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ON FUZZY β-VOLTERRA SPACES

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Abstract. The purpose of this paper is to introduce and study the new class of spaces called the fuzzy β-Volterra spaces with the help of fuzzy β-dense and fuzzy β-G_δ sets. Examples are given to illustrate the concept. Some interesting characterizations of the fuzzy β-Volterra spaces are established in this paper.

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1. Introduction

Fuzzy sets were introduced by L.A.Zadeh [11] in 1965 from his remarkable paper. The theory of topological spaces in fuzzy setting was introduced and studied by C.L.Chang [7] in 1968. M.E.Abd El-Monsef [1] introduced the notion of β-open sets in 1983 and also developed by A.A.Allam et al.[2] in 1989. G.Balasubramanian [5] introduced the notion of fuzzy β-open sets in 1997. G.Thangaraj and S.Soundara Rajan [10] introduced the concept of Volterra spaces in fuzzy setting. The main aim of this paper is to introduce and study the notion of the β-Volterra spaces in fuzzy setting.

2. Preliminaries

We give some basic notions and results used in the sequel.

Definition 2.1. [7] A fuzzy topology is a family 'T' of fuzzy sets in X which satisfies the following conditions:

(1) $\Phi, X \in T$,
(2) If $A, B \in T$, then $A \cap B \in T$,
(3) If $A_i \in T$ for each i, then $\cup A_i \in T$.

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T is called a fuzzy topology for X and the pair (X, T) is a fts or fts in short. Every member of T is called a fuzzy open set or fo set in short. A fuzzy set is a fuzzy closed set or fc set in short if and only if its complement is fo set.

Lemma 2.2. [3] *For a family $\mathcal{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, Then, $\vee\ cl(\lambda_\alpha) \leq cl(\vee\lambda_\alpha)$. In case $\mathcal{A}$ is a finite set, $\vee\ cl(\lambda_\alpha) = cl(\vee\lambda_\alpha)$. Also $\vee\ int\ (\lambda_\alpha) \leq int(\vee\lambda_\alpha)$.*

Definition 2.3. [4] A fuzzy set λ in a fts (X, T) is called a fuzzy G_δ-set or fG_δ-set in short if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fo sets.

Definition 2.4. [4] A fuzzy set λ in a fts (X, T) is called a fuzzy F_σ-set or fF_σ-set in short if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fc sets.

Definition 2.5. [8] A fuzzy set λ in a fts (X, T) is called a fuzzy dense set or fd set in short if there exists no fc set μ such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$.

Definition 2.6. [10] A fts (X, T) is said to be a fuzzy Volterra space or fVs in short if $cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fd and fG_δ-sets.

Definition 2.7. [5] A fuzzy set λ in a fts (X, T) is called a fuzzy β-open set or fβ-o set in short if $\lambda \leq clintcl(\lambda)$.

Definition 2.8. [5] A fuzzy set λ in a fts (X, T) is called a fuzzy β-closed set or fβ-c set in short if $intclint(\lambda) \leq \lambda$.

Definition 2.9. [5] Let λ be a fuzzy set in the fts (X, T). Then we define the fuzzy β-closure and the fuzzy β-interior of λ respectively as follows:
$\beta\text{-}cl(\lambda) = \wedge\{\mu \mid \mu \text{ is a f}\beta - \text{c set and } \mu \geq \lambda\}$;
$\beta\text{-}int(\lambda) = \vee\{\mu \mid \mu \text{ is a f}\beta - \text{o set and } \mu \leq \lambda\}$.

Lemma 2.10. [5] *Let λ be any fuzzy set in the fts (X, T). Then*
$\beta\text{-}cl(1 - \lambda) = 1 - \beta\text{-}int(\lambda)$ and
$\beta\text{-}int(1 - \lambda) = 1 - \beta\text{-}cl(\lambda)$.

Theorem 2.11. [6] *In a fts (X, T), the following are valid:*

(a). *λ is fβ-o $\Leftrightarrow \beta\text{-}int(\lambda) = \lambda$;*
(b). *λ is fβ-c $\Leftrightarrow \beta\text{-}cl(\lambda) = \lambda$.*

Theorem 2.12. [6] *In a fts (X, T), the following hold for fuzzy β-closure. For any two fuzzy sets λ and μ:*

(a). *$\beta\text{-}cl(0) = 0$.*
(b). *$\beta\text{-}cl(\lambda)$ is a fuzzy β-closed.*
(c). *$\beta\text{-}cl(\lambda) \leq \beta - cl(\mu)$ if $\lambda \leq \mu$.*
(d). *$\beta - cl(\beta - cl(\lambda)) = \beta - cl(\lambda)$.*
(e). *$\beta - cl(\lambda \vee \mu) \geq \beta - cl(\lambda) \vee \beta - cl(\mu)$.*
(f). *$\beta - cl(\lambda \wedge \mu) \leq \beta - cl(\lambda) \wedge \beta - cl(\mu)$.*

Similar results hold for fuzzy β-interiors.

3. Fuzzy β-G_δ sets and fuzzy β-F_σ sets

Definition 3.1. A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy β-G_δ set or fβ-G_δ set in short if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-o sets.

Definition 3.2. A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy β-F_σ set or fβ-F_σ set in short if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-c sets.

Proposition 3.3. *A fuzzy set λ is a fβ-o set in a fts (X, T) if and only if $1 - \lambda$ is a fβ-c set.*

Proof. Let λ be a fβ-o set. Then $\lambda \le clintcl(\lambda)$. Then $intclint(1 - \lambda) = 1 - clintcl(\lambda) \le 1 - \lambda$. This implies that $1 - \lambda$ is a fβ-c set.

Conversely, let λ be a fβ-c set. $intclint(\lambda) \le \lambda$. Then $clintcl(1 - \lambda) = 1 - intclint(\lambda) \ge 1 - \lambda$. That is, $1 - \lambda \le clintcl(1 - \lambda)$ and hence $1 - \lambda$ is a fβ-o set. □

Proposition 3.4. *If λ is a fG_δ-set in a fts (X, T), then λ is a fβ-G_δ set.*

Proof. Let λ be a fG_δ-set. Then $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fo sets. Since fo sets (λ_i)'s are fβ-o sets. Thus, $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-o sets and hence λ is a fβ-G_δ set. □

Proposition 3.5. *If λ is a fF_σ-set in a fts (X, T), then λ is a fβ-F_σ set.*

Proof. Let λ be a fF_σ-set. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fc sets. Since fc sets (λ_i)'s are fβ-c sets. Thus, $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-c sets and hence λ is a fβ-F_σ set. □

Proposition 3.6. *A fuzzy set λ is a fβ-G_δ set in a fts (X, T) if and only if $1 - \lambda$ is a fβ-F_σ set.*

Proof. Let λ be a fβ-G_δ set. Then by the Definition 3.1, $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-o sets. Since (λ_i)'s are fβ-o sets and by the Proposition 3.3, $(1-\lambda_i)$'s are fβ-c sets. Now $1 - \lambda = 1 - \wedge_{i=1}^{\infty}(\lambda_i) = \vee_{i=1}^{\infty}(1 - \lambda_i)$. So by the Definition 3.2, $1 - \lambda$ is a fβ-F_σ set.

Conversely, let λ be a fβ-F_σ set. Then by the Definition 3.2, $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-c sets. Since (λ_i)'s are fβ-c sets and by the Proposition 3.3, (λ_i)'s are fβ-o sets. Now $1 - \lambda = 1 - \vee_{i=1}^{\infty}(\lambda_i) = \wedge_{i=1}^{\infty}(1 - \lambda_i)$. So by the Definition 3.1, $1 - \lambda$ is a fβ-G_δ set. □

4. Fuzzy β-dense sets and fuzzy β-nowhere dense sets

Definition 4.1. [9] A fuzzy set λ in a fts (X, T) is called a fuzzy β-dense set or fβ-d set in short if there exists no fβ-c set μ such that $\lambda < \mu < 1$. That is, β-$cl(\lambda) = 1$.

Definition 4.2. [9] Let (X, T) be a fts. A fuzzy set λ in (X, T) is called a fuzzy β-nowhere dense set or fβ-nd set in short if there exists no non zero fβ-o set μ such that $\mu < \beta$-$cl(\lambda)$. That is, β-$int\beta$-$cl(\lambda) = 0$.

Proposition 4.3. *If $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(1-\lambda_i)$'s are fβ-o sets, is a fβ-nd set in a fts (X,T), then (λ_i)'s are fβ-nd sets.*

Proof. Let $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(1-\lambda_i)$'s are fβ-o sets. Then $\beta\text{-}cl(\lambda) = \beta\text{-}cl\,(\vee_{i=1}^{\infty}(\lambda_i)) \geq \vee_{i=1}^{\infty}(\beta\text{-}cl(\lambda_i))$. Since $(1-\lambda_i)$'s are fβ-o sets and by the Proposition 3.3, $(1-(1-\lambda_i))$'s are fβ-c sets and then $\beta\text{-}cl(\lambda_i) = \lambda_i \longrightarrow (1)$ and hence $\beta\text{-}cl(\lambda) \geq \vee_{i=1}^{\infty}(\lambda_i)$. Then $\beta\text{-}int\ \beta\text{-}cl(\lambda) \geq \beta\text{-}int\,(\vee_{i=1}^{\infty}(\lambda_i)) \geq \vee_{i=1}^{\infty}(\beta\text{-}int(\lambda_i))$. Since λ is a fβ-nd set, $\beta\text{-}int\ \beta\text{-}cl(\lambda) = 0$. Thus $0 \geq \vee_{i=1}^{\infty}(\beta\text{-}int(\lambda_i))$. That is, $\vee_{i=1}^{\infty}(\beta\text{-}int(\lambda_i)) = 0$ and hence $\beta\text{-}int(\lambda_i) = 0$ for each i. Now, $\beta\text{-}int\ \beta\text{-}cl(\lambda_i) = \beta\text{-}int(\lambda_i) = 0$ [from (1)]. Hence (λ_i)'s are fβ-nd sets. □

Theorem 4.4. [9] *If λ is a fβ-nd set in a fts (X,T), then $1-\lambda$ is a fβ-d set.*

Theorem 4.5. [9] *If λ is a fβ-d set in a fts (X,T), then λ is a fβ-o set.*

Proposition 4.6. *If λ is a fβ-nd set in a fts (X,T), then λ is a fβ-c set.*

Proof. Let λ is a fβ-nd set. Then by the Theorem 4.4, $1-\lambda$ is a fβ-d set. Then by the Theorem 4.5, the fβ-d set $1-\lambda$ is a fβ-o set and hence λ is a fβ-c set. □

5. Fuzzy β-first category sets and fuzzy β-residual sets

Definition 5.1. [9] A fuzzy set λ in a fts (X,T) is called a fuzzy β-first category set or fβ-fc set in short if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-nd sets. Any other fuzzy set is said to be of fβ-sc.

Definition 5.2. [9] Let λ be a fβ-fc set in a fts (X,T). Then $1-\lambda$ is called a fuzzy β-residual set or fβ-r set in short.

Proposition 5.3. *If λ is a fβ-d and fβ-G_δ set in a fts (X,T), then $1-\lambda$ is a fβ-fc set.*

Proof. Let λ be a fβ-d and fβ-G_δ set. Then $\beta\text{-}cl(\lambda) = 1$ and $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-o sets. This implies that $\beta\text{-}cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$. But $\beta\text{-}cl\,(\wedge_{i=1}^{\infty}(\lambda_i)) \leq \wedge_{i=1}^{\infty}(\beta\text{-}cl(\lambda_i))$. This implies that $1 \leq \wedge_{i=1}^{\infty}(\beta\text{-}cl(\lambda_i))$. That is, $\wedge_{i=1}^{\infty}(\beta\text{-}cl(\lambda_i)) = 1$ and hence $\beta\text{-}cl(\lambda_i) = 1$. Since (λ_i)'s are fβ-o sets, $\beta\text{-}int(\lambda_i) = \lambda_i$ and therefore $\beta\text{-}cl\ \beta\text{-}int(\lambda_i) = 1$. Now $1-\beta\text{-}cl\ \beta\text{-}int(\lambda_i) = 0$, implies that $\beta\text{-}int\ \beta\text{-}cl(1-\lambda_i) = 0$. Thus $(1-\lambda_i)$'s are fβ-nd sets. Then $1-\lambda = 1-\wedge_{i=1}^{\infty}(\lambda_i) = \vee_{i=1}^{\infty}(1-\lambda_i)$. So by the Definition 5.1, $1-\lambda$ is a fβ-fc set. □

In view of the above Proposition 4.3, one will have the following Proposition:

Proposition 5.4. *If a fβ-nd set λ in a fts (X,T) is a fβ-F_σ set, then λ is a fβ-fc set.*

Proof. Let λ be a fβ-nd set such that $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $(1-\lambda_i)$'s are fβ-o sets. Since λ is a fβ-nd set and by the Proposition 4.3, (λ_i)'s are fβ-nd sets. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-nd sets, implies that λ is a fβ-fc set. □

Proposition 5.5. *If β-$int(\lambda) = 0$ for a fβ-F_σ set λ in a fts (X, T), then λ is a fβ-fc set.*

Proof. Let λ be a fβ-F_σ set such that β-$int(\lambda) = 0$. Then $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-c sets. This implies that $0 = \beta$-$int(\lambda) = \beta$-$int(\vee_{i=1}^{\infty}(\lambda_i))$. But $\vee_{i=1}^{\infty}(\beta$-$int(\lambda_i)) \leq \beta$-$int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$ $\vee_{i=1}^{\infty}(\beta$-$int(\lambda_i)) \leq 0$. That is, $\vee_{i=1}^{\infty}(\beta$-$int(\lambda_i)) = 0$ and then β-$int(\lambda_i) = 0$ for each i. Since (λ_i)'s are fβ-c sets and by the Theorem 2.11, β-$cl(\lambda_i) = \lambda_i$. Now β-int β-$cl(\lambda_i) = \beta$-$int(\lambda_i) = 0$ and then by the Definition 4.2, (λ_i)'s are fβ-nd sets. Thus, $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fβ-nd sets. So by the Definition 5.1, λ is a fβ-fc set. □

Proposition 5.6. *If a fβ-G_δ set λ is a fβ-d set in a fts (X, T), then λ is a fβ-r set.*

Proof. Let λ be a fβ-G_δ set with β-$cl(\lambda) = 1$. Then by the Proposition 3.6, $1 - \lambda$ is a fβ-F_σ set with $1 - \beta$-$cl(\lambda) = 0$. That is, $1 - \lambda$ is a fβ-F_σ set with β-$int(1 - \lambda) = 0$. Then by the Proposition 5.5, $1 - \lambda$ is a fβ-fc set. So by the Definition 5.2, λ is a fβ-r set. □

6. Fuzzy β-Volterra spaces

Definition 6.1. A fts (X, T) is called a fuzzy β-Volterra space or fβ-Vs in short if β-$cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where the fuzzy sets (λ_i)'s are fuzzy β-dense and fuzzy β-G_δ sets.

Example 6.2. Let $X = \{a, b, c\}$. The fuzzy sets λ_1, λ_2 and λ_3 are defined on X as follows:

$\lambda_1 : X \to [0, 1]$ is defined as $\lambda_1(a) = 0.8$; $\lambda_1(b) = 0.6$; $\lambda_1(c) = 0.7$,
$\lambda_2 : X \to [0, 1]$ is defined as $\lambda_2(a) = 0.6$; $\lambda_2(b) = 0.9$; $\lambda_2(c) = 0.8$,
$\lambda_3 : X \to [0, 1]$ is defined as $\lambda_3(a) = 0.7$; $\lambda_3(b) = 0.5$; $\lambda_3(c) = 0.9$.

Clearly $T = \{0, \lambda_1, \lambda_2, \lambda_3, \lambda_1 \vee \lambda_2, \lambda_1 \vee \lambda_3, \lambda_2 \vee \lambda_3, \lambda_1 \wedge \lambda_2, \lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_3, \lambda_1 \vee (\lambda_2 \wedge \lambda_3), \lambda_1 \wedge (\lambda_2 \vee \lambda_3), \lambda_2 \vee (\lambda_1 \wedge \lambda_3), \lambda_2 \wedge (\lambda_1 \vee \lambda_3), \lambda_3 \vee (\lambda_1 \wedge \lambda_2), \lambda_3 \wedge (\lambda_1 \vee \lambda_2), \lambda_1 \vee \lambda_2 \vee \lambda_3, 1\}$ is a fuzzy topology on X.

Now consider fuzzy sets

$\mu_1 = \{\lambda_2 \wedge (\lambda_1 \vee \lambda_2) \wedge (\lambda_2 \wedge \lambda_3) \wedge [\lambda_2 \vee (\lambda_1 \wedge \lambda_3)] \wedge [\lambda_3 \wedge (\lambda_1 \vee \lambda_2)] \wedge [\lambda_2 \wedge (\lambda_1 \vee \lambda_3)] \wedge [\lambda_1 \vee (\lambda_2 \wedge \lambda_3)]\}$

$\mu_2 = \{\lambda_1 \wedge (\lambda_1 \wedge \lambda_2) \wedge (\lambda_1 \wedge \lambda_3) \wedge [\lambda_1 \wedge (\lambda_2 \vee \lambda_3)]\}$ and

$\mu_3 = \{\lambda_3 \wedge (\lambda_2 \vee \lambda_3) \wedge (\lambda_1 \vee \lambda_3) \wedge [\lambda_1 \vee (\lambda_2 \wedge \lambda_3)] \wedge [\lambda_3 \vee (\lambda_1 \wedge \lambda_2)]\}$.

Then μ_1, μ_2 and μ_3 are fG_δ-sets and hence are fβ-G_δ sets. On computations, one can see that β-$cl(\mu_1) = 1$, β-$cl(\mu_2) = 1$ and β-$cl(\mu_3) = 1$. Hence μ_1, μ_2 and μ_3 are fβ-d sets. This implies that β-$cl(\mu_1 \wedge \mu_2 \wedge \mu_3) = 1$, where μ_1, μ_2, μ_3 are fβ-d and fβ-G_δ sets. So, the fts (X, T) is a fβ-Vs.

Example 6.3. Let $X = \{a, b, c\}$. The fuzzy sets λ_1, λ_2 and λ_3 are defined on X as follows:

$\lambda_1 : X \to [0, 1]$ is defined as $\lambda_1(a) = 0.4$; $\lambda_1(b) = 0.5$; $\lambda_1(c) = 0.6$,
$\lambda_2 : X \to [0, 1]$ is defined as $\lambda_2(a) = 0.6$; $\lambda_2(b) = 0.4$; $\lambda_2(c) = 0.5$,

$\lambda_3 : X \to [0,1]$ is defined as $\lambda_3(a) = 0.7$; $\lambda_3(b) = 0.6$; $\lambda_3(c) = 0.4$.

Then, $T = \{0, \lambda_1, \lambda_2, \lambda_3, \lambda_1 \vee \lambda_2, \lambda_1 \vee \lambda_3, \lambda_2 \vee \lambda_3, \lambda_1 \wedge \lambda_2, \lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_3, \lambda_1 \wedge (\lambda_2 \vee \lambda_3), \lambda_2 \vee (\lambda_1 \wedge \lambda_3), \lambda_3 \wedge (\lambda_1 \vee \lambda_2), 1\}$ is a fuzzy topology on X. Now there is no fβ-d and fβ-G_δ set such that $cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$ and therefore the fts (X,T) is not a fβ-Vs.

7. Characterizations of fuzzy β-Volterra spaces

The following Propositions give conditions for the fuzzy topological spaces to be the fuzzy β-Volterra spaces.

Proposition 7.1. *If β-int $\left(\vee_{i=1}^{N}(\lambda_i)\right) = 0$, where (λ_i)'s are fβ-nd and fβ-F_σ sets in a fts (X,T), then (X,T) is a fβ-Vs.*

Proof. Let β-*int* $\left(\vee_{i=1}^{N}(\lambda_i)\right) = 0$. Then $1 - \beta$-*int* $\left(\vee_{i=1}^{N}(\lambda_i)\right) = 1$. This implies that β-*cl* $\left(\wedge_{i=1}^{N}(1-\lambda_i)\right) = 1$. Since (λ_i)'s are fβ-nd sets and by the Theorem 4.4, $(1-\lambda_i)$'s are fβ-d sets. Also since (λ_i)'s are fβ-F_σ-sets and by the Proposition 3.6, $(1-\lambda_i)$'s are fβ-G_δ sets. Hence, β-*cl* $\left(\wedge_{i=1}^{N}(1-\lambda_i)\right) = 1$, where $(1-\lambda_i)$'s are fβ-d and fβ-G_δ sets. So by the Definition 6.1, (X,T) is a fβ-Vs. □

Proposition 7.2. *A fts (X,T) is a fβ-Vs if and only if β-int $\left(\vee_{i=1}^{N}(1-\lambda_i)\right) = 0$, where (λ_i)'s are fβ-d and fβ-G_δ sets.*

Proof. Let (X,T) be a fβ-Vs. Then by the Definition 6.1, β-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$ where (λ_i)'s are fβ-d and fβ-G_δ sets. Now β-*int* $\left(\vee_{i=1}^{N}(1-\lambda_i)\right) = 1 - \beta$-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1 - 1 = 0$. Thus, β-*int* $\left(\vee_{i=1}^{N}(1-\lambda_i)\right) = 0$, where (λ_i)'s are fβ-d and fβ-G_δ sets.

Conversely, let β-*int* $\left(\vee_{i=1}^{N}(1-\lambda_i)\right) = 0$ where (λ_i)'s are fβ-d and fβ-G_δ sets. Then $1 - \beta$-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 0$. This implies that β-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$. Therefore β-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fβ-d and fβ-G_δ sets. So by the Definition 6.1, (X,T) is a fβ-Vs. □

Proposition 7.3. *If the fuzzy sets (μ_i)'s, $(i = 1$ to $N)$ are fβ-fc sets formed from the fβ-d and fβ-G_δ sets in a fβ-Vs (X,T), then β-int $\left(\vee_{i=1}^{N}(\mu_i)\right) = 0$.*

Proof. Let (λ_i)'s be the fβ-d and fβ-G_δ sets. Then β-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$. Now $1 - \beta$-*cl* $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 0$. This implies that β-*int* $\left(\vee_{i=1}^{N}(1-\lambda_i)\right) = 0$. Since the fuzzy sets (λ_i)'s are fβ-d and fβ-G_δ sets and by the Proposition 5.3, $(1-\lambda_i)$'s are fβ-fc sets. Let $\mu_i = 1 - \lambda_i$. Hence β-*int* $\left(\vee_{i=1}^{N}(\mu_i)\right) = 0$, where (μ_i)'s are fβ-fc sets. □

Proposition 7.4. *If $\lambda = \wedge_{i=1}^{N}(\lambda_i)$, where (λ_i)'s are fβ-d and fβ-G_δ sets in a fβ-Vs (X,T), then λ is not a fβ-c set.*

Proof. Let $\lambda = \wedge_{i=1}^{N}(\lambda_i)$, where (λ_i)'s are fβ-d and fβ-G_δ sets. Since (X,T) is a fβ-Vs, β-*cl*$(\wedge_{i=1}^{N}(\lambda_i)) = 1$. That is, β-*cl*$(\lambda) = 1 \neq \lambda$. Thus, λ is not a fβ-c set. □

Proposition 7.5. *If $\mu = \vee_{i=1}^{N}(\mu_i)$, where (μ_i)'s are fβ-nd and fβ-F_σ sets in a fβ-Vs (X, T), then μ is not a fβ-o set.*

Proof. Let $\mu = \vee_{i=1}^{N}(\mu_i)$, where (μ_i)'s are fβ-nd and fβ-F_σ sets. Then $1 - \mu = 1 - \vee_{i=1}^{N}(\mu_i) = \wedge_{i=1}^{N}(1 - \mu_i)$. Since (μ_i)'s are fβ-nd sets and by the Theorem 4.4, $(1 - \mu_i)$'s are fβ-d sets. Also since (μ_i)'s are fβ-F_σ sets and by the Proposition 3.6, $(1 - \mu_i)$'s are fβ-G_δ sets. Hence $1 - \mu = \wedge_{i=1}^{N}(1 - \mu_i)$, where $(1 - \mu_i)$'s are fβ-d and fβ-G_δ sets. Then by the Proposition 7.4, $1 - \mu$ is not a fβ-c set. So, μ is not a fβ-o set. □

Proposition 7.6. *If β-int $\left(\vee_{i=1}^{N}(1 - \lambda_i)\right) = 0$, where (λ_i)'s are fβ-G_δ sets in a fts (X, T), then (X, T) is a fβ-Vs.*

Proof. Let (λ_i)'s be the fβ-G_δ sets such that β-*int* $\left(\vee_{i=1}^{N}(1 - \lambda_i)\right) = 0$. But $\vee_{i=1}^{N}(\beta\text{-}int(1 - \lambda_i)) \leq \beta\text{-}int\left(\vee_{i=1}^{N}(1 - \lambda_i)\right)$. Then $\vee_{i=1}^{N}(\beta\text{-}int(1 - \lambda_i)) \leq 0$. That is, $\vee_{i=1}^{N}(\beta\text{-}int(1 - \lambda_i)) = 0$. This implies that $\beta\text{-}int(1 - \lambda_i) = 0$, $(i = 1$ to $N)$. Thus $1 - \beta\text{-}cl(\lambda_i) = 0$ and hence $\beta\text{-}cl(\lambda_i) = 1$. Therefore, (λ_i)'s are fβ-d sets. Since $\beta\text{-}int\left(\vee_{i=1}^{N}(1 - \lambda_i)\right) = 0$, $1 - \beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 0$. Thus, $\beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fβ-d and fβ-G_δ sets. So by the Definition 6.1, (X, T) is a fβ-Vs. □

Proposition 7.7. *If β-cl $\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fβ-G_δ sets in a fts (X, T), then (X, T) is a fβ-Vs.*

Proof. Let (λ_i)'s be the fβ-G_δ sets such that $\beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$. Then $1 - \beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 0$. This implies that $\beta\text{-}int\left(\vee_{i=1}^{N}(1 - \lambda_i\right) = 0$. Then by the Proposition 7.6, (X, T) is a fβ-Vs. □

Proposition 7.8. *If a fts (X, T) is a fβ-Vs, then there exists a fβ-F_σ set μ such that β-int$(\mu) \neq 0$.*

Proof. Let $\lambda = \wedge_{i=1}^{N}(\lambda_i)$, where (λ_i)'s are fβ-d and fβ-G_δ sets. Since (X, T) is a fβ-Vs, $\beta\text{-}cl(\lambda) = \beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$. Then $\beta\text{-}cl(\lambda) = 1 \to (1)$. Now $1 - \beta\text{-}int(\lambda_i)$ is a fβ-c set. Let $\mu = \vee_{i=1}^{\infty}(\mu_i)$, where (μ_i)'s are fβ-c sets in which the first N fβ-c sets as $1 - \beta\text{-}int(\lambda_i)$. Then μ is a fβ-F_σ set. But $\vee_{i=1}^{N}(1 - \beta\text{-}int(\lambda_i)) \leq \vee_{i=1}^{\infty}(\mu_i)$. Then $1 - \wedge_{i=1}^{N}(\beta\text{-}int(\lambda_i)) \leq \vee_{i=1}^{\infty}(\mu_i)$. Now $1 - \wedge_{i=1}^{N}(\lambda_i) < 1 - \wedge_{i=1}^{N}(\beta\text{-}int(\lambda_i)) \leq \vee_{i=1}^{\infty}(\mu_i)$. Then $1 - \lambda < 1 - \wedge_{i=1}^{N}(\beta\text{-}int(\lambda_i)) < \mu$. That is, $1 - \lambda < \mu$. This implies that $\beta\text{-}int(1 - \lambda) < \beta\text{-}int(\mu)$. Then $1 - \beta\text{-}cl(\lambda) < \beta\text{-}int(\mu)$. From (1), $1 - 1 < \beta\text{-}int(\mu)$ and hence $\beta\text{-}int(\mu) > 0$. That is, $\beta\text{-}int(\mu) \neq 0$. Hence if (X, T) is a fβ-Vs, then there exists a fβ-F_σ set μ such that $\beta\text{-}int(\mu) \neq 0$. □

Proposition 7.9. *If each fβ-G_δ set has a fβ-d interior in a fts (X, T), then (X, T) is a fβ-Vs.*

Proof. Let (λ_i)'s be fβ-d and fβ-G_δ sets. Then $\lambda = \wedge_{i=1}^{N}(\lambda_i)$ is a fβ-G_δ set. By hypothesis, λ has a fβ-d interior and hence $\beta\text{-}cl\ \beta\text{-}int(\lambda) = 1$. Now $\beta\text{-}int(\lambda) \leq \lambda$, implies that $\beta\text{-}int(\lambda) \leq \wedge_{i=1}^{N}(\lambda_i)$. Then $\beta\text{-}cl\ \beta\text{-}int(\lambda) \leq \beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right)$. This implies that $1 \leq \beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right)$. Thus $\beta\text{-}cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) \geq 1$. That is,

β-$cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fβ-d and fβ-G_δ sets. So by the Definition 6.1, (X,T) is a fβ-Vs. □

Remark 7.1. A fβ-d set in a fts (X,T) is a fd set since β-$cl(\lambda) \leq cl(\lambda)$, but the converse need not be true. That is, a fd set need not be a fβ-d set. For consider the following example:

Example 7.10. Let $X = \{a,b,c\}$. Then the fuzzy sets α_1, α_2 and α_3 are defined on X as follows:
$\alpha : X \to [0,1]$ defined as $\alpha_1(a) = 0.5, \quad \alpha_1(b) = 0.4, \quad \alpha_1(c) = 0.6;$
$\alpha_2 : X \to [0,1]$ defined as $\alpha_2(a) = 0.6, \quad \alpha_2(b) = 0.5, \quad \alpha_2(c) = 0.7;$
$\alpha_3 : X \to [0,1]$ defined as $\alpha_3(a) = 0.3, \quad \alpha_3(b) = 0.1, \quad \alpha_3(c) = 0.7.$
Clearly $T = \{0, \alpha_1, \alpha_2, 1\}$ is a fuzzy topology on X. By computations, one can see that $cl(\alpha_3) = 1$ and β-$cl(\alpha_3) = \alpha_3 \neq 1$ and α_3 is a fd set and not a fβ-d set.

Proposition 7.11. *If a fts (X,T) is a fVs, then (X,T) is not a fβ-Vs.*

Proof. Let (X,T) be a fVs. Then $cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fd and fG_δ-sets. Since (λ_i)'s are fG_δ-sets and by the Proposition 3.4, (λ_i)'s are fβ-G_δ sets. But by the Remark 7.1 and by the Example 7.10, the fd sets (λ_i)'s are not the fβ-d sets. So by the Definition 6.1, (X,T) is not a fβ-Vs. □

Proposition 7.12. *If a fts (X,T) is a fβ-Vs, then (X,T) is not a fVs.*

Proof. Let (X,T) be a fβ-Vs. Then β-$cl\left(\wedge_{i=1}^{N}(\lambda_i)\right) = 1$, where (λ_i)'s are fβ-d and fβ-G_δ sets. Since (λ_i)'s are fβ-d sets and by the Remark 7.1 and by the Example 7.10, (λ_i)'s are fd sets. But the fβ-G_δ sets (λ_i)'s are not the fG_δ-sets. So by the Definition 2.6, (X,T) is not a fVs. □

Remark 7.2. From the above Propositions 7.11 and 7.12, one can conclude that fuzzy Volterra spaces and fuzzy β-Volterra spaces are independent.

8. Conclusion

The new class of spaces called the fuzzy β-Volterra spaces with the help of fuzzy β-dense and fuzzy β-G_δ sets have been introduced and studied. Examples given to illustrate the concept in this paper. Some interesting characterizations of the fuzzy β-Volterra spaces have established in this paper.

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