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ON COMMUTING CONDITIONS OF SEMIRINGS WITH INVOLUTION

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Abstract. In this research article, we study a class of semirings with involution. Differential identities involving two or three derivations of a semiring with second kind involution are investigated. It is analyzed that how these identities, with a special role for second kind involution, bring commutativity to semirings.

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1. Introduction

The theory of semirings has tremendous and direct applications in the sciences. For instance, idempotent analysis based on additive inverse semirings has interesting applications in quantum physics (see[22, 25]) and the same algebraic structure is used to develop the formal languages and automata theory [11, 17, 12, 7, 10]. One can find the applications of semirings in other fields of science and mathematics such as theoretical computer sciences and engineering, parallel computational systems, optimization theory, combinatorics, functional analysis, topology, graph theory, Euclidean geometry, mathematical modeling of quantum physics (see [13, 6, 14, 15]). Javed et al. [18] defined MA-semiring as an additive inverse semirings S with absorbing zero $'0'$ satisfying $w + w' \in Z(S)$ for all $w \in S$, where $Z(S)$ is the center of S and w' is the pseudo inverse of w. In general, the notion of commutators satisfying Jacobian identities that is not sustainable in semirings, is a peculiarity of MA-semirings. The class of MA-semirings has a significant potential to accommodate the study of derivations satisfying different identities on semirings with involution [2, 4, 3] and without involution [1, 21, 29] for probing commuting conditions. The class of

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MA-semirings properly contains the class of rings. In fact, every ring is an MA-semiring but converse may not be true in general. In the following we present some examples of MA-semirings which are not rings.

Example 1.1. *Let* $(\mathbb{Z}, +, .)$ *be the ring of integers and* $I(\mathbb{Z})$ *be the collection of all ideals of* $\mathbb{Z}$. *Consider the set* $S = M_2(\mathbb{Z}) \times I(\mathbb{Z})$ *and let* $u = (A_1, I), v = (A_2, J) \in S$. *Define addition* $\oplus$ *and multiplication* $\odot$ *by* $u \oplus v = (A_1 + A_2, I + J)$ *and* $u \odot v = (A_1A_2, IJ)$. *Then* $(S, \oplus, \odot)$ *is an example of a proper MA-semiring.*

Example 1.2. *Let* $\mathbb{Z}$ *be the set of integers,* $\mathbb{Z}_0^+$ *be the set of all non-negative integers and* $R = \mathbb{Z} \times \mathbb{Z}_0^+$. *Define addition* $\oplus$ *and multiplication* $\odot$ *by* $(u_1, v_1) \oplus (u_2, v_2) = (u_1 + u_2, v_1 \vee v_2)$ *and* $(u_1, v_1) \odot (u_2, v_2) = (u_1.u_2, v_1.v_2)$, *where* $v_1 \vee v_2 = max\{v_1, v_2\}$. *Then the triplet* $(R, \oplus, \odot)$ *forms an MA-semiring which is not a ring.*

Example 1.3. [30] *Let* $(R, +, .)$ *be a ring and* $\mathfrak{L}$ *be a distributive lattice. Consider the set* $S = R \times \mathfrak{L}$ *and let* $u = (r_1, d_1), v = (r_2, d_2) \in S$. *Define addition* $\oplus$ *and multiplication* $\odot$ *respectively as* $u \oplus v = (r_1 + r_2, d_1 \vee d_2)$ *and* $u \odot v = (r_1r_2, d_1 \wedge d_2)$, *where* $\vee$ *and* $\wedge$ *indicate join and meet respectively. Then* $(S, \oplus, \odot)$ *forms an MA-semiring which is not a ring.*

For the ring theoretical background and motivated sources, we would like to refer [8, 23, 24, 26]). Banach $*$-algebra is a special example of ring with involution in functional analysis (see [19, 20, 27, 28]).

We now state some definitions and basic notions which are pertinent to the main section. Throughout this section S denotes an MA-semiring unless otherwise mentioned. Involution is an additive mapping $* : S \longrightarrow S$ that satisfies $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in S$. The sets of Hermitian and skew Hermitian elements are respectively denoted and defined as $\mathbb{H}(S) = \{a \in S : a^* = a\}$ and $\mathbb{K}(S) = \{a \in S : a^* = a'\}$. Involution is of first kind if $Z(S) \subseteq \mathbb{H}(S)$ otherwise it is of second kind. The examples of first and second kind involution for MA-semirings are presented in the following.

Example 1.4. *Consider the MA-semiring* $(S, \oplus, \odot)$ *as described in Example 1.1. Define a mapping* $* : S \longrightarrow S$ *by* $(A, I)^* = (A^T, I)$, *where* A^T *is the transpose of* A. *The mapping* $*$ *defines an involution on* S. *We further see that* $Z(S) \subseteq \mathbb{H}(S)$, *therefore* $*$ *is an involution of first kind.*

Example 1.5. *Consider the MA-semiring* $(R, \oplus, \odot)$ *as described in Example 1.2. Let*

$$M_R = \left\{ \begin{bmatrix} w & v & u & x \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix} : u, v, w, x \in R \right\},$$

where v' *is the pseudo inverse of* v. *Then* M_R *forms an MA-semiring under matrix addition and multiplication. Next, we define a mapping* $* : M_R \longrightarrow M_R$

by

$$\begin{bmatrix} w & v & u & x \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix}^* = \begin{bmatrix} w & v & u & x' \\ 0 & w & 0 & u \\ 0 & 0 & w & v' \\ 0 & 0 & 0 & w \end{bmatrix}.$$

The mapping $*$ defines an involution on M_R. We further see that

$$\begin{bmatrix} w & 0 & 0 & x \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \in Z(M_R)$$

for all $w, x \in R$. For $\begin{bmatrix} w & 0 & 0 & x \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$,
with $x = (u_1, v_1)$ and $u_1 \neq 0$, we can find

$$\begin{bmatrix} w & 0 & 0 & x \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}^* = \begin{bmatrix} w & 0 & 0 & x' \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix}.$$

This means $\begin{bmatrix} w & 0 & 0 & x \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \notin \mathbb{H}(M_R)$.
Thus $Z(M_R) \nsubseteq \mathbb{H}(M_R)$ and hence $*$ is an involution of second kind.

An additive mapping $\varrho : S \longrightarrow S$ is a derivation if $\varrho(ab) = \varrho(a)b + a\varrho(b)$. The Jordan product or anti-commutator of $a, b \in S$ is defined as $a \circ b = ab + ba$. The commutator of $a, b \in S$ is defined as $[a, b] = ab + b'a$. A mapping $\varrho : S \longrightarrow S$ is commuting (centralizing) if $[\varrho(v), v] = 0$ ($[[\varrho(v), v], t] = 0$), for all $v, t \in S$.
One can find MA-semirings, in which well known properties of rings are not valid in general. For example if S is an MA-semiring and $s, t \in S$, then $st = ts$ does not admit $[s, t] = 0$; $[s, s] \neq 0$ if $s \neq 0$; if ϱ is derivation of S and $s \in Z(S)$, then $\varrho(s)$ may not belong to $Z(S)$.
We now compose some lemmas which will be useful for proving the main results.

Lemma 1.6. *Let S be an MA-semiring and ϱ be a derivation of S. Then for all $a, b, c \in S, z \in Z(S)$, we have*

(1) $[a, ab] = a[a, b]$
(2) $[a, bc] = [a, b]c + b[a, c]$
(3) $[ab, c] = a[b, c] + [a, c]b$
(4) $(ab)' = a'b = ab'$
(5) $[a, b] + [b, a] = b(a + a') = a(b + b')$

(6) $[a,b]' = [a,b'] = [a',b] = [b,a]$
(7) $a \circ (b+c) = a \circ b + a \circ c$
(8) $\varrho(a') = (\varrho(a))'$
(9) $[a,bz] = z[a,b] = [a,b]z$
(10) $[a,a] = [a,a]'$
(11) $a+b=0 \Rightarrow a=b'$, *however the converse may not hold in general.*

For more one can see [18, 29].

Throughout the sequel $h_z \in Z(S) \cap \mathbb{H}(S)$ and $k_z \in Z(S) \cap \mathbb{K}(S)$, for the sake of convenience, unless mentioned otherwise.

Lemma 1.7. [3] *Let S be a semiprime MA-semiring with second kind involution $*$. Then $Z(S) \cap \mathbb{K}(S) \neq \{0\}$ and therefore $Z(S) \cap \mathbb{H}(S) \neq \{0\}$.*

From the definition of Hermitian and the skew Hermitian elements of an MA-semiring with second kind involution, one can observe the following.

Remark 1.8. *If S is an MA-semiring with second kind involution $*$, then*
(1) $k^2 \in \mathbb{H}(S)$.
(2) $hh_z \in \mathbb{H}(S)$.
(3) $kk_z \in \mathbb{H}(S)$.
(4) $hk_z \in \mathbb{K}(S)$.

Lemma 1.9. *Let ϱ be a derivation of a 2-torsion free prime MA-semiring S with second kind involution $*$. If $\varrho(h_z) = 0$ for any $h_z \in Z(S) \cap \mathbb{H}(S)$, then $\varrho(k_z) = 0$ for any $k_z \in Z(S) \cap \mathbb{K}(S)$.*

Proof. By the Observation 1.8, for any $k_z \in Z(S) \cap \mathbb{K}(S)$, we have $k_z^2 \in Z(S) \cap \mathbb{H}(S)$. Then $\varrho(k_z^2) = 2k_z\varrho(k_z) = 0$. As S is 2-torsion free, we find $k_z\varrho(k_z) = 0$, which further implies $k_zS\varrho(k_z) = \{0\}$. As $*$ is of second kind and S is prime, by Lemma 1.7, we obtain $\varrho(k_z) = 0$. □

Lemma 1.10. *Let S be a prime MA-semiring S and ϱ be a nonzero derivation satisfying*

$$[\varrho(w), w] = 0 \tag{1}$$

for all $w \in S$. Then S is commutative.

Proof. Linearizing $[\varrho(w), w] = 0$ and using it again, we get

$$[\varrho(w), u] + [\varrho(u), w] = 0 \tag{2}$$

for all $w, u \in S$. In (2) substituting uw for u, we get

$$[\varrho(w), u]w + u[\varrho(w), w] + [\varrho(u), w]w + u[\varrho(w), w] + [u, w]\varrho(w) = 0$$

for all $w, u \in S$ and therefore

$$([\varrho(w), u] + [\varrho(u), w])w + u[\varrho(w), w] + u[\varrho(w), w] + [u, w]\varrho(w) = 0$$

for all $w, u \in S$. Using (1) and (2) again, we get

$$[u, w]\varrho(w) = 0 \tag{3}$$

for all $w, u \in S$. In (3), substituting sv for u and using (3) again, we get $[s,w]S\varrho(w) = \{0\}$ for all $s, w \in S$. By the primeness of S, we have $[s,w] = 0$ or $\varrho(w) = 0$ for all $s, w \in S$. This means $S = S_1 \cup S_2$, where $S_1 = \{w \in S : [s,w] = 0, \text{ for all } s \in S\}$ and $S_2 = \{w \in S : \varrho(w) = 0\}$. We claim that either $S_1 = S$ or $S_2 = S$. For this we show that either $S_2 \subseteq S_1$ or $S_1 \subseteq S_2$. Assuming on the contrary, let $w_1 \in S_1 \setminus S_2$ and $w_2 \in S_2 \setminus S_1$. One can observe that $w_1 + w_2 \in S_1 + S_2 \subseteq S_1 \cup S_2 = S$, therefore we have either $w_1 + w_2 \in S_1$ or $w_1 + w_2 \in S_2$. If $w_1 + w_2 \in S_1$, then $0 = [w_1 + w_2, s] = [w_1, s] + [w_2, s] = [w_2, s]$ for all $s \in S$, which means that $w_2 \in S_1$, a contradiction. Secondly if $w_1 + w_2 \in S_2$, then $0 = \varrho(w_1 + w_2) = \varrho(w_1) + \varrho(w_2) = \varrho(w_1)$ which implies that $w_1 \in S_2$, a contradiction. Therefore we conclude that either $S_1 = S$ or $S_2 = S$. If $S_2 = S$, then $\varrho = 0$, which contradicts the hypothesis. Secondly S is commutative if $S_1 = S$. □

Shakir et al. [5] investigated $*$-differential identities involving pairs of derivations of prime rings with second kind involution $*$. In the main section of this paper, we establish the results of [5] for a certain class of semirings known as MA-semirings with second kind involution. We also present a generalized version of a result of Herstein [16].

2. Main results

An extended version of Theorem 3.1 of [5] is given in the following.

Theorem 2.1. *Let ϱ_1 and ϱ_2 be two derivations of S such that at least one of ϱ_1 and ϱ_2 is non zero. If*

$$[\varrho_1(w), \varrho_1(w^*)] + \varrho_2(w \circ w^*) = 0 \tag{4}$$

for all $w \in S$, then S is commutative.

Proof. **Case 1:** If $\varrho_1 \neq 0$ and $\varrho_2 = 0$, then from (4), we obtain

$$[\varrho_1(w), \varrho_1(w^*)] = 0 \tag{5}$$

for all $w \in S$. Linearizing (5) and using (5) again, we get

$$[\varrho_1(w), \varrho_1(v^*)] + [\varrho_1(v), \varrho_1(w^*)] = 0 \tag{6}$$

for all $w, v \in S$. Substituting vh_z for v in (6) and employing Lemma 1.6, we get

$$([\varrho_1(w), \varrho_1(v^*)] + [\varrho_1(v), \varrho_1(w^*)])h_z + [\varrho_1(w), v^*\varrho_1(h_z)] + [v\varrho_1(h_z), \varrho_1(w^*)] = 0$$

for all $w, v \in S$. Using (6) again, we get

$$[\varrho_1(w), v^*\varrho_1(h_z)] + [v\varrho_1(h_z), \varrho_1(w^*)] = 0. \tag{7}$$

for all $w, v \in S$. Substituting vk_z for v in (7) and employing Lemma 1.6, we obtain

$$([\varrho_1(w), v^*\varrho_1(h_z)]' + [v\varrho_1(h_z), \varrho_1(w^*)])Sk_z = \{0\}$$

for all $w, v \in S$. As $*$ is of second kind and S is prime, using Lemma 1.7, we obtain

$$[\varrho_1(w), v^*\varrho_1(h_z)]' + [v\varrho_1(h_z), \varrho_1(w^*)] = 0$$

for all $w, v \in S$ and hence by the property of pseudo inverse, we have

$$[\varrho_1(w), v^*\varrho_1(h_z)] = [v\varrho_1(h_z), \varrho_1(w^*)] \tag{8}$$

for all $w, v \in S$. Using (8) into (7), we obtain $[\varrho_1(w), v^*\varrho_1(h_z)] = 0$ and substituting v^* for v, we have

$$[\varrho_1(w), v\varrho_1(h_z)] = 0 \tag{9}$$

for all $w, v \in S$. In (9), substituting rv for v and using Lemma 1.6, we obtain

$$r[\varrho_1(w), v\varrho_1(h_z)] + [\varrho_1(w), r]v\varrho_1(h_z) = 0$$

for all $r, w, v \in S$. Using (9) in the last relation, we obtain

$$[\varrho_1(w), r]S\varrho_1(h_z) = \{0\}.$$

As S is prime, we obtain either $[\varrho_1(w), r] = 0$ or $\varrho_1(h_z) = 0$. Assume that $[\varrho_1(w), r] = 0$, for all $w, r \in S$. Then by Lemma 1.10, S is commutative. In view of Lemma 1.9, from the second possibility, we have $\varrho_1(k_z) = 0$. Substituting vk_z for v in (6) and using the fact that $\varrho_1(k_z) = 0$, we obtain

$$[\varrho_1(w), \varrho_1(v^*)]' + [\varrho_1(v), \varrho_1(w^*)] = 0$$

for all $w, v \in S$ and therefore

$$[\varrho_1(w), \varrho_1(v^*)] = [\varrho_1(v), \varrho_1(w^*)] \tag{10}$$

for all $w, v \in S$. As S is 2-torsion freeness, using (10) into (9), we obtain

$$[\varrho_1(w), \varrho_1(v)] = 0 \tag{11}$$

for all $w, v \in S$. In (11) substituting wv for v and using Lemma 1.6, we obtain

$$w[\varrho_1(w), \varrho_1(v)] + [\varrho_1(w), w]\varrho_1(v) + \varrho_1(w)[\varrho_1(w), v] + [\varrho_1(w), \varrho_1(w)]v = 0$$

for all $w, v \in S$. Using (11) again, we obtain

$$[\varrho_1(w), w]\varrho_1(v) + \varrho_1(w)[\varrho_1(w), v] = 0 \tag{12}$$

for all $w, v \in S$. Substituting rv for v in (12), we have

$$[\varrho_1(w), w]r\varrho_1(v) + [\varrho_1(w), w]\varrho_1(r)v + \varrho_1(w)r[\varrho_1(w), v] + \varrho_1(w)[\varrho_1(w), r]v = 0$$

for all $r, w, v \in S$ and using (12) again, we obtain

$$[\varrho_1(w), w]r\varrho_1(v) + \varrho_1(w)r[\varrho_1(w), v] = 0 \tag{13}$$

for all $r, v, w \in S$. Substituting $\varrho_1(v)$ for v in (13), we get

$$[\varrho_1(w), w]r\varrho_1(\varrho_1(v)) + \varrho_1(w)r[\varrho_1(w), \varrho_1(v)] = 0$$

for all $r, w, v \in S$. Using (11) again, we get

$$[\varrho_1(w), w]S\varrho_1(\varrho_1(v)) = \{0\}$$

and by the primeness, we have either $[\varrho_1(w), w] = 0$ or $\varrho_1^2(v) = 0$ for all $v, w \in S$. For the first possibility, by Lemma 1.10, S is commutative. Secondly if $\varrho_1^2(v) = 0$,

then by Theorem 1 of [1], we get $\varrho_1 = 0$, a contradiction.
Case 2: If $\varrho_1 = 0$ and $\varrho_2 \neq 0$, then we obtain

$$\varrho_2(w \circ w^*) = 0$$

for all $w \in S$, from (4). Then by Theorem 2.6 of [3], S is commutative.
Case 3: If $\varrho_1 \neq 0$ and $\varrho_2 \neq 0$. Substituting w^* for w in (4), we get

$$[\varrho_1(w^*), \varrho_1(w)] + \varrho_2(w^* \circ w) = 0$$

for all $w \in S$. As $w \circ v = v \circ w$ and since $[w, v] = [v, w]'$, for all $v, w \in S$, therefore

$$[\varrho_1(w), \varrho_1(w^*)]' + \varrho_2(w \circ w^*) = 0$$

for all $v, w \in S$ and by the above mentioned identities, we can further write

$$[\varrho_1(w), \varrho_1(w^*)] = \varrho_2(w \circ w^*) \tag{14}$$

for all $w \in S$. Using (14) into (4), we get $2\varrho_2(w \circ w^*) = 0$ for all $w \in S$ and because of the 2-torsion freeness of S, we have $\varrho_2(w \circ w^*) = 0$ for all $w \in S$. The remaining part follows through same arguments of **Case 2**.

□

Theorem 2.2 is an extended form of Theorem 3.2 of [5], which can be established through the similar set of calculations of the proof of Theorem 2.1.

Theorem 2.2. *Let ϱ_1 and ϱ_2 be derivations of S such that at least one of ϱ_1 and ϱ_2 is non zero. If*

$$[\varrho_1(w), \varrho_1(w^*)] + \varrho_2(w' \circ w^*) = 0$$

for all $w \in S$, then S is commutative.

In the following, Theorem 3.3 of [5] is demonstrated for MA-semirings with involution.

Theorem 2.3. *Let ϱ_1 and ϱ_2 be derivations of S such that at least one of ϱ_1 and ϱ_2 is non zero. If*

$$\varrho_1(w) \circ \varrho_1(w^*) + \varrho_2[w, w^*] = 0 \tag{15}$$

for all $w \in S$, then S is commutative.

Proof. **Case 1:** If $\varrho_1 = 0$ and $\varrho_2 \neq 0$, then from (15), we obtain $\varrho_2[w, w^*] = 0$ for all $w \in S$ and hence by Lemma 2.5 of [3], S is commutative.
Case 2: If $\varrho_1 \neq 0$ and $\varrho_2 = 0$, then from (15), we obtain

$$\varrho_1(w) \circ \varrho_1(w^*) = 0 \tag{16}$$

for all $w \in S$. Linearizing (16) and using (16) again, we get

$$\varrho_1(w) \circ \varrho_1(v^*) + \varrho_1(v) \circ \varrho_1(w^*) = 0 \tag{17}$$

for all $w \in S$. In (16) substituting vh_z for v, we find

$$(\varrho_1(w) \circ \varrho_1(v^*) + \varrho_1(v) \circ \varrho_1(w^*))h_z + \varrho_1(w) \circ (v^* \varrho_1(h_z)) + (v\varrho_1(h_z)) \circ \varrho_1(w^*) = 0$$

for all $w, v \in S$ and using (17) again, we have

$$\varrho_1(w) \circ (v^* \varrho_1(h_z)) + (v\varrho_1(h_z)) \circ \varrho_1(w^*) = 0 \tag{18}$$

for all $v, w \in S$. Substituting vk_z for v in (18), we obtain

$$(\varrho_1(w) \circ (v^* \varrho_1(h_z))^{'} + (v\varrho_1(h_z)) \circ \varrho_1(w^*))Sk_z = \{0\}$$

for all $w, v \in S$. As S is prime, employing Lemma 1.7, we have

$$\varrho_1(w) \circ (v^* \varrho_1(h_z))^{'} + (v\varrho_1(h_z)) \circ \varrho_1(w^*) = 0$$

for all $v, w \in S$, which further implies

$$\varrho_1(w) \circ (v^* \varrho_1(h_z)) = (v\varrho_1(h_z)) \circ \varrho_1(w^*) \tag{19}$$

for all $v, w \in S$. In view of the 2-torsion freeness of S, using (19) into (18), and then substituting v^* for v, we have $\varrho_1(w) \circ (v\varrho_1(h_z)) = 0$ and therefore

$$v\varrho_1(h_z)\varrho_1(w) + \varrho_1(w)v\varrho_1(h_z) = 0 \tag{20}$$

for all $w, v \in S$. In view of Lemma 1.6, from (20), we can write

$$v\varrho_1(h_z)\varrho_1(w) = \varrho_1(w)v^{'}\varrho_1(h_z) \tag{21}$$

for all $w, v \in S$. In (20), substituting rv for v, we get

$$rv\varrho_1(h_z)\varrho_1(w) + \varrho_1(w)rv\varrho_1(h_z) = 0 \tag{22}$$

for all $r, w, v \in S$. Multiplying (21) by r from the left, we obtain

$$rv\varrho_1(h_z)\varrho_1(w) = r\varrho_1(w)v^{'}\varrho_1(h_z) \tag{23}$$

for all $r, w, v \in S$. Using (23) into (22), we get $[\varrho_1(w), r]S\varrho_1(h_z) = \{0\}$. In view of the primeness of S, employing Lemma 1.7, we obtain either $[\varrho_1(w), r] = 0$ or $\varrho_1(h_z) = 0$. If $[\varrho_1(w), r] = 0$, then commutativity of S follows by Lemma 1.10. On the other hand if $\varrho_1(h_z) = 0$, then by Lemma 1.9, we find $\varrho_1(k_z) = 0$. Substituting vk_z for v in (17)and hence using the primeness of S, we obtain

$$(\varrho_1(w) \circ \varrho_1(v^*))^{'} + \varrho_1(v) \circ \varrho_1(w^*) = 0$$

for all $v, w \in S$ and hence

$$\varrho_1(w) \circ \varrho_1(v^*) = \varrho_1(v) \circ \varrho_1(w^*) \tag{24}$$

for all $v, w \in S$. As S is 2-torsion free, using (24) into (17), we obtain $\varrho_1(w) \circ \varrho_1(v^*) = 0$ which further gives

$$\varrho_1(w)\varrho_1(v) + \varrho_1(v)\varrho_1(w) = 0 \tag{25}$$

for all $w, v \in S$. From (25), we can write

$$\varrho_1(w)\varrho_1(v) = \varrho_1(v^{'})\varrho_1(w) \text{ for all } w, v \in S. \tag{26}$$

for all $w, v \in S$. Substituting rw for w in (25), we obtain

$$r\varrho_1(w)\varrho_1(v) + \varrho_1(r)u\varrho_1(v) + \varrho_1(v)r\varrho_1(w) + \varrho_1(v)\varrho_1(r)w = 0$$

for all $r, w, v \in S$ and using (25) again

$$r^{'}\varrho_1(v)\varrho_1(w) + \varrho_1(r)w\varrho_1(v) + \varrho_1(v)r\varrho_1(w) + \varrho_1(r)\varrho_1(v)w^{'} = 0$$

for all $r, w, v \in S$ and after the rearrangement of the terms, we obtain

$$r^{'}\varrho_1(v)\varrho_1(w) + \varrho_1(v)r\varrho_1(w) + \varrho_1(r)w\varrho_1(v) + \varrho_1(r)\varrho_1(v)w^{'} = 0$$

for all $r, w, v \in S$. Therefore

$$[\varrho_1(v), r]\varrho_1(w) + \varrho_1(r)[w, \varrho_1(v)] = 0 \tag{27}$$

for all $r, w, v \in S$. In (27) replacing w by $\varrho_1(v)$, we get

$$[\varrho_1(v), r]\varrho_1(\varrho_1(v)) + \varrho_1(r)[\varrho_1(v), \varrho_1(v)] = 0 \tag{28}$$

for all $r, w, v \in S$. By Lemma 1.6, we can write $[w, w] = [w, w]^{'}$ for all $w \in S$, therefore from (28) we have

$$[\varrho_1(v), r]\varrho_1(\varrho_1(v)) + \varrho_1(r)[\varrho_1(v), \varrho_1(v)]^{'} = 0$$

for all $r, v \in S$ which further implies

$$[\varrho_1(v), r]\varrho_1(\varrho_1(v)) = \varrho_1(r)[\varrho_1(v), \varrho_1(v)] \tag{29}$$

for all $r, v \in S$. Using (29) into (28) and we obtain

$$[\varrho_1(v), r]\varrho_1(\varrho_1(v)) = 0,$$

which further implies $[\varrho_1(v), r]S\varrho_1(\varrho_1(v)) = \{0\}$. By the primeness, we find $[\varrho_1(v), r] = 0$ or $\varrho_1^2(v) = 0$ for all $v \in S$. If $\varrho_1^2(v) = 0$, then by Theorem 1 of [1], we get $\varrho_1 = 0$, which contradicts our assumption. On the other hand, if $[\varrho_1(v), r] = 0$, then commutativity of S follows through Lemma 1.10.
Case 3: If $\varrho_1 \neq 0$ and $\varrho_2 \neq 0$. In (15) replacing w by w^*, we obtain

$$\varrho_1(w^*) \circ \varrho_1(w) + \varrho_2[w^*, w] = 0$$

As $w \circ v = v \circ w$ and $[w, v] = [v, w]^{'}$, for all $w, v \in S$, therefore from the last equation, we have

$$\varrho_1(w) \circ \varrho_1(w^*) + \varrho_2[w, w^*]^{'} = 0$$

which further implies

$$\varrho_1(w) \circ \varrho_1(w^*) = \varrho_2[w, w^*] \tag{30}$$

for all $w \in S$. As S is 2-torsion free, using (30) into (15), we obtain $\varrho_2[w, w^*] = 0$ for all $w \in S$. Commutativity of S follows through Lemma 2.5 of [3]. □

Following result presents the Theorem 3.4 of [5] in an extended form.

Theorem 2.4. *Let ϱ_1 and ϱ_2 be derivations of S such that at least one of ϱ_1 and ϱ_2 is non zero. If*

$$\varrho_1(w) \circ \varrho_1(w^*) + \varrho_2[w, w^*]^{'} = 0$$

for all $w \in S$, then S is commutative.

Following result presents an extended form of the Theorem 3.5 of [5].

Theorem 2.5. *Let ϱ be a nonzero derivation of S satisfying*

$$\varrho([w,w^*]) + [\varrho(w),\varrho(w^*)] = 0 \tag{31}$$

for all $w \in S$. Then S is commutative.

Proof. Linearizing (31) and again using (31), we obtain

$$\varrho([w,v^*]) + \varrho([v,w^*]) + [\varrho(w),\varrho(v^*)] + [\varrho(v),\varrho(w^*)] = 0 \tag{32}$$

for all $w, v \in S$. Substituting vh_z for v in (32), and using Lemma 1.6, we obtain

$$\begin{aligned}(\varrho[w,v^*] + \varrho[v,w^*] + [\varrho(w),\varrho(v^*)] + [\varrho(v),\varrho(w^*)])h_z \\ + [w,v^*]\varrho(h_z) + [v,w^*]\varrho(h_z) + [\varrho(w),v^*\varrho(h_z)] + [v\varrho(h_z),\varrho(w^*)] = 0\end{aligned}$$

for all $w, v \in S$. Using (32) in the last identity and then after the replacement of v by v^*, we obtain

$$[w,v]\varrho(h_z) + [\varrho(w),v\varrho(h_z)] + [v^*,w^*]\varrho(h_z) + [v^*\varrho(h_z),\varrho(w^*)] = 0 \tag{33}$$

for all $w, v \in S$. In (33) substituting vk_z for v, we get

$$([w,v]\varrho(h_z) + [\varrho(w),v\varrho(h_z)] + [v^*,w^*]'\varrho(h_z) + [v^*\varrho(h_z),\varrho(w^*)]')Sk_z = \{0\}$$

for all $w, v \in S$. In view of Lemma 1.7, using the primeness of S, we have

$$([w,v]\varrho(h_z) + [\varrho(w),v\varrho(h_z)] + [v^*,w^*]'\varrho(h_z) + [v^*\varrho(h_z),\varrho(w^*)]') = 0$$

for all $w, v \in S$ and therefore

$$[w,v]\varrho(h_z) + [\varrho(w),v\varrho(h_z)] = [v^*,w^*]\varrho(h_z) + [v^*\varrho(h_z),\varrho(w^*)] \tag{34}$$

for all $w, v \in S$. In view of the 2-torsion freeness of S, using (34) into (33), we obtain

$$[w,v]\varrho(h_z) + [\varrho(w),v\varrho(h_z)] = 0 \tag{35}$$

for all $w, v \in S$. In (35), substituting wv for v and employing Lemma 1.6, we obtain

$$w[w,v]\varrho(h_z) + w[\varrho(w),v\varrho(h_z)] + [\varrho(w),w]v\varrho(h_z) = 0$$

for all $w, v \in S$ and using (35) again, we get $[\varrho(w),w]S\varrho(h_z) = \{0\}$. As S is prime, from the last relation we conclude that either $[\varrho(w),w] = 0$ for all $w \in S$ or $\varrho(h_z) = 0$. If $[\varrho(w),w] = 0$ for all $w \in S$, then by Lemma 1.10, S is commutative. Secondly if $\varrho(h_z) = 0$, then by Lemma 1.9, we have $\varrho(k_z) = 0$. Substituting vk_z for v in (32) and then using $\varrho(k_z) = 0$, we have

$$\varrho([w,v^*])' + \varrho([v,w^*]) + [\varrho(w),\varrho(v^*)]' + [\varrho(v),\varrho(w^*)] = 0$$

for all $w, v \in S$, which further gives

$$\varrho[w,v^*] + [\varrho(w),\varrho(v^*)] = \varrho([v,w^*]) + [\varrho(v),\varrho(w^*)] \tag{36}$$

for all $w, v \in S$. As S is 2-torsion free, using (36) into (32), we obtain

$$\varrho[w,v^*] + [\varrho(w),\varrho(v^*)] = 0$$

and by replacing v by v^*, it further gives

$$\varrho[w,v] + [\varrho(w), \varrho(v)] = 0 \tag{37}$$

for all $w, v \in S$. In view of Lemma 1.6, replacing v by vw in (37) and then using (37) again, we get

$$[w,v]\varrho(w) + [\varrho(w), v]\varrho(w) + \varrho(v)[\varrho(w), w] = 0 \tag{38}$$

for all $w, v \in S$. In (38) replacing v by rv and using Lemma 1.6, we obtain

$$[w,r]v\varrho(w) + r[w,v]\varrho(w) + [\varrho(w), r]v\varrho(w) + r[\varrho(w), v]\varrho(w) + r\varrho(v)[\varrho(w), w] + \varrho(r)v[\varrho(w), w] = 0$$

for all $r, w, v \in S$ and using (38) again, we have

$$[w,r]v\varrho(w) + [\varrho(w), r]v\varrho(w) + \varrho(r)v[\varrho(w), w] = 0$$

and replacing r by w, we get

$$[w,w]v\varrho(w) + [\varrho(w), w]v\varrho(w) + \varrho(w)v[\varrho(w), w] = 0 \tag{39}$$

for all $w, v \in S$. Using Lemma 1.6 in (39), we can write

$$[w,w]'v\varrho(w) + [\varrho(w), w]v\varrho(w) + \varrho(w)v[\varrho(w), w] = 0$$

for all $w, v \in S$, which further implies

$$[w,w]v\varrho(w) = [\varrho(w), w]v\varrho(w) + \varrho(w)v[\varrho(w), w] \tag{40}$$

for all $w, v \in S$. Using (40) into (39) and then by the 2-torsion freeness, we have

$$[\varrho(w), w]v\varrho(w) + \varrho(w)v[\varrho(w), w] = 0$$

for all $w, v \in S$. Let $a = [\varrho(w), w]$ and $b = \varrho(w)$. Then we can write

$$avb + bva = 0$$

and by Lemma 5 of [2], for all $w \in S$, we have either $[\varrho(w), w] = a = 0$ or $\varrho(w) = b = 0$. If $\varrho(w) = 0$, then $\varrho = 0$, a contradiction. Secondly if $[\varrho(w), w] = 0$ for all $w \in S$, then commutativity of S follows through Lemma 1.10. □

Theorem 2.6 is an extended form of Theorem 3.6 of [5], which can be demonstrated using the same reasoning as Theorem 2.5.

Theorem 2.6. *Let ϱ be a nonzero derivation of a 2-torsion free prime MA-semiring S satisfying*

$$\varrho(u \circ u^*) + \varrho(u) \circ \varrho(u^*) = 0$$

for all $u \in S$. Then S is commutative.

From the aforementioned results, we can derive the following corollaries.

Corollary 2.7. *Let ϱ_1 and ϱ_2 be derivations of a 2-torsion free prime MA-semiring S such that at least one of ϱ_1 and ϱ_2 is nonzero. If one of the following identities holds*

(1) $[\varrho_1(w), \varrho_1(v)] + \varrho_2(w \circ v) = 0$

(2) $[\varrho_1(w), \varrho_1(v)] + \varrho_2(w^{'} \circ v) = 0$

for all $v, w \in S$, *then* S *is commutative.*

Corollary 2.8. *Let* ϱ_1 *and* ϱ_2 *be derivations of a 2-torsion free prime MA-semiring* S *such that at least one of* ϱ_1 *and* ϱ_2 *is nonzero. If one of the following identities holds*

(1) $\varrho_1(w) \circ \varrho_1(v) + \varrho_2[w, v] = 0$

(2) $\varrho_1(w) \circ \varrho_1(v) + \varrho_2[w, v]^{'} = 0$

for all $v, w \in S$, *then* S *is commutative.*

Corollary 2.9. *Let* S *be 2-torsion free prime MA-semiring and* ϱ *be a nonzero derivation such that*

$$[\varrho(w), \varrho(v)] = 0$$

for all $w, v \in S$. *Then* S *is commutative.*

In the following, a generalized version of Herstein's theorem [16] and its extended formats established in [5, 9] is presented.

Corollary 2.10. *Let* S *be 2-torsion free prime MA-semiring with second kind involution* $*$ *and* ϱ *be a nonzero derivation such that*

$$[\varrho(w), \varrho(w^*)] = 0$$

for all $w \in S$. *Then* S *is commutative.*

Corollary 2.11. *Let* ϱ *be a nonzero derivation of a 2-torsion free prime MA-semiring* S *with second kind involution* $*$ *satisfying*

$$\varrho(w) \circ \varrho(w^*) = 0$$

for all $w \in S$. *Then* S *is commutative.*

Corollary 2.12. *Let* ϱ *be a nonzero derivation of a 2-torsion free prime MA-semiring* S *with second kind involution* $*$ *satisfying*

$$[\varrho(w), \varrho(w^*)] = 0$$

for all $w \in S$. *Then* S *is commutative.*

Corollary 2.13. *Let* ϱ *be a nonzero derivation of a 2-torsion free prime MA-semiring* S *with second kind involution* $*$ *satisfying*

$$\varrho(w \circ w^*) = 0$$

for all $w \in S$. *Then* S *is commutative.*

Corollary 2.14. *Let* ϱ *be a nonzero derivation of a 2-torsion free prime MA-semiring* S *with second kind involution* $*$ *satisfying*

$$\varrho(ww^*) = 0 \tag{41}$$

for all $w \in S$. *Then* S *is commutative.*

Proof. In (41) replacing w by w^*, we obtain

$$\varrho(w^*w) = 0 \tag{42}$$

for all $w \in S$. Adding (41) and (42), we get

$$\varrho(w \circ w^*) = 0$$

for all $w \in S$. By Corollary 2.13 S is commutative. □

Corollary 2.15. *Let ϱ be a nonzero derivation of a 2-torsion free prime MA-semiring S with second kind involution $*$ satisfying*

$$\varrho(ww^*) + \varrho(w)\varrho(w^*) = 0 \tag{43}$$

for all $w \in S$. Then S is commutative.

Proof. In (43) replacing w by w^* and taking pseudo inverse we get

$$\varrho(w^*w^{'}) + \varrho(w^*)\varrho(w^{'}) = 0, \text{ for all } w \in S. \tag{44}$$

Adding (43) and (44), we get

$$\varrho[w, w^*] + [\varrho(w), \varrho(w^*)] = 0,$$

for all $w \in S$. Thus by Theorem 2.5, S is commutative. □

3. Conclusions

We have studied a certain class of semirings (known as MA-semirings) with involution of second kind satisfying various identities involving two or three derivations. We have established commutativity and other interesting features of semirings through differential identities, with a key role for second-kind involution. The research work presented in this paper motivates others to investigate and prove the results for semiprime semirings. Furthermore, investigating the differential identities for Lie or Jordan ideals of semirings would also be an interesting open problem for the researchers.

Conflicts of interest : The authors declare that they have no conflict of interest.

Data availability : The data is available on the request.

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