ-set.*

Proof. Let ν be a pfsG_δ-set. Then, $\nu = \wedge_{k=1}^{\infty}(\nu_k)$, where (ν_k)'s are pfso sets. Since (ν_k)'s are pfso sets and by the Proposition 3.4, $(1-\nu_k)$'s are pfsc sets. Now $1-\nu = 1-\wedge_{k=1}^{\infty}(\nu_k) = \vee_{k=1}^{\infty}(1-\nu_k)$. Thus, $1-\nu$ is a pfsF_σ-set.

Conversely, let ν be a pfsF_σ-set. Then, $\nu = \vee_{k=1}^{\infty}(\nu_k)$, where (ν_k)'s are pfsc sets. Since (ν_k)'s are pfsc sets and by the Proposition 3.4, $(1-\nu_k)$'s are pfso sets. Now $1-\nu = 1-\vee_{k=1}^{\infty}(\nu_k) = \wedge_{k=1}^{\infty}(1-\nu_k)$. Thus, $1-\nu$ is a pfsG_δ-set. □

The following Proposition gives the condition for a pfsσ-nd set to be a pfsnd set in a pfsGID-s.

Proposition 3.6. *In a pfsGID-s (X, T_1, T_2), every pfsσ-nd set is a pfsnd set.*

Proof. Let ν be a pfsσ-nd set. Then, ν is a pfsF_σ-set such that $sint_{T_i}sint_{T_j}(\nu) = 0$, $(i \neq j$ and $i, j = 1, 2)$. Since ν is a pfsF_σ-set and by the Proposition 3.5, $1-\nu$ is a pfsG_δ-set. Also, since $sint_{T_i}sint_{T_j}(\nu) = 0$, $1 - sint_{T_i}sint_{T_j}(\nu) = 1$. This implies that $scl_{T_i}scl_{T_j}(1-\nu) = 1$ and hence $1-\nu$ is a pfsd set. Since (X, T_1, T_2) is a pfsGID-s, $scl_{T_i}sint_{T_j}(1-\nu) = 1$, for the pfsd and pfs$G_\delta$-set $1-\nu$. This implies that $1 - sint_{T_i}scl_{T_j}(\nu) = 1$ and hence $sint_{T_i}scl_{T_j}(\nu) = 0$. So, ν is a pfsnd set. □

The following Proposition shows the inter-relations among the three fbts namely, pfsGID-s, pfsσ-Bs and pfsBs.

Proposition 3.7. *If a pfsGID-s (X, T_1, T_2) is a pfsσ-Bs, then (X, T_1, T_2) is a pfsBs.*

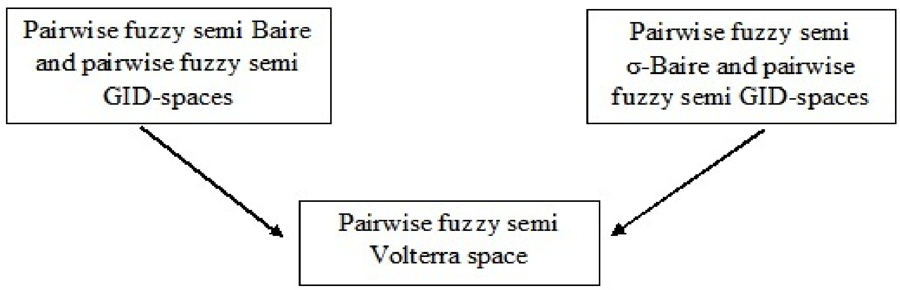
Proof. Let (ν_k)'s be the pfsσ-nd sets. Since (X, T_1, T_2) is a pfsσ-Bs, $sint_{T_i}(\vee_{k=1}^{\infty}(\nu_k)) = 0$, $(i = 1, 2)$. Also, since (X, T_1, T_2) is a pfsGID-s and by the Proposition 3.6, the pfsσ-nd sets (ν_k)'s are pfsnd sets. Now $sint_{T_k}(\vee_{k=1}^{\infty}(\nu_k)) = 0$, where (ν_k)'s are pfsnd sets. So, (X, T_1, T_2) is a pfsBs. □

The following Proposition shows the inter-relations among the three fbts namely, pfsGID-s, pfsσ-Bs and pfsVs.

Proposition 3.8. *If a pfsGID-s (X, T_1, T_2) is a pfsσ-Bs, then (X, T_1, T_2) is a pfsVs.*

Proof. Since (X, T_1, T_2) is a pfsGID-s, pfsσ-Bs and by the Proposition 3.7, (X, T_1, T_2) is a pfsBs. Also since (X, T_1, T_2) is a pfsBs, pfsGID-s and by the Proposition 3.3, (X, T_1, T_2) is a pfsVs. □

Remark 3.1. The inter-relations between the pairwise fuzzy semi Volterra space and the other fuzzy bitopological spaces such as pairwise fuzzy semi Baire space, pairwise fuzzy semi σ-Baire space, pairwise fuzzy semi GID-space can be summarized as follows:



4. Pairwise fuzzy semi Volterra spaces and pairwise fuzzy semi door spaces

Definition 4.1. A fbts (X, T_1, T_2) is said to a pairwise fuzzy semi door space or pfsds in short if each fuzzy set is either a pfso set or a pfsc set.

Proposition 4.2. *Every pfsds* (X, T_1, T_2) *is a pfssis.*

Proof. Let (X, T_1, T_2) be a pfsds. Then each fuzzy set is either a pfso set or a pfsc set.

Case (i).: Let ν be a pfso set. Then, $\nu \leq scl_{T_i} sint_{T_j}(\nu)$, $(i \neq j$ and $i, j = 1, 2)$. This implies that $scl_{T_i}(\nu) \leq scl_{T_i} scl_{T_i} sint_{T_j}(\nu)$ and hence $scl_{T_i}(\nu) \leq scl_{T_i} sint_{T_j}(\nu)$. If $scl_{T_i}(\nu) = 1$, then $1 \leq scl_{T_i} sint_{T_j}(\nu)$. That is, $scl_{T_i} sint_{T_j}(\nu) = 1$, for a pfsd set ν. So, (X, T_1, T_2) is a pfssis.

Case (ii).: Let ν be a pfsc set. Then, $sint_{T_i} scl_{T_j}(\nu) \leq \nu$. This implies that $1 - sint_{T_i} scl_{T_j}(\nu) \geq 1 - \nu$ and hence $scl_{T_i} sint_{T_j}(1 - \nu) \geq 1 - \nu$. If $scl_{T_i}(\nu) = 1$, then $1 - scl_{T_i}(\nu) = 0$ and hence $sint_{T_i}(1 - \nu) = 0$. Now, $scl_{T_i}(0) \geq 1 - \nu$, implies that $0 \geq 1 - \nu$. That is, $1 - \nu = 0$. This implies that $\nu = 1$. Clearly, $scl_{T_i} sint_{T_j}(\nu) = scl_{T_i} sint_{T_j}(1) = scl_{T_i}(1) = 1$, for a pfsd set ν. So, (X, T_1, T_2) is a pfssis.

□

The following Proposition gives a condition for a pairwise fuzzy semi door space to be a pairwise fuzzy semi Volterra space.

Proposition 4.3. *If each pfsσ-fc set is a pfsσ-nd set in a pfsds* (X, T_1, T_2), *then* (X, T_1, T_2) *is a* $pfsVs$.

Proof. Let (ν_k)'s ($k = 1$ to N) be the pfsd and pfsG_δ-sets. Then by the Theorem 2.17, $(1-\nu_k)$'s are pfsσ-nd sets. Let (γ_α)'s ($\alpha = 1$ to ∞) be the pfsσ-nd sets in which the first N pfsσ-nd sets of (γ_α)'s be $(1-\nu_k)$'s. Now $\vee_{\alpha=1}^{\infty}(\gamma_\alpha)$ is a pfsσ-fc set. By hypothesis, $\vee_{\alpha=1}^{\infty}(\gamma_\alpha)$ is a pfsσ-nd set. Then, $sint_{T_i} int_{T_j}\left(\vee_{\alpha=1}^{\infty}(\gamma_\alpha)\right) = 0$, ($i \neq j$ and $i,j = 1,2$). Now $sint_{T_i} sint_{T_j}\left(\vee_{k=1}^{N}(1-\nu_k)\right) \leq sint_{T_i} sint_{T_j}\left(\vee_{\alpha=1}^{\infty}(\gamma_\alpha)\right)$. This implies that $sint_{T_i} sint_{T_j}\left(\vee_{k=1}^{N}(1-\nu_k)\right) \leq 0$. That is, $sint_{T_i} sint_{T_j}\left(\vee_{k=1}^{N}(1-\nu_k)\right) = 0$ and then $1 - scl_{T_i} scl_{T_j}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 0$. This implies that $scl_{T_i} scl_{T_j}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1$. Since (X, T_1, T_2) is a pfsds and by the Proposition 4.2, (X, T_1, T_2) is a pfssis. Then by the Theorem 2.19, $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1$. Thus, $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1$, where (ν_k)'s are pfsd and pfsG_δ-sets. So, (X, T_1, T_2) is a pfsVs. □

Proposition 4.4. *If a pfsVs (X, T_1, T_2) is a pfsds, then $sint_{T_i}\left(\vee_{k=1}^{N}(\gamma_k)\right) = 0$, $(i = 1, 2)$ where (γ_k)'s are pfsfc sets which are formed from the pfsd and pfsG_δ-sets (ν_k)'s.*

Proof. Let (ν_k)'s ($k = 1$ to N) be the pfsd and pfsG_δ-sets. Since (X, T_1, T_2) is a pfsVs, $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1$, $(i = 1, 2)$. Then $1 - scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 0$. This implies that $sint_{T_i}\left(\vee_{k=1}^{N}(1-\nu_k)\right) = 0 \longrightarrow (A)$. Since (X, T_1, T_2) is a pfsds and by the Proposition 4.2, (X, T_1, T_2) is a pfssis. Since (ν_k)'s are pfsd and pfsG_δ-sets and by the Theorem 2.20, $(1-\nu_k)$'s are pfsfc sets. Let $\gamma_k = 1 - \nu_k$. So from (A), $sint_{T_i}\left(\vee_{k=1}^{N}(\gamma_k)\right) = 0$, where (γ_k)'s are pfsfc sets which are formed from the pfsd and pfsG_δ-sets (ν_k)'s. □

The following Proposition gives a condition for a pairwise fuzzy semi Volterra and pairwise fuzzy semi door space to be a pairwise fuzzy semi Baire space.

Proposition 4.5. *If each pfsfc set γ_k ($k = 1$ to N) is formed from the pfsd and pfsG_δ-sets (ν_k)'s in a pfsVs and pfsds (X, T_1, T_2), then (X, T_1, T_2) is a pfsBs.*

Proof. Since (X, T_1, T_2) is a pfsds and by the Proposition 4.2, (X, T_1, T_2) is a pfssis. Now $\vee_{k=1}^{N}\left(sint_{T_i}(\gamma_k)\right) \leq sint_{T_i}\left(\vee_{k=1}^{N}(\gamma_k)\right)$, $(i = 1, 2)$. Since (X, T_1, T_2) is a pfsVs, pfsds and by the Proposition 4.4, $sint_{T_i}\left(\vee_{k=1}^{N}(\gamma_k)\right) = 0$, where (γ_k)'s are pfsfc sets which are formed from the pfsd and pfsG_δ-sets (ν_k)'s. Then $\vee_{k=1}^{N}\left(sint_{T_i}(\gamma_k)\right) \leq 0$ and hence $\vee_{k=1}^{N}\left(sint_{T_i}(\gamma_k)\right) = 0$ implies that $sint_{T_i}(\gamma_k) = 0$, where (γ_k)'s are pfsfc sets. So by the Theorem 2.18, (X, T_1, T_2) is a pfsBs. □

Proposition 4.6. *If a pfsds (X, T_1, T_2) is a pfsBs, then (X, T_1, T_2) is a pfsVs.*

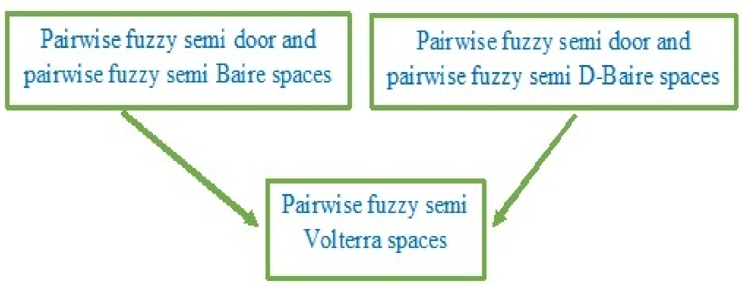
Proof. Let (ν_k)'s ($k = 1$ to N) be the pfsd and pfsG_δ-sets. Since (X, T_1, T_2) is a pfsds and by the Proposition 4.2, (X, T_1, T_2) is a pfssis. Since (ν_k)'s are pfsd sets in the pfssis (X, T_1, T_2), $scl_{T_i} sint_{T_j}(\nu_k) = 1$, ($i \neq j$ and $i,j = 1,2$). Now $sint_{T_i} scl_{T_j}(1-\nu_k) = 1 - scl_{T_i} sint_{T_j}(\nu_k) = 1 - 1 = 0$ and hence $(1-\nu_k)$'s are pfsnd sets. But $\vee_{k=1}^{N}(1-\nu_k) \leq \vee_{k=1}^{\infty}(\gamma_k)$, where (γ_k)'s are pfsnd sets in which the first N pfsnd sets of (γ_k)'s are $(1-\nu_k)$'s. Since (X, T_1, T_2) is a pfsBs, $sint_{T_i}\left(\vee_{k=1}^{\infty}(\gamma_k)\right) = 0$. Now $sint_{T_i}\left(\vee_{k=1}^{N}(1-\nu_k)\right) \leq sint_{T_i}\left(\vee_{k=1}^{\infty}(\gamma_k)\right)$ and then $sint_{T_i}\left(\vee_{k=1}^{N}(1-\nu_k)\right) \leq 0$. Thus, $sint_{T_i}\left(\vee_{k=1}^{N}(1-\nu_k)\right) = 0$. This implies that

$1 - scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 0$ and hence $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1$, where (ν_k)'s are pfsd and pfsG_δ-sets. So, (X, T_1, T_2) is a pfsVs. □

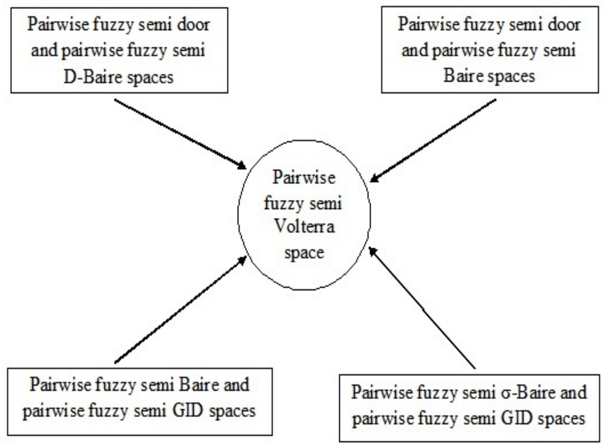
Proposition 4.7. *If a pfsds (X, T_1, T_2) is a pfsD-Bs, then (X, T_1, T_2) is a pfsVs.*

Proof. Let (ν_k)'s $(k = 1$ to $N)$ be the pfsG_δ-sets such that $scl_{T_i}(\nu_k) = 1$, $(i = 1, 2)$. Then, by the Theorem 2.21, $(1 - \nu_k)$'s are pfsfc sets. Since (X, T_1, T_2) is a pfs$D$-Bs, $(1 - \nu_k)$'s are pfsnd sets. Then by the Theorem 2.16, (ν_k)'s are pfsd sets. Let (γ_α)'s $(\alpha = 1$ to $\infty)$ be pfsnd sets in which the first N pfsnd sets of (γ_α)'s are $(1 - \nu_k)$'s. Now $\vee_{\alpha=1}^{\infty}(\gamma_\alpha)$ is a pfsfc set. Since (X, T_1, T_2) is a pfs$D$-Bs, $\vee_{\alpha=1}^{\infty}(\gamma_\alpha)$ is a pfsnd set. Then by the Theorem 2.16, $1 - \vee_{\alpha=1}^{\infty}(\gamma_\alpha)$ is a pfsd set and hence $\wedge_{\alpha=1}^{\infty}(1 - \gamma_\alpha)$ is a pfsd set. Then, $scl_{T_i} scl_{T_j}\left(\wedge_{\alpha=1}^{\infty}(1 - \gamma_\alpha)\right) = 1$, $(i \neq j$ and $i, j = 1, 2)$. Since (X, T_1, T_2) is a pfsds and by the Proposition 4.2, (X, T_1, T_2) is a pfssis. Then by the Theorem 2.19, $scl_{T_i}\left(\wedge_{\alpha=1}^{\infty}(1 - \gamma_\alpha)\right) = 1$. But $1 = scl_{T_i}\left(\wedge_{\alpha=1}^{\infty}(1 - \gamma_\alpha)\right) \leq scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right)$. This implies that $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) \geq 1$. Thus $scl_{T_i}\left(\wedge_{k=1}^{N}(\nu_k)\right) = 1$, where (ν_k)'s are pfsd and pfsG_δ-sets. So, (X, T_1, T_2) is a pfsVs. □

Remark 4.1. The inter-relations between the pairwise fuzzy semi Volterra spaces and the other fuzzy bitopological spaces such as pairwise fuzzy semi door space, pairwise fuzzy semi Baire space, pairwise fuzzy semi D-Baire space can be summarized as follows:



Remark 4.2. The inter-relations between the pairwise fuzzy semi Volterra space and the other fuzzy bitopological spaces such as pairwise fuzzy semi Baire space, pairwise fuzzy semi σ-Baire space, pairwise fuzzy semi D-Baire space, pairwise fuzzy semi GID-space, pairwise fuzzy semi door space can be summarized as follows:



5. Conclusion

The concept of pairwise fuzzy semi door spaces were introduced and studied its properties and applications in this paper. The conditions for a pairwise fuzzy semi door space to become a pairwise fuzzy semi Volterra space and for a pairwise fuzzy semi Volterra space together with a pairwise fuzzy semi door space to become a pairwise fuzzy semi Baire space were established. Also, the inter-relations between pairwise fuzzy semi Volterra spaces and other fuzzy bitopological spaces such as pairwise fuzzy semi Baire space, pairwise fuzzy semi σ-Baire space, pairwise fuzzy semi D-Baire space, pairwise fuzzy semi GID-space, pairwise fuzzy semi door space were also discussed in this paper.

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