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ON SOME NEW SOLITONS SOLUTIONS OF NONLINEAR COMPLEX GINZBURG-LANDAU EQUATION SOLVED BY MODIFIED JACOBI ELLIPTIC FUNCTIONS METHOD

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ABSTRACT. This article explains how solitons propagate when there is a detuning factor involved. The explanation is based on the nonlinear complex Ginzburg-Landau equation, and we first consider this equation before systematically deriving its solutions using Jacobian elliptic functions. We illustrate that one specific ellipticity modulus is on the verge of occurring. The findings from this study can contribute to the understanding of previous research on the Ginzburg-Landau equation. Additionally, we utilize Jacobi's elliptic functions to define specific solutions, especially when the ellipticity modulus approaches either unity or zero. These solutions correspond to particular periodic wave solitons, which have been previously discussed in the literature.

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1. Introduction

In recent decades, the investigation of the traveling wave solutions to linear and nonlinear equations has played a significant role in mathematics, engineering science, and other nonlinear sciences. Nonlinear problems are far more difficult to solve than linear ones, which play a major role in many scientific fields of physics, such as plasma physics, solid-state physics, quantum mechanics, nonlinear optics, fluid mechanics, etc. Solitons, known as solitary waves, have been the subject of theoretical and experimental study in numerous domains since 1834 [7, 5]. In 1971, optical solitons were first studied. The nonlinear complex Ginzburg-Landau equation GLE [3], which may be found in many fields

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of applied mathematics, theoretical physics, nonlinear optics, and engineering, describes the optical soliton. When, it is a fundamental model that encompasses a wide range of phenomena, including wave propagation, pattern formation, and phase transitions. While the complex GLE has been extensively studied, finding exact analytical solutions, especially soliton solutions, has proven to be a challenging task. Solitons are fascinating entities that exhibit remarkable properties, such as self-sustaining stability, localized energy, and the ability to retain their shape and speed during propagation. However, despite their ubiquity in nature, solitons are often elusive and difficult to characterize mathematically.

Obtaining soliton solutions to the complex GLE is a topic of great significance in scientific research and technological advancement. By unraveling the nature of solitons, we gain valuable insights into the behavior of complex nonlinear systems and pave the way for novel applications in diverse fields. While challenges remain in finding new styles of analytical solutions, the pursuit of soliton solutions is a compelling endeavor that promises to enhance our understanding of many physical interpretations [3].

Various modified versions of the GLE model have been the subject of several articles [6, 8, 10]. The improved Kudryashov approach has been used in [2]. The main advantage of using the current modified Jacobi elliptic functions (JEFs) method is that it provides an analytical solution more general for GLE, that can be used to study the behavior of superconductors, superfluids, and solitons under different conditions. However, this method has some limitations, and the solution obtained may not be accurate in certain situations. Therefore, it is important to validate the solution obtained from the modified JEFs approach with numerical simulations or other analytical methods. The pursuit of an accurate nonlinear partial differential equation (NLPDE) is one of the most popular study areas nowadays. As is common knowledge, accurate solutions to NLPDEs are important in a wide range of applied disciplines and offer useful details about the physical phenomena that GLE attempts to explain. Also, several systematic approaches [8] have been put forth to get precise solutions to complex GLE with the introduction of symbolic computation packages. Numerous of these techniques are founded on the idea that the answer may be thought of as a finite series when compared to the solutions of well-known ordinary differential equations ODEs such as the Bernoulli, Riccati, and Jacobi equations. One way to resolve this equation is by using the Jacobi elliptic function JEFs method. In this method, GLE is transformed into a nonlinear Schrödinger equation NLSE using a suitable substitution as an auxiliary equation method. The solution of nonlinear Schrödinger equation NLSE can then be expressed in terms of JEFs. The modified JEFs approach [1] has garnered the most interest among these techniques. These findings prompted the authors of the current research to use the Jacobi elliptic expansion approach to the following complex GLE [4] in order to construct complex wave patterns.

In this study, we implement another improved JEFs method [1] to investigate the optical solitons of the GLE with detuning factor, which is considered as [4].

The structure of this paper is as follows: The modified JEFs technique is used as an auxiliary equation method in Section 1. Exact explicit solutions of the complex GLE are provided in Section 2. Plotting solutions are done in the last Section; when we conclude to summarize the findings.

2. Mathematical Results

In this work, we use a modified JEFs method to construct exact wave solutions for the complex GLE. The governing model is read as:

$$\begin{gathered} i\frac{\partial q(x,t)}{\partial t}+\alpha_1\frac{\partial^2 q(x,t)}{\partial x^2}+\alpha_2\left|q\left(x,t\right)\right|^2 q\left(x,t\right) \\ -\frac{\alpha_3}{|q(x,t)|^2 q^*}\left[2\left|q\left(x,t\right)\right|^2\frac{\partial^2|q(x,t)|^2}{\partial x^2}-\left(\frac{\partial|q(x,t)|^2}{\partial x}\right)^2\right]-, \\ -\alpha_4 q\left(x,t\right)=0, \end{gathered} \tag{1}$$

where $q\left(x,t\right)$ represents the complex wave structures that describe the soliton propagation, $q^*\left(x,t\right)$ is the conjugate of $q\left(x,t\right), x$ and t are the spatial and temporal variables, respectively. Furthermore, α_1 and α_2 are the coefficients of the group velocity dispersion and the Kerr law non-linearity, while α_3 and α_4 denote the coefficients of the perturbation effects, especially α_4, which comes from the detuning effect [4]. For extracting optical solutions, the wave profile is divided into amplitude and phase components, respectively, as follows:

$$q\left(x,t\right)=u\left(\xi\right)e^{i\psi(\xi)}, \xi=x-\nu t, \tag{2}$$

where $q\left(x,t\right)$ is the amplitude component of the wave profiles and the phase factor is:

$$\psi\left(\xi\right)=-cx+\omega t+\theta, \tag{3}$$

when ν is the velocity of the soliton, whereas c,ω and θ represent the wave number, the frequency, and the phase constant, respectively.
The nonlinear complex GLE (1) is transformed into a one-dimensional nonlinear ordinary differential equation NLODE by taking the required values of (2) and (3) for (1). Nonlinear GLE (1) turns into an NLODE; we decompose this last one into real and imaginary parts, which results in a pair of relations. The imaginary part results in a constraint relation between the soliton parameters as follows:

$$\nu=-2\alpha_1 c, \tag{4}$$

that is a constraint condition for solutions to exist. The real part of ODE is the following formula:

$$\left(\alpha_1-4\alpha_3\right)u''-\left(\omega+c^2\alpha_1+\alpha_4\right)u+\alpha_2 u^3=0. \tag{5}$$

The balance rule detailed in [9] gives $N=1$.
Solitons that emerge from the limiting process are presented in the next section.

3. Optical Solitons-Solutions

Applying the modified auxiliary equation method to the nonlinear GLE (1) and using the balance rule of [9] (when $N = 1$), we get to write the solution of (5) as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i F^i(\xi) = a_0 + a_1 F(\xi), \tag{6}$$

where a_0, a_1 are arbitrary constants such that $a_1 \neq 0$ and $F(\xi)$ is a Jacobian elliptic function [1], when that last satisfying the following formula:

$$\left(F^{'}(\xi)\right)^2 = A_2 F^2(\xi) + A_4 F^4(\xi) + A_6 F^6(\xi), \tag{7}$$

where A_2, A_4 and A_6 are arbitrary constants determined by a modified JEFs method [1].
Substituting (6) and the derivative of (7) in (5) and collecting all terms with the same power and setting them to zero, we get the following algebraic system:

$$\begin{cases} 3a_1 A_6 (\alpha_1 - 4\alpha_3) = 0, \\ 2a_1 A_4 (\alpha_1 - 4\alpha_3) + \alpha_2 a_1^3 = 0, \\ 3\alpha_2 a_0 a_1^2 = 0, \\ A_2 (\alpha_1 - 4\alpha_3) + 3\alpha_2 a_0^2 - \left(\omega + c^2\alpha_1 + \alpha_4\right) = 0, \\ \alpha_2 a_0^2 - \left(\omega + c^2\alpha_1 + \alpha_4\right) = 0. \end{cases} \tag{8}$$

Solving algebraic system (8) by using any computer software (Matlab, Maple, Wolfram, Mathematica,...) yields the following values:

$$a_0 = 0, a_1^2 = -2(\alpha_1 + \alpha_3)\frac{A_4}{\alpha_2} > 0, \omega = A_2(\alpha_1 - 4\alpha_3) - c^2\alpha_1 - \alpha_4, A_6 = 0.$$

From $A_6 = 0$ and the cases detailed in [1]; we can deduce that all Jacobian elliptic solutions of nonlinear GLE (1) have only one modulus k with $0 \leq k \leq 1$. To the best of our current state of knowledge, we think that result may have been obtained here for the first time in the literature.
According to the work [1]; we can find three cases of solutions as follows:
Case 1: For $A_2 = -\left(1 + k^2\right), A_4 = k^2$ and $A_6 = 0$, we can obtain the following new complex Jacobi sine function solution for equation (1):

$$q_1(x, t, k) = \sqrt{-2(\alpha_1 + \alpha_3)\frac{k^2}{\alpha_2}} e^{i\left[-cx - \left(1+k^2\right)(\alpha_1 - 4\alpha_3)t - c^2\alpha_1 t - \alpha_4 t + \theta\right]} sn(x + 2\alpha_1 ct, k). \tag{9}$$

Case 2: For $A_2 = 2k^2 - 1, A_4 = -k^2$ and $A_6 = 0$, we can obtain the following new complex Jacobi cosine function solution for equation (1):

$$q_2(x, t, k)$$

$$= \sqrt{-2\left(\alpha_1 + 4\alpha_3\right)\frac{k^2}{\alpha_2}} e^{i\left[-cx+\left(2k^2-1\right)\left(\alpha_1-4\alpha_3\right)t-c^2\alpha_1 t-\alpha_4 t+\theta\right]} cn\left(x + 2\alpha_1 ct, k\right). \tag{10}$$

Case 3: For $A_2 = 2 - k^2, A_4 = -1$, and $A_6 = 0$, we can obtain the following new complex Jacobi function solution of the third kind for equation (1):

$$q_3\left(x, t, k\right)$$
$$= \sqrt{-2\left(\alpha_1 + 4\alpha_3\right)\frac{k^2}{\alpha_2}} e^{i\left[-cx+\left(2-k^2\right)\left(\alpha_1-4\alpha_3\right)t-c^2\alpha_1 t-\alpha_4 t+\theta\right]} dn\left(x + 2\alpha_1 ct, k\right). \tag{11}$$

We see that the new optical solitons-solutions (9)–(10)–(11) have generally different analytical solutions compared to optical solitons that have already been published. However, for $k \to 0$ or $k \to 1$, several of the specific solutions in [2] and [4] have become special solutions of (9)-(10)-(11).

4. Concluding Results and Perspectives

We used a modified Jacobi elliptic functions (JEFs) method along with the complex Ginzburg-Landau equation (GLE) and a tuning factor to find optical solitons in this study. When the unique ellipticity modulus gets close to one or zero, these solutions become very useful because they show periodic wave solutions for solitons. To provide a visual representation, we have included wave profile plots for soliton solutions denoted as $q_1\left(x, t\right), q_2\left(x, t\right)$ and $q_3\left(x, t\right)$ are presented; see Figures 1, 2, and 3, respectively. The obtained solitons hold significant value in enhancing our understanding of various intricate physical phenomena, given the GLE's importance in the telecommunications industry, where it is employed to describe pulse propagation in nonlinear optical media. There is no prior literature that mentions the solitons presented in this study, which have a distinctive modulus. We anticipate that these methodologies and findings will serve as a valuable resource for researchers grappling with various nonlinear problems, contributing to a better comprehension of the dynamics underlying physical phenomena. Perspectives For future study, we can find new solitons solutions to the complex Ginzburg-Landau equation with Kerr law nonlinearity according to the new extended direct algebraic method and for particular solutions.

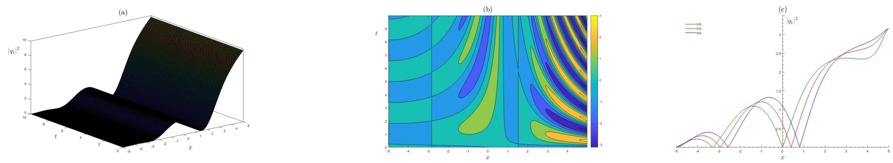

FIGURE 1. Represents q_1 soliton solution (9) with appropriate values of parameters $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 1, c = -0.1, \theta = 0$ with $0 \leq k \leq 1$. (a) is a 3D graph describing the structure of the soliton, and (b) is the contour that can describe the soliton's propagation, while (c) is a 2D graph depicting the propagation of the soliton waves along x-direction for different times.

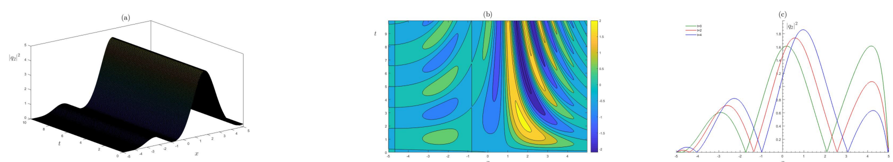

FIGURE 2. Represents q_2 soliton solution (10) with appropriate values of parameters $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 1, c = -0.1, \theta = 0$ with $0 \leq k \leq 1$. (a) is a 3D graph describing the structure of the soliton, and (b) is the contour that can describe the soliton's propagation, while (c) is a 2D graph depicting the propagation of the soliton waves along x-direction for different times.

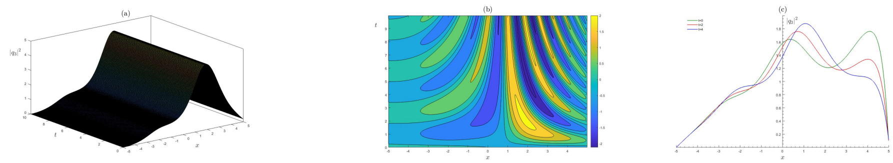

FIGURE 3. Represents q_3 soliton solution (11) with appropriate values of parameters $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 1, c = -0.1, \theta = 0$ with $0 \leq k \leq 1$. (a) is a 3D graph describing the structure of the soliton, and (b) is the contour that can describe the soliton's propagation, while (c) is a 2D graph depicting the propagation of the soliton waves along x-direction for different times.

Conflicts of interest : The author declare no conflict of interest.

Data availability : Not applicable

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