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A NOTE ON THE NONLOCAL CONTROLLABILITY OF HILFER FRACTIONAL DIFFERENTIAL EQUATIONS VIA MEASURE OF NONCOMPACTNESS

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Abstract. We looked at nonlocal controllability for Hilfer fractional differential equations with almost sectorial operator in this manuscript. We show certain necessary criteria for nonlocal controllability using the measure of noncompactness and the *Mönch* fixed point theorem. Finally, we provided theoretical and practical applications are given to demonstrate how the abstract results might be applied.

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1. Introduction

Nowadays, the principles of fractional calculation and the fractional differential equation have played the main role in Mathematics. Currently, the concept of fractional calculation has been powerfully tested in many social, physical, signal, image processing, biological, control theory, and engineering problems, etc. On the other hand, it has been proved that fractional differential equations are a useful tool in modeling several events. Fractional-order models are better than integer-order models for several sorts of realistic applications. The extension of differential equations and inequalities known as differential inclusions, which is sometimes referred to as control theory, has several users and applications. When one is adept at employing differential inclusions, dynamical systems with velocities that aren't solely determined by the system's state are easier to analyse. Numerous studies have been undertaken on boundary value problems.

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Numerous investigations have been conducted to ascertain if fractional differential inclusions and systems have solutions. The following research articles can be referenced to support the theory and its application discussed in relation to fractional calculus: [1, 2, 7, 16, 17, 21, 23, 31, 37].

Controllability plays a vital role in both pure and applied mathematics and is a key concept in mathematical control theory. Today, controllability plays a significant role in fractional calculus. As a result, scholars are very interested in this field and working to develop a new idea and concept connected to control theory, specifically how to apply control theory to fractional differential systems. Researchers have made great strides in recent years in understanding the precise and approximate controllability of many types of dynamical systems, including delay or not. Discussions of theory and practice related to controllability can be supported by the research publications [3, 9, 13, 18, 19, 24, 25, 26, 33]. For further details, see [22, 28, 34, 27].

Another sort of fractional derivative was introduced by Hilfer [11], which included both the R-L derivative and the Caputo fractional derivative. Hilfer fractional calculus is now widely used by scholars. Recently, a lot of academics have shown a considerable interest in this area, which has led to the work in [8, 10, 14, 35]. Researchers at [12, 4]came to their results using virtually sectorial operators using Schauder's fixed point theorem. The author of [13, 32] established their findings using the $Mönch$ fixed point theory and a noncompactness metric. Recently, [29] studied the exact controllability of the Hilfer fractional system with almost sectorial operator, by using Mönch's fixed point theorem. In [30] established the sufficient condition of the existence of mild solution of Hilfer fractional differential equation on an infinite interval via generalized Arzela-Ascoli's theorem.

Following are our article's significant contributions:

(i) For the Hilfer fractional differential system, we show the necessary and sufficient conditions for the nonlocal controllability of the Cauchy problem 1.
(ii) In this work, we study how a fractional differential System 1 has a mild solution existence and controllable.
(iii) Our System 1 is defined with a nonlocal condition.
(iv) We show that our result is consistent with the concept of measure of noncompactness.
(v) Firstly, we proved the nonlocal controllability of the system via the measure of noncompactness by $Mönch's$ 2.11 fixed point theorem.
(vii) In Example part, first we give a theoretical problem to illustrate our result.

(viii) Finally, we study a digital filter system related to our System 1 and the output given.

In this article, we will look at the following subject: Hilfer fractional differential equations have almost sectorial operators.

$$\begin{cases} D_{0^+}^{\eta,\zeta}\mathtt{z}(\mathfrak{s}) = \mathsf{A}\mathtt{z}(\mathfrak{s}) + \mathfrak{F}\big(\mathfrak{s},\mathtt{z}(\mathfrak{s})\big) + \mathtt{B}v(\mathfrak{s}), \qquad \mathfrak{s}\in\mathcal{J}'=(0,b], \\ I_{0^+}^{(1-\eta)(1-\zeta)}\big[\mathtt{z}(0)+N(\mathtt{z})\big] = \mathtt{z}_0, \end{cases} \tag{1}$$

where A denote the almost sectorial operator, which generate an analytic semigroup $\{\mathtt{T}(\mathfrak{s}), \mathfrak{s}\geq 0\}$ on Y. $D_{0^+}^{\eta,\zeta}$ denotes the Hilfer fractional derivative of order $\eta,\ 0<\eta<1$ and type $\zeta,\ 0\leq\zeta\leq 1$. Let $\mathtt{z}(\cdot)$ be the state in a Banach space Y with norm $\|\cdot\|$ and $v(\cdot)$ be the control function in $L^2(\mathcal{I},U)$, where U be the Banach space. Here $\mathtt{B}$ is the bounded linear operator from U into Y. Set $\mathcal{J}=[0,b]$, and let $\mathfrak{F}:\mathcal{I}\times Y\to Y$ be the Y-valued function and nonlocal term $N:C(\mathcal{I},Y)\to Y$.

The following is an outline of the article's structure. Section 2, discusses the fundamentals of fractional system, semigroups, sectorial operators, and the measure of noncompactness (MNC). We discussed the system's nonlocal controllability in Section 3. In Section 4, we presented theoretical and practical applications. Finally, conclusions are provided.

2. Main results

In this section, some definitions, theorems and lemma that are used every part of the article.

Consider $\mathfrak{F}$ is function defined by $\mathfrak{F}:[b,\infty)\to\mathbb{R}$:

Definition 2.1. [37] Let $\mathfrak{F}$ be a function, then the $R-L$ fractional derivative has order $\eta>0$, $k-1\leq\eta<k,\ k\in\mathbb{N}$, presented by

$${}^{L}D_{b+}^{\eta}\mathfrak{F}(\mathfrak{s}) = \frac{1}{\Gamma(k-\eta)}\frac{d^k}{d\mathfrak{s}^k}\int_b^{\mathfrak{s}}\frac{\mathfrak{F}(\rho)}{(\mathfrak{s}-\rho)^{\eta+1-k}}d\rho,\ \mathfrak{s}>b, \rho\in\mathbb{R}^+.$$

Definition 2.2. [37] The Caputo fractional derivative has order $\eta>0,\ k-1\leq\eta<k,\ k\in\mathbb{N}$ for a function $\mathfrak{F}$, is defined as

$${}^{C}D_{b^+}^{\eta}\mathfrak{F}(\mathfrak{s}) = \frac{1}{\Gamma(k-\eta)}\int_b^{\mathfrak{s}}\frac{\mathfrak{F}^k(\rho)}{(\mathfrak{s}-\rho)^{\eta+1-k}}d\rho = I_{b+}^{k-\eta}\mathfrak{F}^k(\mathfrak{s}),\ \mathfrak{s}>b,\ \rho\in\mathbb{R}^+.$$

Definition 2.3. [11] The Hilfer fractional derivative of order $0<\eta<1$ and type $\zeta\in[0,1]$ for the function $\mathfrak{F}:[b,+\infty)\to\mathbb{R}$, presented by

$$D_{b^+}^{\eta,\zeta}\mathfrak{F}(\mathfrak{s}) = [I_{b^+}^{(1-\eta)\zeta}D(I_{b^+}^{(1-\eta)(1-\zeta)}\mathfrak{F})](\mathfrak{s}).$$

Definition 2.4. [22] Let $0<\vartheta<1,\quad 0<\varphi<\frac{\pi}{2}$, we define the family of closed linear operators $\Theta_{\varphi}^{-\vartheta}$, the sector $S_{\varphi}=\{\theta\in\mathbb{C}\backslash\{0\}$ with $|arg\ \theta|\leq\varphi\}$ and $\mathsf{A}:D(\mathsf{A})\subset Y\to Y$ that satisfy

(a) $\sigma(\mathsf{A})\subseteq S_{\varphi}$;

(b) $\left\|(\theta I - \mathsf{A})^{-1}\right\| \leq \mathcal{K}_\delta |\nu|^{-\vartheta}$, for every $\varphi < \delta < \pi$ and there exists $\mathcal{K}_\delta$ be a constant,

then $\mathsf{A} \in \Theta_\varphi^{-\vartheta}$ is known as almost sectorial operator on Y.

Let $\{\mathcal{S}_\eta(\mathfrak{s})\}_{\mathfrak{s}\in S_{\frac{\pi}{2}-\varphi}}, \{\mathcal{Q}_\eta(\mathfrak{s})\}_{\mathfrak{s}\in S_{\frac{\pi}{2}-\varphi}}$ be the operator families defined as follows:

$$\mathcal{S}_\eta(\mathfrak{s}) = \int_0^\infty \mathfrak{W}_\eta(\xi)\mathtt{T}(\mathfrak{s}^\eta \xi)d\xi,$$
$$\mathcal{Q}_\eta(\mathfrak{s}) = \int_0^\infty \eta\xi\mathfrak{W}_\eta(\xi)\mathtt{T}(\mathfrak{s}^\eta \xi)d\xi,$$

where $\mathfrak{W}_\eta(\beta)$ be the Wright-type function:

$$\mathfrak{W}_\eta(\beta) = \sum_{k\in\mathbb{N}} \frac{(-\beta)^{k-1}}{\Gamma(1-\eta k)(k-1)!},\ \beta \in \mathbb{C}. \tag{2}$$

Let $-1 < \iota < \infty,\ p > 0$, the given properties are hold.

(a) $\mathfrak{W}_\eta(\theta) \geq 0, \quad \mathfrak{s} > 0$;
(b) $\int_0^\infty \theta^\iota \mathfrak{W}_\eta(\theta)d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\eta\iota)}$;
(c) $\int_0^\infty \frac{\eta}{\theta^{(\eta+1)}} e^{-p\theta}\mathfrak{W}_\eta(\frac{1}{\theta^\eta})d\theta = e^{-p^\eta}$.

Lemma 2.5. [10] *Equation (1) is equivalent to an integral equation given by*

$$\mathsf{z}(\mathfrak{s}) = \frac{\mathsf{z}(0) - N(\mathsf{z}))}{\Gamma(\zeta(1-\eta)+\eta)}\mathfrak{s}^{(1-\eta)(\zeta-1)} + \frac{1}{\Gamma(\eta)}\int_0^\mathfrak{s} (\mathfrak{s}-\rho)^{\eta-1}\left[\mathsf{A}\mathsf{z}(\rho) + \mathsf{B}v(\rho) + \mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)\right]d\rho.$$

Proof. The proof of the Lemma is similar to Lemma 2.12 in [10], so we omit it. □

From the above Lemma, we get the mild solution of the Cauchy problem 1:

Definition 2.6. The mild solution of the Cauchy problem (1), is a function $\mathsf{z}(\mathfrak{s}) \in C(\mathcal{I}', Y)$, that satisfies

$$\mathsf{z}(\mathfrak{s}) = \mathcal{S}_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}(0) - N(\mathsf{z})\big] + \int_0^\mathfrak{s} \mathcal{K}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)d\rho + \int_0^\mathfrak{s} \mathcal{K}_\eta(\mathfrak{s}-\rho)\mathsf{B}v(\rho)d\rho,\ \mathfrak{s} \in \mathcal{I},$$

where $\mathcal{S}_{\eta,\zeta}(\mathfrak{s}) = I_0^{\zeta(1-\eta)}\mathcal{K}_\eta(\mathfrak{s}),\ \mathcal{K}_\eta(\mathfrak{s}) = \mathfrak{s}^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s})$.
i.e.,

$$\mathsf{z}(\mathfrak{s}) = \mathcal{S}_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}(0) - N(\mathsf{z})\big] + \int_0^\mathfrak{s} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)d\rho + \int_0^\mathfrak{s} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathsf{B}v(\rho)d\rho,\ \mathfrak{s} \in \mathcal{I}. \tag{3}$$

Lemma 2.7. [38] *Assume* $\mathtt{T}(\mathfrak{s})$ *be an equicontinuous, then* $\mathcal{Q}_\eta(\mathfrak{s})$, $\mathcal{K}_\eta(\mathfrak{s})$, *and* $\mathcal{S}_{\eta,\sigma}(\mathfrak{s})$ *are the strongly continuous,* $\mathtt{z} \in Y$ *and* $\mathtt{e}_2 > \mathtt{e}_1 > 0$,

$$\left\|\mathcal{Q}_\eta(\mathtt{e}_2)\mathtt{z} - \mathcal{Q}_\eta(\mathtt{e}_1)\mathtt{z}\right\| \to 0,\ \left\|\mathcal{K}_\eta(\mathtt{e}_2)\mathtt{z} - \mathcal{K}_{\mathfrak{p}}(\mathtt{e}_1)\mathtt{z}\right\| \to 0$$
$$\left\|\mathcal{S}_{\eta,\zeta}(\mathtt{e}_2)\mathtt{z} - \mathcal{S}_{\eta,\zeta}(\mathtt{e}_1)\mathtt{z}\right\| \to 0,\ \textit{as } \mathtt{e}_2 \to \mathtt{e}_1.$$

Lemma 2.8. [38] *For every* $\mathfrak{s} > 0$, *the linear operators* $\mathcal{Q}_\eta(\mathfrak{s})$, $\mathcal{K}_\eta(\mathfrak{s})$ *and* $\mathcal{S}_{\eta,\zeta}(\mathfrak{s})$ *are satisfied following,*

$$\left\|\mathcal{Q}_\eta(\mathfrak{s})\mathtt{z}\right\| \le L'\mathfrak{s}^{-\eta+\eta\vartheta},\ \left\|\mathcal{K}_\eta(\mathfrak{s})\mathtt{z}\right\| \le L'\mathfrak{s}^{-1+\eta\vartheta}\|\mathtt{z}\|,\ \left\|\mathcal{S}_{\eta,\sigma}(\mathfrak{s})\mathtt{z}\right\| \le L''\mathfrak{s}^{-1+\zeta-\eta\zeta+\eta\vartheta}\|\mathtt{z}\|,$$

where

$$L' = \kappa_0\frac{\Gamma(\vartheta)}{\Gamma(\eta\vartheta)},\ L'' = \kappa_0\frac{\Gamma(\vartheta)}{\Gamma(\sigma(1-\eta)+\eta\vartheta)},\ \textit{for every } \mathtt{z} \in Y.$$

Definition 2.9. The system (1) is called nonlocally controllable on the interval $\mathcal{I}$ if and only if, for every $\mathtt{z}_0, \mathtt{z}_1 \in Y$, there exists a control $v \in L(\mathcal{J}, U)$ such that the mild solution $\mathtt{z}(\cdot)$ of the System (1) satisfies $\mathtt{z}(b) + N(\mathtt{z}) = \mathtt{z}_1$.

Theorem 2.10. [32] *if* $\{\mathtt{z}_k\}_{k=1}^{\infty}$ *is a sequence of Bochner integrable functions from* $\mathcal{I}$ *into* Y *with the estimation* $\|\mathtt{z}_k(\mathfrak{s})\| \le \mu(\mathfrak{s})$ *for almost all* $\mathfrak{s} \in \mathcal{I}$ *and every* $k \ge 1$, *where* $\mu \in L^1(\mathcal{I}, \mathbb{R})$, *then the function* $\varphi(\mathfrak{s}) = \beta\big(\{\mathtt{z}_k(\mathfrak{s}) : k \ge 1\}\big)$ *be in* $L^1(\mathcal{I}, \mathbb{R})$ *and satisfies*

$$\beta\left(\left\{\int_0^{\mathfrak{s}} \mathtt{z}_k(\rho)d\rho :\ k \ge 1\right\}\right) \le 2\int_0^{\mathfrak{s}} \varphi(\rho)d\rho.$$

Lemma 2.11. [20] *Let* $\mathtt{D}$ *be a closed convex subset of a Banach space* Y *and* $0 \in \mathtt{D}$. *Assume that* $\mathfrak{F} : \mathtt{D} \to Y$ *continuous map which satisfies Mönch's condition, i.e., if* $G_1 \subset \mathtt{D}$ *is countable and,* $\mathtt{D}_1 \subset conv(\{0\} \cup \mathfrak{F}(\mathtt{D}_1))$ *implies* $\overline{\mathtt{D}}_1$ *is compact. Then* $\mathfrak{F}$ *has a fixed point in* G.

For our convenience, we introduce $K_{\eta_n} = \left[\left(\frac{1-\eta_n}{-\eta\vartheta-1}\right)b^{\left(\frac{-\eta\vartheta-1}{1-\eta_n}\right)}\right]$, $n = 1, 2$, $\mathcal{K}_4 = K_{\eta_1}\left\|K_W\right\|_{L^{\frac{1}{\eta_1}}(\mathcal{I},\mathbb{R}^+)}$ *and* $\mathcal{K}_5 = K_{\eta_2}\left\|h\right\|_{L^{\frac{1}{\eta_2}}(\mathcal{I},\mathbb{R}^+)}$.

3. Controllability

Consider the succeeding hypotheses:

- (H_1) Let $\mathtt{T}(\mathfrak{s})$ be the analytical semigroup with $\|\mathtt{T}(\mathfrak{s})\| \le \mathcal{K}_1$ where the constant $\mathcal{K}_1 \ge 0$.
- (H_2) The function $\mathfrak{F} : \mathcal{I} \times Y \to Y$ such that:
  - (a) $\mathfrak{F}(\cdot, \mathtt{z})$ is strongly measurable for every $\mathtt{z} \in Y$ and $\mathfrak{F}(\mathfrak{s}, \cdot)$ is continuous for a.e. $\mathfrak{s} \in \mathcal{I}$;
  - (b) there exists a constant $0 < \eta_1 < \eta$ and $m \in L^{\frac{1}{\eta_1}}(\mathcal{I}, [0, \infty))$ and continuous increasing function $f : [0, \infty) \to [0, \infty)$ such that $\left\|\mathfrak{F}(\mathfrak{s}, \mathtt{z})\right\| \le m(\mathfrak{s})f(\|\mathtt{z}\|)$, $\mathtt{z} \in Y$, $\mathfrak{s} \in \mathcal{I}$ where f satisfies $\lim\inf\limits_{k\to\infty} \frac{f(k)}{k} = 0$;

(c) there exists a constant $0 < \eta_2 < \eta$ and $h \in L^{\frac{1}{\eta_2}}(\mathcal{I}, \mathbb{R}^+)$ such that, for every bounded subset $M \subset Y$, $\beta(\mathfrak{F}(\mathfrak{s}, M)) \leq h(\mathfrak{s})\beta(M)$ for a.e. $\mathfrak{s} \in \mathcal{I}$.

(H_3) (a) The linear operator $\mathsf{B} : L^2(\mathcal{I}, U) \to L^1(\mathcal{I}, Y)$ is bounded, $W : L^2(\mathcal{I}, U) \to Y$ defined by $\mathsf{W}v = \int_0^b (b-w)^{\eta-1}\mathcal{Q}_\eta(b-\rho)\mathsf{B}v(\rho)d\rho$ has an inverse operator W^{-1} which take the values in $L^2(\mathcal{I}, U)/\ker \mathsf{W}$ and there exists two positive values $\mathcal{K}_2$ *and* $\mathcal{K}_3$ such that $\|\mathsf{B}\|_{L_b(U,Y)} \leq \mathcal{K}_2$, $\|\mathsf{W}^{-1}\|_{L_b(Y, U/\ker \mathsf{W})} \leq \mathcal{K}_3$;

(b) there exists a constant $\eta_0 \in (0, \eta)$ and $K_{\mathsf{W}} \in L^{\frac{1}{\eta_0}}(\mathcal{I}, \mathbb{R}^+)$ such that, for every bounded set $Q \subset Y$, $\beta((\mathsf{W}^{-1}Q)(\mathfrak{s})) \leq K_{\mathsf{W}}(\mathfrak{s})\beta(Q)$.

(H_4) The function $N : C(\mathcal{I}, Y) \to Y$ is continuous, compact operator and there exists $L_1 > 0$ be the value such that $\|N(\mathsf{z}_1) - N(\mathsf{z}_2)\| \leq L_1\|\mathsf{z}_1 - \mathsf{z}_2\|$.

Theorem 3.1. *Assume $(H_1) - (H_4)$ holds, then the Hilfer fractional system (1) has a solution on $\mathcal{J}$ provided,* $\widehat{\mathcal{K}} = \left[1 + \kappa_{\mathsf{q}}\mathcal{K}_2\mathcal{K}_4\right]2\kappa_{\mathsf{q}}\mathcal{K}_5 b^{1-\zeta+\eta\zeta-\eta\vartheta} < 1$, *and* $\mathsf{z}(0) \in D(\mathsf{A}^\theta)$ *with* $\theta > 1 - \vartheta$.

Proof. Consider the operator $\Psi : \mathcal{X} \to \mathcal{X}$, defined

$$\Psi(\mathsf{z}(\mathfrak{s})) = \Big\{ z \in \mathcal{X} : z(\mathfrak{s}) = \mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta}\Big[\mathcal{S}_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}_0 - N(\mathsf{z})\big] + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho) \times \mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)d\rho + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathsf{B}v(\rho)d\rho\Big],\ \mathfrak{s} \in (0, d]\Big\}.$$

Show that Ψ has a fixed point.
Using (H_3), for an arbitrary function $\mathsf{z} \in \mathcal{X}$, we define the control $v_{\mathsf{z}}(\mathfrak{s})$ by

$$v_{\mathsf{z}}(\mathfrak{s}) = \mathsf{W}^{-1}\left(\mathsf{z}_1 - N(\mathsf{z}) - \mathcal{S}_{\eta,\zeta}(b)(\mathsf{z}_0 - N(\mathsf{z})) - \int_0^b (b-r)^{\eta-1}\mathcal{Q}_\eta(b-r)\mathfrak{F}(r, \mathsf{z}(r))dr\right)(\mathfrak{s}).$$

As we can see $\Psi\big(\mathsf{z}(b)\big) = \mathsf{z}_1 - N(\mathsf{z})$ which means that v_{z} steer the Hilfer fractional system (1) z_0 to z_1 in the finite time b. This implies that the Equations (1) become nonlocally controllable on $\mathcal{I}$.

Now, we define $\Psi = \Psi_1 + \Psi_2$ where

$$\Psi_1\mathsf{z}(\mathfrak{s}) = \mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta}\big(\mathcal{S}_{\eta,\zeta}(\mathfrak{s})(\mathsf{z}_0 - N(\mathsf{z}))\big),$$

$$\Psi_2\mathsf{z}(\mathfrak{s}) = \mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta}\left(\int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)d\rho + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathsf{B}v(\mathfrak{s})d\rho\right).$$

Step:1 Prove there exists $\mathsf{q} > 0$ such that $\Psi(\mathsf{S}_{\mathsf{q}}(\mathcal{I})) \subseteq \mathsf{S}_{\mathsf{q}}(\mathcal{I})$. Suppose that statement is not true i.e., for all $\mathsf{q} > 0$, there exits $\mathsf{z}^{\mathsf{q}} \in \mathsf{S}_{\mathsf{q}}(\mathcal{I})$, but $\Psi(\mathsf{z}^{\mathsf{q}})$ not in

$\mathsf{S}_{\mathsf{q}}(\mathcal{I})$, that is,

$$\begin{aligned}
\left\|\mathsf{z}^{\mathsf{q}}\right\| &\leq \mathsf{q} < \left\|\left(\Psi \mathsf{z}^{\mathsf{q}}(\mathfrak{s})\right)\right\| \\
&\leq \Bigg\| \mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta}\Bigg[\mathcal{S}_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}_0 - N(\mathsf{z}^{\mathsf{q}})\big] \\
&\quad + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}^{\mathsf{q}}(\rho)\big)d\rho \\
&\quad + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathsf{B}W^{-1}\Bigg(\mathsf{z}_1 - N(\mathsf{z}^{\mathsf{q}}) - \mathcal{S}_{\eta,\zeta}(b)(\mathsf{z}_0 - N(\mathsf{z}^{\mathsf{q}})) \\
&\quad - \int_0^b (b-r)^{\eta-1}\mathcal{Q}_\eta(b-r)\mathfrak{F}(r, \mathsf{z}^{\mathsf{q}}(r))dr\Bigg)(\mathfrak{s})d\rho\Bigg\| \\
&\leq b^{(1-\zeta+\eta\zeta-\eta\vartheta}\Bigg[\sup\left\|\mathcal{S}_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}_0 - N(\mathsf{z}^{\mathsf{q}})\big]\right\| \\
&\quad + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\Bigg\|\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}^{\mathsf{q}}(\rho)\big)\Bigg\|d\rho \\
&\quad + \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\Bigg\|\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathsf{B}W^{-1}\Bigg(\mathsf{z}_1 - N(\mathsf{z}^{P}) - \mathcal{S}_{\eta,\zeta}(b)(\mathsf{z}_0 - N(\mathsf{z}^{\mathsf{q}})) \\
&\quad - \int_0^b (b-r)^{\eta-1}\mathcal{Q}_\eta(b-r)\mathfrak{F}(r, \mathsf{z}^{\mathsf{q}}(r))dr\Bigg)(\rho)\Bigg\|d\rho\Bigg] \\
&\leq b^{1-\zeta+\eta\zeta-\eta\vartheta}\left[M^* + \frac{b^{\eta\vartheta}}{\eta\vartheta}\kappa_p\mathcal{K}_2\mathcal{K}_3\left[\|\mathsf{z}_1\| + L_1\|\mathsf{z}\| + \|N(0)\| - M^*\right]\right]
\end{aligned}$$

where $M^* = \Bigg(\dfrac{\Gamma(\eta\zeta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)}\kappa_p b^{\zeta(1-\eta)-\eta\vartheta-1}\|\mathsf{z}_0\| + L_1\|\mathsf{z}\| + \|N(0)\|$
$$+ \frac{b^{\eta\vartheta}}{\eta\vartheta}\kappa_p m(b) f(\|\mathsf{z}\|)\Bigg).$$

Dividing both sides by $\left\|\mathsf{z}^{\mathsf{q}}\right\|$ and $\left\|\mathsf{z}^{\mathsf{q}}\right\| \to \infty$, we get $0 \geq 1$, which is the contradiction. Thus, $\Psi(\mathsf{S}_{\mathsf{q}}(\mathcal{I})) \subset \mathsf{S}_{\mathsf{q}}(\mathcal{I})$.

Step 2: The operator Ψ is continuous on $\mathsf{S}_{\mathsf{q}}(\mathcal{I})$. For that, consider the sequence $\mathsf{z}_k \to \mathsf{z}$ in $\mathsf{S}_{\mathsf{q}}(\mathcal{I})$. From hypotheses (H_4) and Lemma 2.8, we get

$$\|\Psi_1(\mathsf{z}_k) - \Psi_1(\mathsf{z})\| \leq b^{1-\zeta+\eta\zeta-\eta\vartheta}\|\mathcal{S}_{\eta,\zeta}(\mathfrak{s})\|\|N(\mathsf{z}_k) - N(\mathsf{z})\| \to 0 \ as \ k \to \infty. \quad (1)$$

From (H_2) and Lebesgue dominated convergence theorem, we write

$$\int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\left\|\mathfrak{F}(\rho, \mathsf{z}_k(\rho)) - \mathfrak{F}(\rho, \mathsf{z}(\rho))\right\| \to 0 \ as \ k \to \infty.$$

Therefore

$$\left\|\Psi_2(\mathsf{z}_k) - \Psi_2(\mathsf{z})\right\| \leq \frac{b^{\eta\vartheta}}{\eta\vartheta}\kappa_p\left\|\mathfrak{F}(\rho, \mathsf{z}_k(\rho)) - \mathfrak{F}(\rho, \mathsf{z}(\rho))\right\| + \frac{b^{\eta\vartheta}}{\eta\vartheta}\kappa_p\mathcal{K}_2\|v_{\mathsf{z}_k} - v_{\mathsf{z}}\| \quad (2)$$

where

$$\begin{aligned}\left\|v_{\mathsf{z}_k}-v_{\mathsf{z}}\right\| \leq \mathcal{K}_3\left(1-\kappa_p b^{\zeta(1-\eta)-\eta\vartheta-1}\frac{\Gamma(\eta\zeta)}{\Gamma(\zeta(1-\eta)-\eta\vartheta)}\right)\left\|N(\mathsf{z}_k)-N(\mathsf{z})\right\| \\ -\frac{b^{\eta\vartheta}}{\eta\vartheta}\kappa_p\left\|\mathfrak{F}(\rho,\mathsf{z}_k(\rho))-\mathfrak{F}(\rho,\mathsf{z}(\rho))\right\|\end{aligned}$$

From the above equations, we get $\|\Psi_2(\mathsf{z}_k)-\Psi_2(\mathsf{z})\| \to 0 \ as \ k \to \infty$. So Ψ_2 is continuous on $B_P(\mathcal{I})$. Hence

$$\left\|\Psi(\mathsf{z}_k)-\Psi(\mathsf{z})\right\| \leq \left\|\Psi_1(\mathsf{z}_k)-\Psi_1(\mathsf{z})\right\| + \left\|\Psi_2(\mathsf{z}_k)-\Psi_2(\mathsf{z})\right\| \to 0 \ as \ n \to \infty.$$

Step 3: To show $M\ddot{o}nch's$ conditions.
Consider $M \subseteq \mathsf{S}_{\mathfrak{q}}(\mathcal{I})$ is countable and $M \subseteq conv(\{0\}\cup\Psi(M))$, we prove that $\beta(M)=0$, where β is the Hausdorff MNC. We can assume, without losing generality, $M=\{\mathsf{z}_k\}_{k=1}^{\infty}$. If we can show that $\{\Psi\mathsf{z}_k\}$ is equicontinuous on $\mathcal{I}$, then $M \subseteq conv(\{0\}\cup\Psi(M))$ is also equicontinuous on $\mathcal{I}$. Consider $z(\mathfrak{s}) \in \Psi(M)$, and $0 \leq \mathfrak{s}_1 < \mathfrak{s}_2 \leq b$, there is $\mathsf{z} \in M$ such that

$$\begin{aligned}\|z(\mathfrak{s}_2)-z(\mathfrak{s}_1)\| \leq & \left\|\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathfrak{s}_2)-\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathfrak{s}_1)\right\|\left\|\mathsf{z}_0-N(\mathsf{z})\right\| \\ & +\left\|\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho\right. \\ & +\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho \\ & \left.-\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_1-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho\right\| \\ & +\left\|\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathsf{B}v(\rho)d\rho\right. \\ & +\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_{\mathfrak{s}_1}^{\mathfrak{s}_2}(\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathsf{B}v(\rho)d\rho \\ & \left.-\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_1-\rho)\mathsf{B}v(\rho)d\rho\right\| \\ \leq & \left\|\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathfrak{s}_2)-\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathfrak{s}_1)\right\|\left\|\mathsf{z}_0-N(\mathsf{z})\right\| \\ & +\left\|\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho\right. \\ & \left.-\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho\right\| \\ & +\left\|\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho\right. \\ & \left.-\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}\int_0^{\mathfrak{s}_1}(\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_1-\rho)\mathfrak{F}(\rho,\mathsf{z}(\rho))d\rho\right\|\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} (\mathfrak{s}_2-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathfrak{F}(\rho, \mathsf{z}(\rho)) d\rho \right\| \\
&+ \left\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_2-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathsf{B}v(\rho) d\rho \right. \\
&\left. - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathsf{B}v(\rho) d\rho \right\| \\
&+ \left\| \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathsf{B}v(\rho) d\rho \right. \\
&\left. - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_1-\rho) \mathsf{B}v(\rho) d\rho \right\| \\
&+ \left\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} (\mathfrak{s}_2-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathsf{B}v(\rho) d\rho \right\| \\
= &\sum_{i=1}^{7} I_i
\end{aligned}
$$

By the strong continuity of $\mathcal{S}_{\eta,\zeta}(\mathfrak{s})$, we get
I_1 tends to zero as $\mathfrak{s}_2 \to \mathfrak{s}_1$.

$$
\begin{aligned}
I_2 = &\left\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_2-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathfrak{F}(\rho, \mathsf{z}(\rho)) d\rho \right. \\
&\left. - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathfrak{F}(\rho, \mathsf{z}(\rho)) d\rho \right\| \\
\le &\kappa_p \int_0^{\mathfrak{s}_1} (\mathfrak{s}_2-\rho)^{-\eta-\eta\vartheta} \left| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathfrak{s}_2-\rho)^{\eta-1} - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} (\mathfrak{s}_1-\rho)^{\eta-1} \right| \\
&\times m(\rho) f(\|\mathsf{z}\|) d\rho.
\end{aligned}
$$

From Lebesgue's dominated convergence theorem, we get $\lim_{\mathfrak{s}_2 \to \mathfrak{s}_1} I_2 = 0$.

$$
\begin{aligned}
I_3 = &\left\| \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathfrak{F}(\rho, \mathsf{z}(\rho)) d\rho \right. \\
&\left. - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_1-\rho) \mathfrak{F}(\rho, \mathsf{z}(\rho)) d\rho \right\| \\
\le &\mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \left\| \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) - \mathcal{Q}_\eta(\mathfrak{s}_1-\rho) \right\| m(\rho) f(\|u\|) d\rho
\end{aligned}
$$

By Theorem 2.7, $\mathcal{Q}_\eta(\mathfrak{s})$ is uniformly continuous in operator norm topology, we get $I_3 \to 0$ as $\mathfrak{s}_2 \to \mathfrak{s}_1$.

$$
\begin{aligned}
I_4 = &\left\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} (\mathfrak{s}_2-\rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) \mathfrak{F}(\rho, \mathsf{z}(\rho)) d\rho \right\| \\
\le &\kappa_p \left| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_2} (\mathfrak{s}_2-\rho)^{-\eta\vartheta-1} - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{-\eta\vartheta-1} \right|
\end{aligned}
$$

$$\times m(\rho)f(\|u\|)d\rho.$$

Then $I_4 \to 0$ as $\mathfrak{s}_2 \to \mathfrak{s}_1$, by using dominated convergence theorem.

$$\begin{aligned}
I_5 &= \Big\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathsf{B}v(\rho)d\rho \\
&\quad - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathsf{B}v(\rho)d\rho \Big\| \\
&\leq \kappa_p\mathcal{K}_2 \int_0^{\mathfrak{s}_1} \Big(\mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta}(\mathfrak{s}_2-\rho)^{\eta-1} - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta}(\mathfrak{s}_1-\rho)^{\eta-1} \Big) \\
&\quad \times (\mathfrak{s}_2-\eta)^{-\eta-\eta\vartheta} v(\rho)d\rho. \\
I_6 &= \Big\| \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathsf{B}v(\rho)d\rho \\
&\quad - \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_1-\rho)\mathsf{B}v(\rho)d\rho \Big\| \\
&\leq \mathcal{K}_2\ \mathfrak{s}_1^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}_1} (\mathfrak{s}_1-\rho)^{\eta-1} \big\| \mathcal{Q}_\eta(\mathfrak{s}_2-\rho) - \mathcal{Q}_\eta(\mathfrak{s}_1-\rho) \big\| v(\rho)d\rho \\
I_7 &= \Big\| \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} (\mathfrak{s}_2-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}_2-\rho)\mathsf{B}v(\rho)d\rho \Big\| \\
&\leq \kappa_p\mathcal{K}_2\ \mathfrak{s}_2^{1-\zeta+\eta\zeta-\eta\vartheta} \Big\| \int_{\mathfrak{s}_1}^{\mathfrak{s}_2} (\mathfrak{s}_2-\rho)^{-\eta\vartheta-1} v(\rho)d\rho \Big\|.
\end{aligned}$$

Similar proof of I_2 and I_3, we get I_5 and I_6 are tend to zero, also I_7 tends to zero as $\mathfrak{s}_2 \to \mathfrak{s}_1$. Therefore, $\Psi(M)$ is equicontinuous on $\mathcal{I}$.

Now, we need to prove the relatively compactness of $\Psi(M)$ in Y for every $\mathfrak{s} \in \mathcal{I}$. Using the compactness of the function N, we get

$$\beta\big(\{(\Psi_1\mathsf{z}_k)(\mathfrak{s})\}_{k=1}^\infty\big) \leq \beta\big(\{\mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta}\mathcal{S}_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}_0 - N(\mathsf{z}_k)\big]\}_{k=0}^\infty\big) = 0.$$

By Lemma 2.7 and Theorem 2.10, we have

$$\begin{aligned}
\beta\big(\{\Psi_2\mathsf{z}_k\}_{k=0}^\infty\big) &\leq \beta\Big(\{\mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{\eta-1}\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}(\rho,\mathsf{z}_k(\rho))\}_{k=1}^\infty\Big) \\
&\quad + \beta\Big(\{\mathfrak{s}^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} \mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathsf{B}v_{\mathsf{z}_k}(\rho)\}_{k=1}^\infty\Big) \\
&\leq 2\kappa_p b^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{-\eta\vartheta-1}h(\rho)d\rho\beta\big(M(\mathfrak{s})\big) \\
&\quad + 2\kappa_p\mathcal{K}_2 b^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{-\eta\vartheta-1}d\rho\beta\big(v_{\mathsf{z}_k}\big) \\
&= J_1 + J_2.
\end{aligned}$$

where

$$J_1 = 2\kappa_p b^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{-\eta\vartheta-1}h(\rho)d\rho\beta\big(M(\mathfrak{s})\big)$$

$$\leq 2\kappa_p b^{1-\zeta+\eta\zeta-\eta\vartheta} K_{\eta_2} \|h\|_{L^{\frac{1}{\eta_2}}(\mathcal{I},\mathbb{R}^+)} \beta(M)$$
$$\leq 2\kappa_p \mathcal{K}_5 b^{1-\zeta+\eta\zeta-\eta\vartheta} \beta(M).$$

and

$$\begin{aligned}
J_2 &= 2\kappa_p \mathcal{K}_2 b^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{-\eta\vartheta-1} d\rho \beta\big(v_{\mathsf{z}_k}\big) \\
&\leq 2\kappa_p \mathcal{K}_2 b^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{-\eta\vartheta-1} d\rho \\
&\quad \times \beta\bigg(W^{-1}\bigg(\big\{\mathsf{z}_1 - N(\mathsf{z}_k) - \mathcal{S}_{\eta,\zeta}(b)(\mathsf{z}_0 - N(\mathsf{z})) \\
&\quad - \int_0^b (b-r)^{\eta-1} \mathcal{Q}_\eta(b-r) \mathfrak{F}(r, \mathsf{z}_k(r)) dr\big\}_{k=1}^{\infty}\bigg)(\rho)\bigg) \\
&\leq 2\kappa_p \mathcal{K}_2 b^{1-\zeta+\eta\zeta-\eta\vartheta} \int_0^{\mathfrak{s}} (\mathfrak{s}-\rho)^{-\eta\vartheta-1} K_{\mathsf{W}}(\rho) d\rho \kappa_p \int_0^b (b-r)^{-\eta\vartheta-1} h(r) dr \beta(D(\mathfrak{s})) \\
&\leq 2(\kappa_p)^2 \mathcal{K}_2 b^{1-\zeta+\eta\zeta-\eta\vartheta} K_{\eta_1} \|K_{\mathsf{W}}\|_{L^{\frac{1}{\eta_1}}(\mathcal{I},\mathbb{R}^+)} K_{\eta_2} \|h\|_{L^{\frac{1}{\eta_2}}(\mathcal{I},\mathbb{R}^+)} \beta(M) \\
&\leq 2(\kappa_p)^2 \mathcal{K}_2 \mathcal{K}_4 \mathcal{K}_5 b^{1-\zeta+\eta\zeta-\eta\vartheta} \beta(M).
\end{aligned}$$

Then,

$$\begin{aligned}
\beta\big(\Psi(M)(\mathfrak{s})\big) &\leq \beta\big(\Psi_1(M)(\mathfrak{s})\big) + \beta\big(\Psi_2(M)(\mathfrak{s})\big) \\
&\leq \big[1 + \kappa_p \mathcal{K}_2 \mathcal{K}_4\big] 2\kappa_p \mathcal{K}_5 b^{1-\zeta+\eta\zeta-\eta\vartheta} \beta(M), \\
\beta\big(\Psi(M)\big) &\leq \widehat{\mathcal{K}} \beta(M)
\end{aligned}$$

where $\widehat{\mathcal{K}} = \big[1 + \kappa_p \mathcal{K}_2 \mathcal{K}_4\big] 2\kappa_p \mathcal{K}_5 b^{1-\zeta+\eta\zeta-\eta\vartheta} < 1$. Therefore, by $M\ddot{o}nch's$ condition, we obtain

$$\begin{aligned}
&\beta(M) \leq \beta\big(conv\big(\{0\} \cup \Psi(M)\big) = \beta\big(\Psi(M)\big) \leq \widehat{\mathcal{K}} \beta(M), \\
&\beta(M) = 0.
\end{aligned}$$

Therefore, M is relatively compactness.

Hence, from Lemma 2.11, Ψ has a fixed point z on $\mathsf{S}_{\mathsf{q}}(\mathcal{I})$, which is the mild solution of the System (1) satisfying $\mathsf{z}(b) - N(\mathsf{z}) = \mathsf{z}_1$. Hence completed the proof. □

4. Applications

4.1. Application 1. Consider the Hilfer fractional differential systems given below:

$$\begin{cases} D_{0^+}^{\frac{2}{3},\zeta} \mathsf{z}(\mathfrak{s}, y) = \mathsf{z}_y(\mathfrak{s}, y) + \chi\varphi(\mathfrak{s}, y) + \frac{e^{-\mathfrak{s}}}{\lambda + e^{\mathfrak{s}}} \sin\big(\mathsf{z}(\mathfrak{s}, y)\big), \ \mathfrak{s} \in \mathcal{I} = [0, 1], \\ \mathsf{z}(\mathfrak{s}, 0) = \mathsf{z}(\mathfrak{s}, 1) = 0, \\ I_{0^+}^{(1-\frac{2}{3})(1-\zeta)}[\mathsf{z}(0, y)] + \int_0^b h(\rho) \ln\big(1 + |\mathsf{z}(\rho, y)|^{\frac{1}{2}}\big) d\rho = \mathsf{z}_0, \ 0 < y < 1, \end{cases} \tag{1}$$

where $\chi > 0,\ \lambda \geq 1$ and $D_{0^+}^{\frac{2}{3};\zeta}$ is the Hilfer fractional derivative of order $\frac{2}{3}$ and type ζ, $I_{0^+}^{(1-\frac{2}{3})(1-\zeta)}$ is the R-L integral and the function $\varphi : \mathcal{I} \times (0,1) \to (0,1)$ is continuous in $\mathfrak{s}$ and $h \in L^1(\mathcal{I}, \mathbb{R})$.

Consider $Y = C([0,1]),\ U = C([0,1])$, the operator $\mathsf{A} : D(\mathsf{A}) \subset Y \to Y$ is defined as

$$\mathsf{A}x = x',\ x \in D(\mathsf{A}) = \{x \in Y : x' \in Y, x(0) = x(1) = 0\}.$$

Here A is the almost sectorial operator of the semigroup $\{T(\mathfrak{s}),\ \mathfrak{s} \geq 0\}$ in Y, such that $\sup_{\mathfrak{s} \in \mathcal{I}} \|T(\mathfrak{s})\| \leq \mathcal{K}_1$. Furthermore, $\mathfrak{s} \to x(\mathfrak{s}^{\frac{2}{3}}\theta + \rho)\mathsf{z}$ is equicontinuous for $\mathfrak{s} > 0$ and $\theta \in (0, \infty)$.

Set $\mathsf{z}(x)(y) = \mathsf{z}(\mathfrak{s}, y)$,

$$\mathfrak{F}(\mathfrak{s}, \mathsf{z}(\mathfrak{s}))(y) = \frac{e^{-\mathfrak{s}}}{\lambda + e^{\mathfrak{s}}} \sin\big(\mathsf{z}(\mathfrak{s}, y)\big).$$

We have $\mathfrak{F}$ is Lipschitz continuous and satisfies (H_2). Let $\mathbb{B} : U \to Y$ be defined by $(\mathbb{B}v)(\mathfrak{s})(y) = \chi\varphi(\mathfrak{s}, y),\ 0 < y < 1$ and $N : C(\mathcal{I}, Y) \to Y$ given by:

$$N(\mathsf{z})(y) = \int_0^b h(\rho) \ln\big(1 + |\mathsf{z}(\rho)(y)|^{\frac{1}{2}}\big) d\rho,$$

which satisfy the hypotheses (H_4). For $y \in (0,1)$, the operator W is defined as

$$(\mathsf{W}v)(y) = \int_0^1 (1-\rho)^{\frac{-1}{3}} \mathcal{Q}_{\frac{2}{3}}(1-\rho)\chi\varphi(\rho, y) d\rho.$$

For $\mathfrak{s} \in [0,1]$,

$$\mathcal{Q}_{\frac{2}{3}}(\mathsf{z}(\rho)) = \frac{2}{3} \int_0^\infty \theta \mathfrak{W}_{\frac{2}{3}}(\theta) x(\mathfrak{s}^{\frac{2}{3}} + \rho) d\theta,$$

with the Wright type function.

Assume that W satisfies (H_3), then Theorem 3.1 are satisfied. Hence, the System (1) is nonlocal controllable on $\mathcal{I}$.

4.2. Application 2. Suppose the mild solution of the system 1,

$$\mathsf{z}(\mathfrak{s}) = \S_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}(0) - N(\mathsf{z})\big] + \int_0^{\mathfrak{s}} (\mathfrak{s} - \rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s} - \rho)\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big) d\rho$$
$$+ \int_0^{\mathfrak{s}} (\mathfrak{s} - \rho)^{\eta-1} \mathcal{Q}_\eta(\mathfrak{s} - \rho)\mathsf{B}v(\rho) d\rho,\ \mathfrak{s} \in \mathcal{I}. \tag{2}$$

We offer the digital filter system matching to the mild solution in 1 which was inspired by the filter system described in [5, 36]. Any signal processing application relies on digital filters as its foundation. Nowadays, a lot of bio-medical signals relating to the human body are collected for a variety of useful feature extractions. The majority of the aforementioned signals are typically low frequency in nature. These signals describe data relating to various illnesses and conditions for which accuracy is of major relevance. Any digital signal processing filtering system's effectiveness depends on its capacity to reject noise.

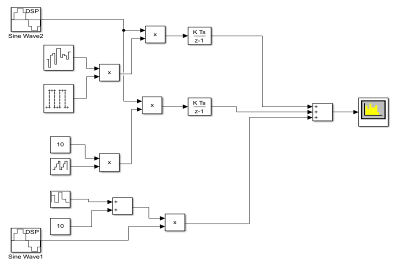

Figure 1

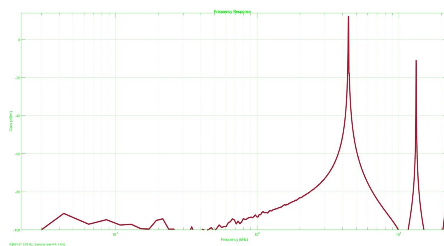

Figure 2

FIGURE 1. Digital filter system model

The Figure 1 describes the input of the filter system, we defined theoretically given by:

(1) Product modulator ($\mathcal{PM}$) 1 receive the input $\mathsf{z}(\mathfrak{s})$ and $\mathfrak{F}$ give the output as $\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)$.
(2) $\mathcal{PM}$ 2 receives the input $\mathcal{Q}_\eta(\mathfrak{s}-\rho)$ and $\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)$ produces the output as $\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)$.
(3) $\mathcal{PM}$ 3 receives the input $v(\rho)$ and $\mathtt{B}$ produces $\mathtt{B}v(\rho)$.
(4) $\mathcal{PM}$ 4 receives the input $\mathcal{Q}_\eta(\mathfrak{s}-\rho)$ and $\mathtt{B}v(\rho)$ produces the output as $\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathtt{B}v(\rho)$.
(5) $\mathcal{PM}$ 5 receives $\big[\mathsf{z}(0)+N(\mathsf{z})\big]$ and $\S_{\eta,\zeta}(\mathfrak{s})$ give the output $\S_{\eta,\zeta}(\mathfrak{s})\big[\mathsf{z}(0)+N(\mathsf{z})\big]$.
(6) Then the integrator execute each value $\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathfrak{F}\big(\rho, \mathsf{z}(\rho)\big)$ and $\mathcal{Q}_\eta(\mathfrak{s}-\rho)\mathtt{B}v(\rho)$ get its integrals.

Finally, we get the output from the integrator to summer network, i.e., the output become the mild solution 2.

The resonant band-pass digital filter implemented as shown in the Figure 1 using Mat lab-simulink through the proposed Equation 2. For the single frequency signal processing application where the selection with accuracy is if great concern this model would shows the better noise rejection ability. Further the above shall also be evident through the frequency response obtained through the simulation as depicted in the Figure 1. The fractional differentiation model based designed resonant band-pass digital filter system exhibits better tenability to single frequency along with significant noise rejection ability.

5. Conclusion

In this paper, we primarily focused on the nonlocal controllability of the Hilfer fractional differential equation via Mönch's fixed point theorem. Application of the findings and concepts from almost sectorial operators, fractional calculus, measure of noncompactness, and the fixed point technique leads to the main conclusions. For nonlocal controllability, we developed the necessary requirements. After that, we provided a theoretical example to illustrate the result, and another example explained a digital filter system for corresponding system. In the future, we will investigate the exact controllability of the Hilfer fractional systems on an infinite interval and also study the existence and controllability of the Hilfer fractional differential system with higher-order derivatives.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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