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TOTAL COLORING OF MIDDLE GRAPH OF CERTAIN SNAKE GRAPH FAMILIES

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ABSTRACT. A total coloring of a graph G is an assignment of colors to both the vertices and edges of G, such that no two adjacent or incident vertices and edges of G are assigned the same colors. In this paper, we have discussed the total coloring of $M(T_n), M(D_n), M(DT_n), M(AT_n), M(DA(T_n))$, $M(Q_n), M(AQ_n)$ and also obtained the total chromatic number of $M(T_n)$, $M(D_n), M(DT_n), M(AT_n), M(DA(T_n)), M(Q_n), M(AQ_n)$.

AMS Mathematics Subject Classification : 05C15.
Key words and phrases : Total coloring, total chromatic number, middle graph, triangular snake, quadrilateral snake.

1. Introduction

All graphs consider here are finite, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph with vertex set V(G) and edge set E(G) respectively. A coloring of a graph G is an assignment of colors to the vertices or edges or both. A vertex-coloring(edge coloring) is called a proper coloring if no two adjacent vertices or edges receive the same colors. A total coloring of G is a function $f : V(G) \cup E(G) \to C$, where C is the set of colors to satisfies the following conditions.

i) no two adjacent vertices receive the same colors
ii) no two adjacent edges receive the same colors
iii) no edges and its incident vertices receive the same colors

Bezhad [1] and Vizing [11] introduced the concept of total coloring. Also, they have proposed the conjecture for every simple graph G has $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G)+2$, where $\Delta(G)$ is the maximum degree of G. This conjecture is known as the Total Coloring Conjecture (TCC). Bezhad et al.[2] computed the total chromatic number of complete graphs. Rosenfeld[9] and Vijayaditya[10]

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examined the TCC, for any graph G with maximum degree ≤ 3 and Kostochka [7] for maximum degree ≤ 5. In Borodin[4] verified the Total Coloring Conjecture (TCC) for maximum degree ≥ 9 in planar graphs. Jayaraman et al.[5] proved that the total chromatic number of double star graph families. Jayaraman et al.[6] proved that the total coloring of middle, total graph of bistar, double wheel and double crown graph.

The Middle graph [6] of a graph G is formed by subdividing each edge exactly once and connecting these newly obtained vertices of adjacent edges of G. A Triangular snake graph T_n[8] is obtained from the path by replacing every K_2 by C_3. A Double triangular snake DT_n[8] consists of two triangular snakes that have a common path. A Diamond triangular snake graph D_n [8] is obtained from a path by replacing every K_2 by $2C_3$. An Alternate triangular snake AT_n [8] is obtained from a path by replacing every K_2 by C_3 alternatively. A Double alternate triangular snake DAT_n[8] consists of two alternate triangular snakes that have a common path. A Quadrilateral snake Q_n is obtained from a path by replacing every edge by a cycle C_4. An Alternate quadrilateral snake AQ_n is obtained from a path by replacing every alternate edge by a cycle C_4.

2. Main results

Theorem 2.1. *Let $M(T_n)$ be the middle graph of triangular snake graph, then $\chi''(M(T_n)) = 9$.*

Proof. Let $V(M(T_n)) = \{u_i; x_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n+1\} \cup \{y_i : 1 \leq i \leq 2n\}$ and $E(M(T_n)) = \{u_iy_i; u_iy_{2i}; v_iy_{2i-1}; v_{i+1}y_{2i}; x_iy_{2i-1}; x_iy_{2i}; v_ix_i; x_iv_{i+1} : 1 \leq i \leq n\} \cup \{x_iy_{2i+1}; x_{i+1}y_{2i}; x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{y_iy_{i+1} : 1 \leq i \leq 2n-1\}$. Define a total coloring $f : V(M(T_n)) \cup E(M(T_n)) \to \{1, 2, 3, \ldots, 9\}$ as follows:

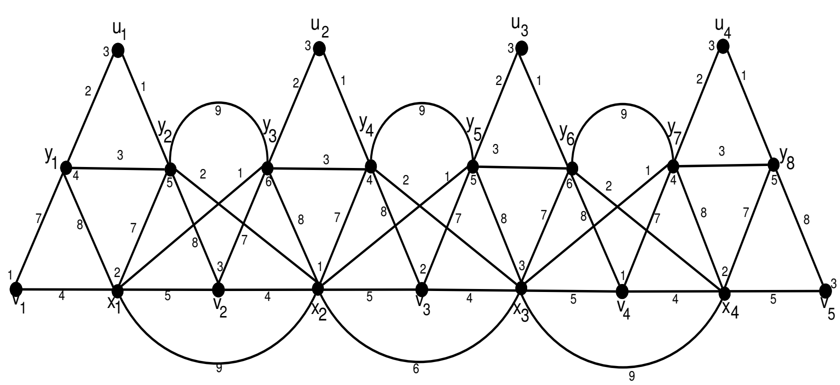

Figure 1. Total coloring for $M(T_5)$

The assigning of colors to each vertices and edges as follows:

For $1 \leq i \leq n$

$$f(u_i) = 3;$$

$$f(x_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(u_i y_i) = 2; f(u_i y_{2i}) = 9; f(v_i y_{2i-1}) = 7; f(v_{i+1} y_{2i}) = f(x_i y_{2i-1}) = 8;$$

$$f(x_i y_{2i}) = 7; f(v_i x_i) = 5; f(x_i v_{i+1}) = 6$$

For $1 \leq i \leq n+1$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

For $1 \leq i \leq n-1$

$$f(x_i y_{2i+1}) = 9; f(x_{3i-1} y_{6i-4}) = 2; f(x_{3i} y_{6i-2}) = 2; f(x_{3i+1} y_{6i}) = 4$$

$$f(x_i x_{i+1}) = \begin{cases} 3, & \text{if } i \equiv \ 1 \ (mod\ 3) \\ 4, & \text{if } i \equiv \ 2 \ (mod\ 3) \\ 1, & \text{if } i \equiv \ 0 \ (mod\ 3) \end{cases}$$

For $1 \leq i \leq 2n-1$

$$f(y_i y_{i+1}) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$$

For $1 \leq i \leq 2n$

$$f(y_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

Hence f is a total coloring of $M(T_n)$ and therefore $\chi''(M(T_n)) \leq 9$. By conjecture, $\chi''(M(T_n)) \geq \Delta(M(T_n)) + 1 = 8 + 1 \geq 9$ and $\chi''(M(T_n)) = 9$. □

Theorem 2.2. *Let $M(DT_n)$ be the middle graph of double triangular snake, then $\chi''(M(DT_n)) = 13$.*

Proof. Let $V(M(DT_n)) = \begin{cases} \{u_i; x_i; z_i : 1 \le i \le n\} \cup \{v_i; y_i : 1 \le i \le 2n\} \cup \\ \{w_i : 1 \le i \le n+1\} \end{cases}$ and

$$E(M(DT_n)) = \begin{cases} \{u_i v_{2i-1}; u_i v_{2i}; v_{2i-1} w_i; v_{2i} x_i; v_{2i-1} x_i; v_{2i} w_{2i}; w_i x_i; x_i w_{i+1}; v_i y_i; \\ w_i y_{2i-1}; x_i y_{2i}; y_{2i-1} z_i; z_i y_{2i}; x_{3i} y_{6i-1}; w_{3i+1} y_{6i} : 1 \le i \le n\} \cup \\ \{x_i v_{2i+1}; x_{2i} v_{2i}; x_i x_{i+1}; x_i y_{2i+1}; x_{i+1} y_{2i}; x_{3i-1} y_{6i-3}; x_{3i-2} y_{6i-5}; \\ w_{3i-1} y_{6i-4}; w_{3i} y_{6i-2} : 1 \le i \le n-1\} \cup \{x_1 y_1\} \cup \\ \{v_i v_{i+1}; y_i y_{i+1} : 1 \le i \le 2n-1\} \end{cases}$$

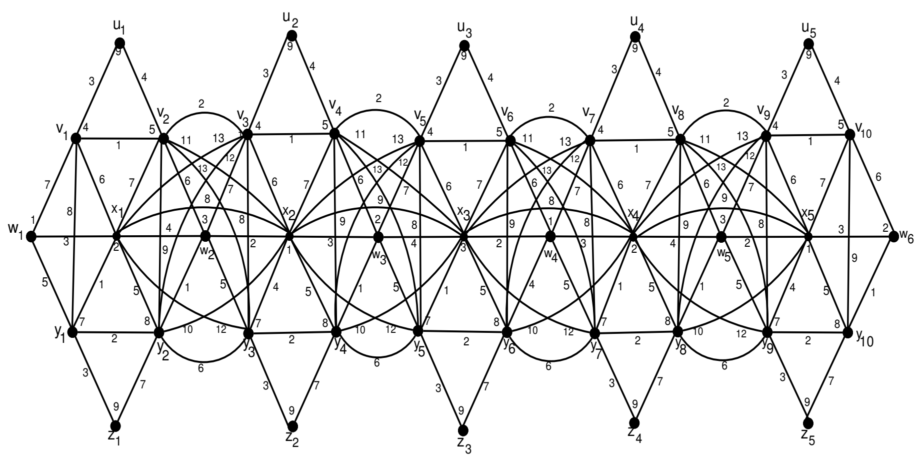

FIGURE 2. Total coloring for $M(DT_5)$

Define a total coloring $f : V(M(DT_n)) \cup E(M(DT_n)) \to \{1, 2, 3, \ldots, 13\}$ as follows: The assigning of colors to each vertices and edges as follows:
For $1 \le i \le n$

$$f(u_i) = 9;$$

$$f(x_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(w_i x_i) = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 4, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(x_i w_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(v_iy_i) = \begin{cases} 8, & \text{if } i \text{ is odd} \\ 9, & \text{if } i \text{ is even} \end{cases}$$

$$f(z_i) = 9; f(u_iv_{2i-1}) = 3; f(u_iv_{2i}) = 4; f(v_{2i-1}w_i) = 7;$$
$$f(v_{2i}x_i) = 7; f(v_{2i-1}x_i) = 6; f(v_{2i}w_{2i}) = 6; f(w_iy_{2i-1}) = 5;$$
$$f(x_iy_{2i}) = 5; f(y_{2i-1}z_i) = 3; f(z_iy_{2i}) = 7; f(w_{3i+1}y_{6i}) = 4; f(x_{3i}y_{6i-1}) = 1$$

For $1 \le i \le n-1$

$$f(x_iv_{2i+1}) = 13; f(x_{2i}v_{2i}) = 11;$$
$$f(x_ix_{i+1}) = \begin{cases} 8, & \text{if } i \text{ is odd} \\ 9, & \text{if } i \text{ is even} \end{cases}$$

$$f(x_iy_{2i+1}) = 12; f(x_{i+1}y_{2i}) = 10; f(x_{3i-1}y_{6i-3}) = 4;$$
$$f(x_{3i-2}y_{6i-5}) = 1; f(w_{3i-1}y_{6i-4}) = 1; f(w_{3i}y_{6i-2}) = 1$$

For $1 \le i \le n+1$

$$f(w_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

For $1 \le i \le 2n-1$

$$f(v_iv_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

$$f(y_iy_{i+1}) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 6, & \text{if } i \text{ is even} \end{cases}$$

For $1 \le i \le 2n$

$$f(v_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

$$f(y_i) = \begin{cases} 7, & \text{if } i \text{ is odd} \\ 8, & \text{if } i \text{ is even} \end{cases}$$

Hence f is a total coloring of $M(DT_n)$ and therefore $\chi''(M(DT_n)) \le 13$. By conjecture, $\chi''(M(DT_n)) \ge \Delta(M(DT_n)) + 1 = 12 + 1 \ge 13$ and $\chi''(M(DT_n)) = 13$. □

Theorem 2.3. *Let $M(D_n)$ be the middle graph of diamond triangular snake, then $\chi''(M(D_n)) = 7$.*

Proof. Let $V(M(D_n)) = \begin{cases} \{u_i; y_i : 1 \leq i \leq n\} \cup \{v_i; x_i : 1 \leq i \leq 2n\} \cup \\ \{w_i : 1 \leq i \leq n+1\} \end{cases}$ and

$$E(M(D_n)) = \begin{cases} \{u_iv_i; u_iv_{i+1}; w_iv_{2i-1}; w_{i+1}v_{2i}; v_{2i-1}x_{2i-1}; v_{2i}x_{2i}; w_ix_i; x_{2i}w_{i+1}; \\ y_ix_{2i-1}; y_ix_{2i} : 1 \leq i \leq n\} \cup \{v_iv_{i+1}; x_ix_{i+1} : 1 \leq i \leq 2n\} \cup \\ \{x_{2i}v_{2i+1}; v_{2i}x_{2i+1} : 1 \leq i \leq n-1\} \end{cases}$$

Define a total coloring $f : V(M(D_n)) \cup E(M(D_n)) \to \{1, 2, 3, \ldots, 7\}$ as follows:

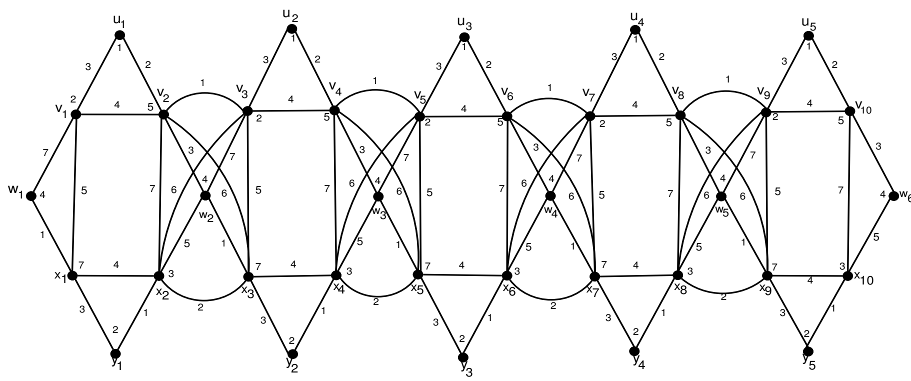

FIGURE 3. Total coloring for $M(D_5)$

The assigning of colors to each vertices and edges as follows:
For $1 \leq i \leq n$

$$f(u_i) = 1; f(y_i) = 2; f(u_iv_{2i-1}) = 3; f(u_iv_{2i}) = 2; f(w_iv_{2i-1}) = 7;$$
$$f(w_{i+1}v_{2i}) = 3; f(v_{2i-1}x_{2i-1}) = 5; f(v_{2i}x_{2i}) = 7; f(w_ix_{2i-1}) = 1;$$
$$f(x_{2i}w_{i+1}) = 5; f(y_ix_{2i-1}) = 3; f(y_ix_{2i}) = 1$$

For $1 \leq i \leq 2n$

$$f(x_i) = \begin{cases} 7, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$$

$$f(v_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$$

$$f(x_i x_{i+1}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

For $1 \leq i \leq n-1$

$$f(x_{2i} v_{2i+1}) = 6; f(v_{2i} x_{2i+1}) = 6$$

For $1 \leq i \leq n+1$

$$f(w_i) = 4.$$

Hence f is a total coloring of $M(D_n)$ and therefore $\chi''(M(D_n)) \leq 7$. By conjecture, $\chi''(M(D_n)) \geq \Delta(M(D_n)) + 1 = 6 + 1 \geq 7$ and $\chi''(M(D_n)) = 7$. □

Theorem 2.4. *Let $M(AT_n)$ be the middle graph of alternate triangular snake, then $\chi''(M(AT_n)) = 7$.*

Proof. Let $V(M(AT_n)) = \{v_i; w_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\} \cup \{x_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ and $E(M(AT_n)) = \{w_i u_i : 1 \leq i \leq n\} \cup \{v_i u_i; u_i v_{i+1}; u_i w_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-2\} \cup \{x_i w_{2i-1}; x_i w_{2i}; w_{2i-1} w_{2i}; v_{2i-1} w_{2i-1}; v_{2i} w_{2i} : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.

Define a total coloring $f : V(M(AT_n)) \cup E(M(AT_n)) \to \{1, 2, \ldots, 7\}$ as follows.

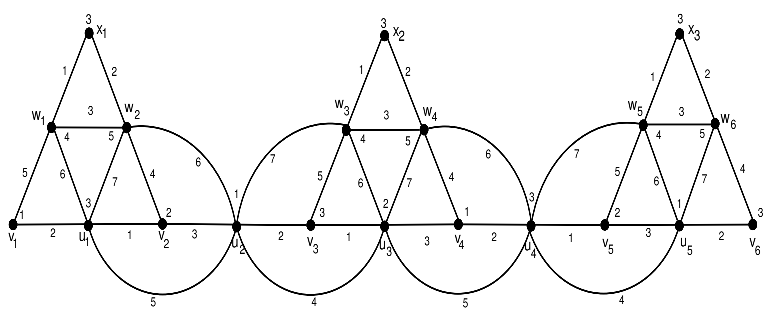

FIGURE 4. Total coloring for $M(AT_6)$

The assigning of colors to each vertices and edges as follows:
For $1 \leq i \leq n$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(w_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

$$f(w_i u_i) = 6$$

For $1 \leq i \leq n-1$

$$f(u_i) = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(v_i u_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 1, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(u_i v_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(u_i w_{i+1}) = 7$$

For $1 \leq i \leq n-2$

$$f(u_i u_{i+1}) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$$

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$f(x_i) = 3; f(x_i w_{2i-1}) = 1; f(x_i w_{2i}) = 2;$$

$$f(w_{2i-1} w_{2i}) = 3; f(v_{2i-1} w_{2i-1}) = 5; f(v_{2i} w_{2i}) = 4$$

Hence f is a total coloring of $M(AT_n)$ and therefore $\chi''(M(AT_n)) \leq 7$. By conjecture, $\chi''(M(AT_n)) \geq \Delta(M(AT_n)) + 1 = 6 + 1 \geq 7$ and $\chi''(M(AT_n)) = 7$. □

Theorem 2.5. *Let $M(DA(T_n))$ be the middle graph of double alternate triangular snake, then $\chi''(M(DA(T_n)) = 9$.*

Proof. Let $V(M(DA(T_n)) = \{v_i; w_i; y_i : 1 \leq i \leq n\} \cup \{x_i : 1 \leq i \leq n-1\} \cup \{z_i : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}$ and $E(M(DA(T_n)) = \{v_i x_i; v_i w_i; v_i y_i; w_i y_i : 1 \leq i \leq n\} \cup \{w_i x_i; x_i w_{i+1}; x_i v_{i+1}; x_i y_i; x_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-2\} \cup \{u_i v_{2i-1}; u_i v_{2i}; v_{2i-1} v_{2i}; y_{2i-1} z_i; z_i y_{2i}; y_{2i-1} y_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}$.
Define a total coloring $f : V(M(DA(T_n)) \cup E(M(DA(T_n)) \rightarrow \{1, 2, 3, \ldots, 9\}$ as follows. The assigning of colors to each vertices and edges as follows:
For $1 \leq i \leq n$

$$f(v_i) = \begin{cases} 8, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$$

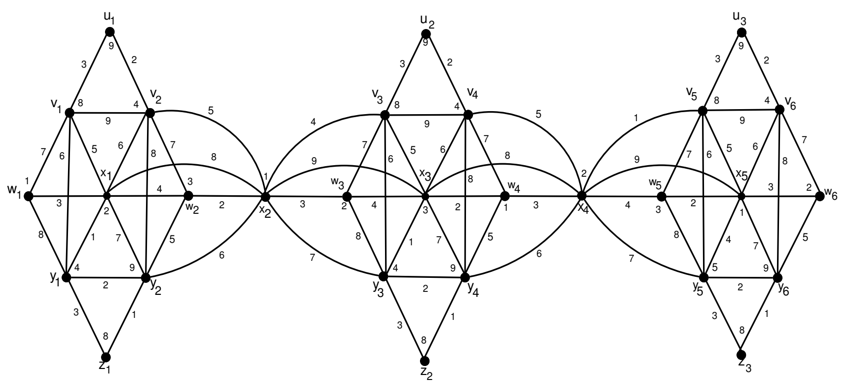

FIGURE 5. Total coloring for $M(DA(T_6))$

$$f(w_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(y_i) = \begin{cases} 4, & \text{if } i \text{ is odd,except } i = 6i-1 \\ 9, & \text{if } i \text{ is even} \\ 5, & \text{if } i = 6i-1 \end{cases}$$

$$f(v_i x_i) = 5; f(v_i w_i) = 7;$$

$$f(v_i y_i) = \begin{cases} 6, & \text{if } i \text{ is odd} \\ 8, & \text{if } i \text{ is even} \end{cases}$$

$$f(w_i y_i) = \begin{cases} 8, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

For $1 \leq i \leq n-1$

$$f(x_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(w_i x_i) = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 4, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(x_i w_{i+1}) = \begin{cases} 4, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(x_i v_{i+1}) = \begin{cases} 6, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$$

$$f(x_i y_i) = \begin{cases} 1, & \text{if } i \text{ is odd,except } i = 6i-1 \\ 6, & \text{if } i \text{ is even} \\ 4, & \text{if } i = 6i-1 \end{cases}$$

$$f(x_i y_{i+1}) = 7$$

For $1 \leq i \leq n-2$

$$f(x_i x_{i+1}) = \begin{cases} 8, & \text{if } i \text{ is odd} \\ 9, & \text{if } i \text{ is even} \end{cases}$$

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$f(u_i) = 9; f(z_i) = 8; f(u_i v_{2i-1}) = 3; f(u_i v_{2i}) = 2;$$

$$f(v_{2i-1} v_{2i}) = 9; f(y_{2i-1} z_i) = 3; f(z_i y_{2i}) = 1; f(y_{2i-1} y_{2i}) = 2$$

Hence f is a total coloring of $(M(DA(T_n))$ and therefore $\chi''(M(DA(T_n)) \leq 9$. By conjecture, $\chi''(M(DA(T_n)) \geq \Delta(M(DA(T_n)) + 1 = 8 + 1 \geq 9$ and $\chi''(M(DA(T_n)) = 9$. □

Theorem 2.6. *Let $M(QS_n)$ be the middle graph of quadrilateral snake, then $\chi''(M(QS_n)) = 9$.*

Proof. Let $V(M(QS_n)) = \{u_i; y_i : 1 \leq i \leq n\} \cup \{w_i; x_i : 1 \leq i \leq 2n\} \cup \{v_i : 1 \leq i \leq n+1\}$ and $E(M(QS_n)) = \{v_i u_i; u_i v_{i+1}; v_i w_{2i-1}; v_{i+1} w_{2i}; u_i w_{2i-1}; u_i w_{2i}; y_i w_{2i-1}; y_i w_{2i}; x_{2i-1} y_i; y_i x_{2i} : 1 \leq i \leq n\} \cup \{u_i u_{i+1}; u_i w_{2i+1}; w_{2i} w_{2i+1}; w_{2i} u_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i w_i : 1 \leq i \leq 2n\}$.
Define a total coloring $f : V(M(QS_n)) \cup EV(M(QS_n)) \to \{1, 2, \ldots, 9\}$ as follows. The assigning of colors to each vertices and edges as follows:
For $1 \leq i \leq n$

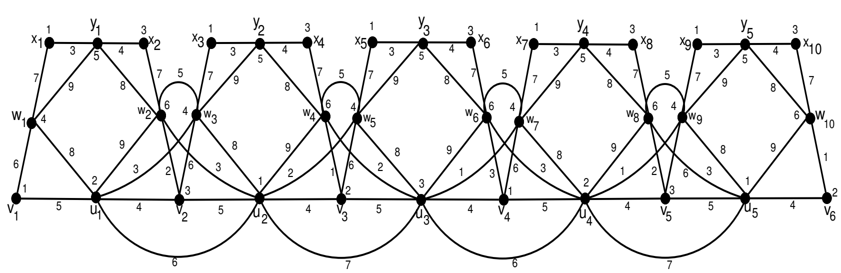

FIGURE 6. Total coloring for $M(QS_5)$

$$f(u_i) = f(v_{i+1}w_{2i}) = \begin{cases} 2, & \text{if } \ i \equiv 1 \ (mod \ 3) \\ 1, & \text{if } \ i \equiv 2 \ (mod \ 3) \\ 3, & \text{if } \ i \equiv 0 \ (mod \ 3) \end{cases}$$

$$f(y_i) = 5; f(v_iu_i) = 5; f(u_iv_{i+1}) = 4; f(v_iw_{2i-1}) = 6;$$

$$f(u_iw_{2i-1}) = 8; f(u_iw_{2i}) = 9; f(y_iw_{2i-1}) = 9;$$

$$f(y_iw_{2i}) = 8; f(x_{2i-1}y_i) = 3; f(y_ix_{2i}) = 4$$

For $1 \leq i \leq n-1$

$$f(u_iu_{i+1}) = \begin{cases} 6, & \text{if } \ i \ \text{ is odd} \\ 7, & \text{if } \ i \ \text{ is even} \end{cases}$$

$$f(w_{2i}u_{i+1}) = \begin{cases} 3, & \text{if } \ i \ \text{ is odd} \\ 2, & \text{if } \ i \ \text{ is even} \end{cases}$$

$$f(u_iw_{2i+1}) = \begin{cases} 3, & \text{if } \ i \equiv 1 \ (mod \ 3) \\ 2, & \text{if } \ i \equiv 2 \ (mod \ 3) \\ 1, & \text{if } \ i \equiv 0 \ (mod \ 3) \end{cases}$$

$$f(w_{2i}w_{2i+1}) = 5$$

For $1 \leq i \leq 2n$

$$f(w_i) = \begin{cases} 4, & \text{if } \ i \ \text{ is odd} \\ 6, & \text{if } \ i \ \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 1, & \text{if } \ i \ \text{ is odd} \\ 3, & \text{if } \ i \ \text{ is even} \end{cases}$$

$$f(x_iw_i) = 7$$

For $1 \leq i \leq n+1$

$$f(v_i) = \begin{cases} 1, & \text{if } \ i \equiv 1 \ (mod \ 3) \\ 3, & \text{if } \ i \equiv 2 \ (mod \ 3) \\ 2, & \text{if } \ i \equiv 0 \ (mod \ 3) \end{cases}$$

Hence f is a total coloring of $M(QS_n)$ and therefore $\chi''(M(QS_n)) \leq 9$. By conjecture, $\chi''(M(QS_n)) \geq \Delta(M(QS_n)) + 1 = 8 + 1 \geq 9$ and $\chi''(M(QS_n)) = 9$. □

Theorem 2.7. *Let $M(AQ_n)$ be the middle graph of alternate quadrilateral snake, then $\chi''(M(AQ_n)) = 7$.*

Proof. Let $V(M(AQ_n)) = \{v_i; w_i; x_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\} \cup \{y_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ and
$E(M(AQ_n)) = \{x_iw_i; w_iv_i; w_iu_i : 1 \leq i \leq n\} \cup \{u_iw_{i+1}; v_iu_i; u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-2\} \cup \{x_{2i-1}y_i; y_ix_{2i}; w_{2i-1}y_i; y_iw_{2i} : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$.
Define a total coloring $f : V(M(AQ_n)) \cup EV(M(AQ_n)) \rightarrow \{1, 2, \ldots, 7\}$ as follows. The assigning of colors to each vertices and edges as follows:

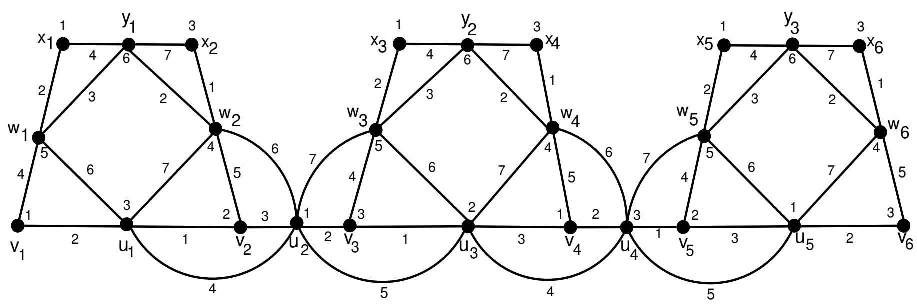

FIGURE 7. Total coloring for $M(AQ_6)$

For $1 \leq i \leq n$

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(w_i) = \begin{cases} 5, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$$

$$f(x_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 3, & \text{if } i \text{ is even} \end{cases}$$

$$f(x_iw_i) = \begin{cases} 2, & \text{if } i \text{ is odd} \\ 1, & \text{if } i \text{ is even} \end{cases}$$

$$f(w_iv_i) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

$$f(w_iu_i) = 6$$

For $1 \leq i \leq n-1$

$$f(u_i) = \begin{cases} 3, & \text{if } i \equiv 1 \ (mod\ 3) \\ 1, & \text{if } i \equiv 2 \ (mod\ 3) \\ 2, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(v_iu_i) = \begin{cases} 2, & \text{if } i \equiv 1 \ (mod\ 3) \\ 3, & \text{if } i \equiv 2 \ (mod\ 3) \\ 1, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(u_iv_{i+1}) = \begin{cases} 1, & \text{if } i \equiv 1 \ (mod\ 3) \\ 2, & \text{if } i \equiv 2 \ (mod\ 3) \\ 3, & \text{if } i \equiv 0 \ (mod\ 3) \end{cases}$$

$$f(u_iw_{i+1}) = 7$$

For $1 \leq i \leq n-2$

$$f(u_iu_{i+1}) = \begin{cases} 4, & \text{if } i \text{ is odd} \\ 5, & \text{if } i \text{ is even} \end{cases}$$

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

$$f(y_i) = 6; f(x_{2i-1}y_i) = 4; f(y_ix_{2i}) = 7;$$
$$f(w_{2i-1}y_i) = 3; f(y_iw_{2i}) = 2$$

Hence f is a total coloring of $M(AQ_n)$ and therefore $\chi''(M(AQ_n)) \leq 7$. By conjecture, $\chi''(M(AQ_n)) \geq \Delta(M(AQ_n)) + 1 = 6 + 1 \geq 7$ and $\chi''(M(AQ_n)) = 7$. □

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