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ON MALCEV ALGEBRA BUNDLES

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Abstract. In this paper, we study Malcev algebra bundles and Malcev algebra bundles of finite type. Lie algebra bundles and Lie transformation algebra bundles are defined using given Malcev algebra bundle and we conclude some results for finite type.

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1. Introduction

While discussing on a class of non-associative algebras, Arthur A. Sagle [10] generalised the class of Lie algebras. Certain identities proved in [10] give raise to the definition of Malcev algbera. When the "Lie-ness" of a Malcev algebra is being measured, important linear transformations are defined which yields some tool to define Lie algebras. Many results on Lie algebras and semisimple Lie algebras are adapted to bundles (See [1, 3–8]). We have seen the discussions on vector bundles of finite type in [11], from which R. Kumar et al. have defined Lie algebra bundles of finite type in [9]. In this paper, we define Malcev algebra bundles and Malcev algebra bundles of finite type. We shall study semisimple Malcev algebra bundles using the results on semisimple Malcev algebras [10]. All underlying vector spaces are considered to be real and finite dimensional unless otherwise stated.

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2. Malcev Algebra Bundles

Definition 2.1. An algebra M over a field $\mathbb{F}$ which is a nonassociative algebra, that is a vector space over a field $\mathbb{F}$ with a distributive multiplication defined on it, is said to be a Malcev algebra if it satisfies the following

(1) $x^2 = 0$, $xy = -yx$,
(2) $xy \cdot xz = (xy \cdot z)x + (yz \cdot x)x + (zx \cdot x)y$, for all $x, y, z \in M$.

We observe that every Lie algebra is a Malcev algebra (Lemma 2.3 in [10]).

Definition 2.2. Let x be any arbitrary element in a Malcev algebra M. A linear transformation R_x, is defined by the equation $aR_x = ax = -xa$ where $a \in M$.

Definition 2.3. Let M be a Malcev algebra. For any $x, y \in M$, we define

$$\Delta(x, y) = (R_x, R_y) - R_{xy}.$$

Also, we define $\Delta(M, M)$ as the linear space spanned by all the $\Delta(x, y)$ where x, y are in M.

Remark 2.1. From Theorem 2.36 in [10] we obsevre that $\Delta(M, M)$ is a Lie algebra under commutation.

Definition 2.4. Consider a vector bundle $\xi = (\xi, p, X)$ over a topological space X. When there exists a morphism, $\Theta : \xi \oplus \xi \to \xi$ which induces a Malcev (Lie) algebra structure on each ξ_x, ξ is called a weak Malcev (Lie) algebra bundle.

Definition 2.5. If $\xi = (\xi, p, X)$ is a vector bundle where each fibre is a Malcev (Lie) algebra and for each x in X there is an open set U in X containing x, a Malcev (Lie) algebra M and a homeomorphism $\phi : U \times M \to p^{-1}(U)$ such that for each x in U, $\phi_x : \{x\} \times M \to p^{-1}(x)$ is a Malcev (Lie) algebra isomorphism, then ξ is called a locally trivial Malcev (Lie) algebra bundle.

Definition 2.6. A semisimple Malcev (Lie) algebra bundle is a vector bundle ξ in which the morphism $\Theta : \xi \oplus \xi \to \xi$ induces a semisimple Malcev (Lie) algebra structure on each fibre ξ_x.

Definition 2.7. A Malcev (Lie) algebra bundle ξ over an arbitrary space X is of finite type if there is a a finite partition of unity S on X (that is, a finite set S of non-negative continuous functions on X whose sum is 1) such that the bundle restricted to the set $\{x \in X \,|\, \alpha(x) \neq 0\}$ is a trivial Malcev (Lie) algebra bundle for each α in S.

Theorem 2.8. *Let ζ and η be ideal bundles of a Malcev algebra bundle ξ. Then $J(\zeta, \eta, \xi) = \bigcup_{x \in X} J(\zeta_x, \eta_x, \xi_x)$ is an ideal bundle of ξ.*

Proof. For each $x \in X$, $J(\zeta_x, \eta_x, \xi_x)$ is ideal of ξ_x (By Theorem 3.5 in [10]). By local triviality of ξ, ζ and η, for any $x \in X$ we have, $U \times I \cong \zeta|_U$, $U \times J \cong \eta|_U$ and $U \times M \cong \xi|_U$, where U is a neighborhood of x, I and J are ideals of a Malcev algebra M. But then $U \times J(I,J,M) \cong \bigcup_{x\in U} J(\zeta_x,\eta_x,\xi_x)$. Note that $J(I,J,M)$ is an ideal of M. Hence $J(\zeta,\eta,\xi) = \bigcup_{x\in X} J(\zeta_x,\eta_x,\xi_x)$ is an ideal bundle of ξ. □

The following corollary is immediate from the above Theorem.

Corollary 2.9. *Suppose ζ and η are finite type ideal bundles of a Malcev algebra bunlde ξ of finite type. Then $J(\zeta,\eta,\xi) = \bigcup_{x\in X} J(\zeta_x,\eta_x,\xi_x)$ is a finite type ideal bundle of ξ.*

Proposition 2.10. *If $\xi = (\xi, p, X)$ is a Malcev algebra bundle, then $\Delta(\xi,\xi) = \bigcup_{x\in X} \Delta(\xi_x,\xi_x)$ is a Lie algebra bundle.*

Proof. By local triviality of ξ, for every $x \in X$, there is a neighborhood of U of x and a Malcev algebra M such that $\phi : U \times M \to \xi|_U$ is an isomorphism. But then

$$\Delta\phi : U \times \Delta(M,M) \to \Delta(\xi,\xi)|_U \text{ given by } x \times \Delta(a,b) \mapsto \Delta(\phi_x(a),\phi_x(b)),$$

is an isomorphism. Thus $\Delta(\xi,\xi)$ is locally trivial. □

The following corollary is immediate from the above Proposition.

Corollary 2.11. *If ξ is a Malcev algebra bundle of finite type, then $\Delta(\xi,\xi)$ is Lie algebra bundle of finite type.*

Definition 2.12. Let M be a Malcev algebra and $R(M)$, $L(M)$ be the linear spaces spanned by all right and left multiplications R_x and L_x respectively, where $x \in M$. Consider $A = R(M) + L(M)$. Then the Lie transformation algebra of M, $\mathfrak{L}(M)$ is the intersection of all Lie algebras containing A.

Proposition 2.13. *Consider $\xi = (\xi, p, X)$, a Malcev algebra bundle. Let $\mathfrak{L}(\xi) = \bigcup_{x\in X} \mathfrak{L}(\xi_x)$. Then $\mathfrak{L}(\xi)$ is a Lie algebra bundle (It is called the Lie transformation algebra bundle).*

Proof. Since ξ is locally trivial, for any element x in X, there is a neighborhood U of x and a Malcev algebra M such that $\psi : U \times M \to \xi|_U$ is an isomorphism. Then

$$\psi_{\mathfrak{L}} : U \times \mathfrak{L}(M) \to \mathfrak{L}(\xi)|_U \text{ given by } x \times R_a + L_b \mapsto R_{\psi_x(a)} + L_{\psi_x(b)}$$

is an isomorphism. Hence, $\mathfrak{L}(\xi)$ is a Lie algebra bundle. □

The following corollary is immediate from the above Proposition.

Corollary 2.14. *Suppose that ξ is a Malcev algebra bundle of finite type. Then $\mathfrak{L}(\xi)$ is a Lie transformation algebra bundle of finite type.*

Theorem 2.15. *Let η be an ideal bundle of a Malcev algebra bundle ξ. For each $x \in X$, let $\zeta_x = R(\eta_x) + (R(\eta_x), R(\xi_x))$ be generarted by all the elements of the form $R_a + (R_{a'}, R_b)$, where $a, a' \in \eta_x$, $b \in \xi_x$. Then $\zeta = \bigcup_{x \in X} \zeta_x$ is an ideal bundle of $\mathfrak{L}(\xi)$.*

Proof. By Theorem 5.2 in [10], each ζ_x is an ideal in $\mathfrak{L}(\xi_x)$. Under local triviality these ideals are mapped into ideals. Hence $\zeta = \bigcup_{x \in X} \zeta_x$ is an ideal bundle of $\mathfrak{L}(\xi)$. □

The following corollary is immediate from the above Theorem.

Corollary 2.16. *Let η be an ideal bundle of a Malcev algebra bundle ξ. If η and ξ are of finite type, then $\zeta = \bigcup_{x \in X} \zeta_x$ is also an ideal bundle of finite type.*

Theorem 2.17. *If ξ is a Malcev algebra bundle such that $\xi_x = J(\xi_x, \xi_x, \xi_x)$, then $\Delta(\xi, \xi) = \mathfrak{L}(\xi)$.*

Proof. By theorem 5.9 in [10], we have $\Delta(\xi_x, \xi_x) = \mathfrak{L}(\xi_x)$, for all $x \in X$. Hence $\Delta(\xi, \xi) = \mathfrak{L}(\xi)$. □

Remark 2.2. For any $x \in X$, let $N_x = \{a \in \xi_x : a\Delta(\xi_x, \xi_x) = 0\}$, called a J-nucleus of the Malcev algebra ξ_x. Consider $\mathfrak{N} = \bigcup_{x \in X} N_x$. By local triviality of ξ, we have $\phi : U \times M \to \xi|_U$ is an isomorphism, where U is a neighborhood of any element x in X and M is a Malcev algebra. But then $\phi_{\mathfrak{N}} : U \times N \to \bigcup_{x \in U} N_x$, where N is the J-nucleus of the Malcev algebra M, defined by $\phi_{\mathfrak{N}}(x, a) = \phi(x, a)$ is an isomorphism. As each N_x is an ideal of ξ_x, $\mathfrak{N}$ is an ideal bundle of ξ, which we call the J-nucleus bundle of ξ.

Lemma 2.18. *Let η be an ideal bundle of a semisimple Malcev algebra ξ such that $\eta \cap \mathfrak{N} = 0$. Then $\eta = J(\eta, \eta, \eta) = J(\eta, \eta, \xi) = J(\eta, \xi, \xi)$.*

Proof. For each $x \in X$, we have $\eta_x = J(\eta_x, \eta_x, \eta_x) = J(\eta_x, \eta_x, \xi_x) = J(\eta_x, \xi_x, \xi_x)$ (by Lemma 5.16 in [10]). Hence the result follows. □

Note: The Whitney sum of two Malcev algebra bundles (ξ, p) and (η, q) is again a Malcev algebra bundle $\xi \oplus \eta$ with the Malcev algebra morphism

$$p \oplus q : (\xi \oplus \eta) \oplus (\xi \oplus \eta) = (\xi \oplus \xi) \oplus (\eta \oplus \eta) \to \xi \oplus \eta,$$

defined by

$$p \oplus q[(a, b) \oplus (a', b')] = p(a, a') \oplus q(b, b'),$$

$a, a' \in \xi$ and $b, b' \in \eta$.

Theorem 2.19. *Let ξ be a semisimple Malcev algebra bundle. Then $\xi = \mathfrak{N} \oplus J(\xi, \xi, \xi)$.*

Proof. By Theorem 5.17 in [10], we have for each $x \in \xi_x$, $\xi_x = N_x \oplus J(\xi_x, \xi_x, \xi_x)$. It follows that $\xi = \mathfrak{N} \oplus J(\xi, \xi, \xi)$. □

Corollary 2.20. *If η is a nonzero Lie algebra ideal bundle in a semisimple Malcev algebra bundle ξ, then $\eta \subset \mathfrak{N}$.*

Proof. By corollary 5.18 in [10], we have $\eta_x \subset N_x$. Hence $\eta \subset \mathfrak{N}$. $\square$

Lemma 2.21. *Let η be an ideal bundle of a Malcev algebra bundle ξ. Then $\Delta(\eta, \xi)$ is an ideal bundle of the Lie algebra bundle $\Delta(\xi, \xi)$.*

Proof. As η_x is an ideal of ξ_x, by Lemma 5.19 in [10], each $\Delta(\eta_x, \xi_x)$ is an ideal in $\Delta(\xi_x, \xi_x)$. Let us consider the local trivialities $U \times M \cong \xi|_U$ and $U \times L \cong \eta|_U$, where U is a neigborhood of x and L is an ideal of the Malcev algebra M. These give the required isomorphism, $U \times \Delta(L, M) \cong \bigcup\limits_{x \in U} \Delta(\eta_x, \xi_x)$. We observe that $\Delta(L, M)$ is an ideal in $\Delta(M, M)$ (by lemma 5.19 in [10]). Hence $\Delta(\eta, \xi) = \bigcup\limits_{x \in X} \Delta(\eta_x, \xi_x)$ is an ideal bundle of $\Delta(\xi, \xi)$. $\square$

The following corollary is immediate from the above lemma.

Corollary 2.22. *Let η be a finite type ideal bundle of a Malcev algebra bundle of finite type ξ. Then $\Delta(\eta, \xi)$ is an ideal bundle of finite type in $\Delta(\xi, \xi)$.*

Theorem 2.23. *Let ξ be a non-associative algebra bundle over a field of characteristic zero. Then ξ is semisimple if and only if $\mathfrak{L}(\xi)$ is completely reducible in ξ if and only if $\mathfrak{L}(\xi) = \mathcal{C}(\mathfrak{L}(\xi)) \oplus \mathfrak{L}_1(\xi)$ where $\mathfrak{L}_1(\xi)$ is a semisimple ideal bundle of $\mathfrak{L}(\xi)$ and $\mathcal{C}(\mathfrak{L}(\xi))$ is the center of $\mathfrak{L}(\xi)$ which consists of semisimple elements.*

Proof. We observe that for each $x \in X$, ξ_x is semisimple if and only if $\mathfrak{L}(\xi_x)$ is completely reducible in ξ_x which is if and only if $\mathfrak{L}(\xi_x) = \mathcal{C}(\mathfrak{L}(\xi_x)) \oplus \mathfrak{L}_1(\xi_x)$, where $\mathfrak{L}_1(\xi_x)$ is a semisimple ideal of $\mathfrak{L}(\xi_x)$ (By Theorem 7.2 in [10]). Thus ξ is semisimple if and only if $\mathfrak{L}(\xi)$ is completely reducible if and only if $\mathfrak{L}(\xi) = \mathcal{C}(\mathfrak{L}(\xi)) \oplus \mathfrak{L}_1(\xi)$ where $\mathcal{C}(\mathfrak{L}(\xi)) = \bigcup\limits_{x \in X} \mathcal{C}(\mathfrak{L}(\xi_x))$ is the center of $\mathfrak{L}(\xi)$ and $\mathfrak{L}_1(\xi) = \bigcup\limits_{x \in X} \mathfrak{L}_1(\xi_x)$ is locally trivial Lie algebra bundle being semisimple bundle (By Lemma 2.1 in [4]). $\square$

The following corollary is immediate from the above Theorem.

Corollary 2.24. *Let ξ be a non-associative algebra over a field of characteristic zero such that ξ being semisimple implies $\mathcal{C}(\mathfrak{L}(\xi)) = 0$. Then ξ is semisimple if and only if $\mathfrak{L}(\xi)$ is a semisimple Lie algebra bundle.*

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